



DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM PART II

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<https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start>

Based on [Berg], [Reiberg] and [Nandy]

Version from 12.11.2019

Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

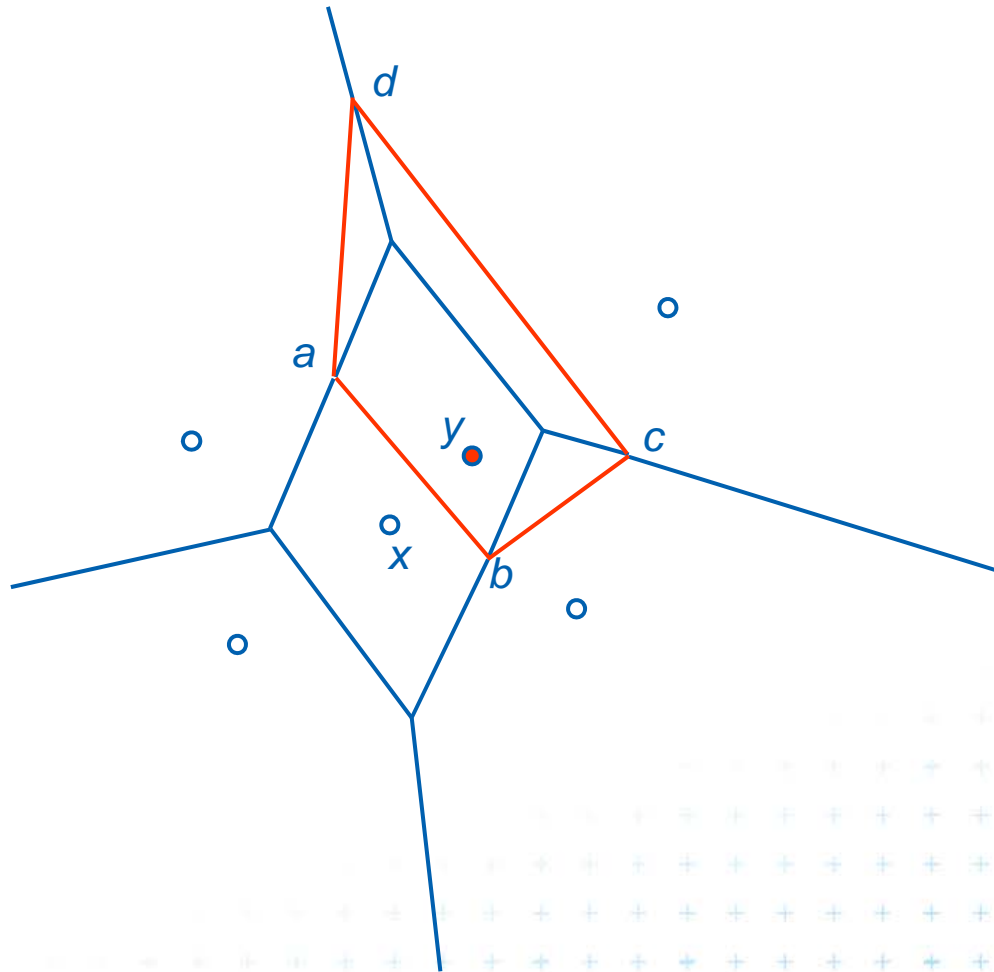


Summary of the VD terms

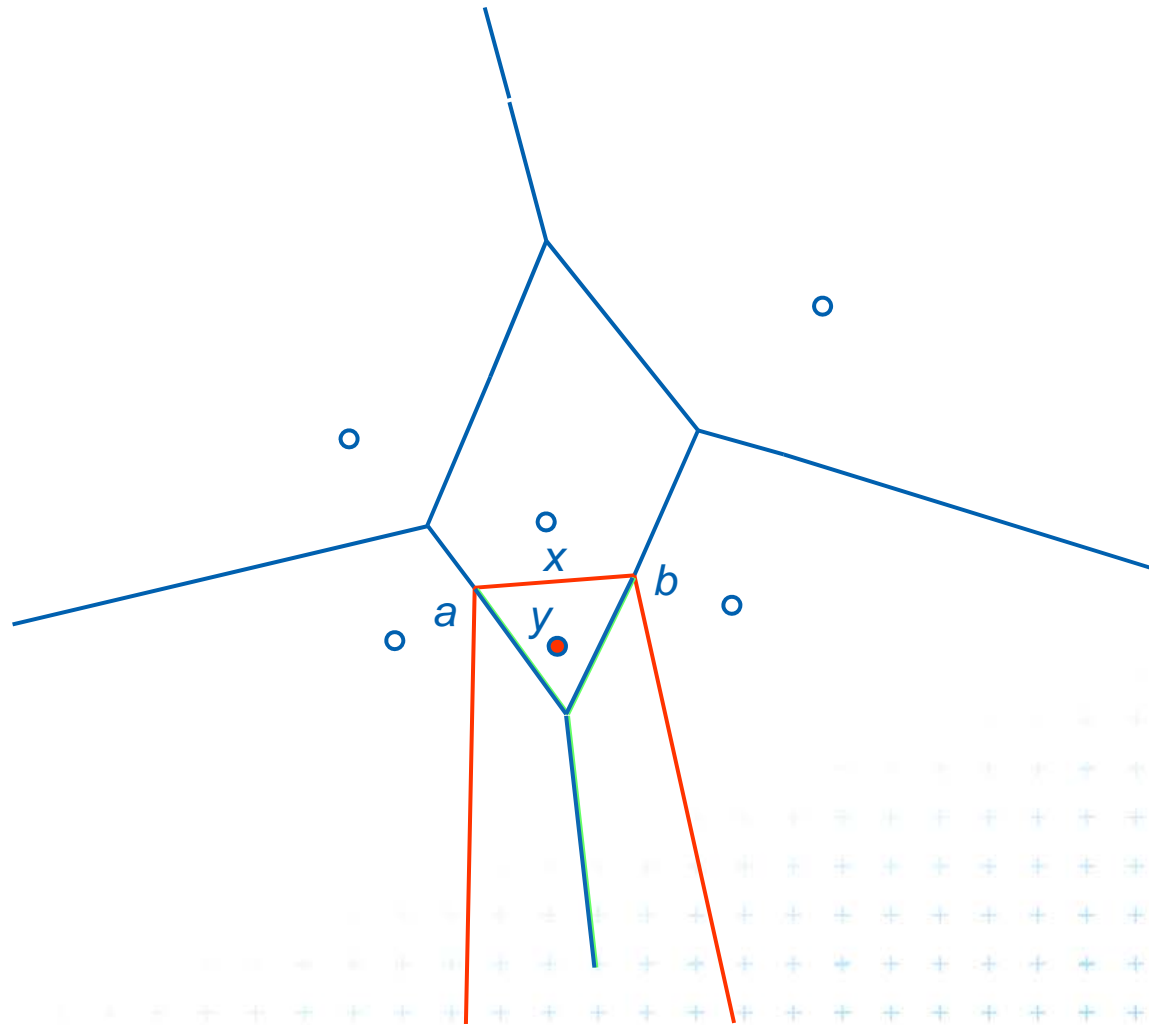
- Site = input point, line segment, ...
- Cell = area around the site, in VD_1 the nearest to site
- Edge, arc = part of Voronoi diagram (border between cells)
- Vertex = intersection of VD edges



Incremental construction – bounded cell



Incremental construction – unbounded cell



Incremental construction algorithm

InsertPoint(S, Vor(S), y) ... **y = a new site**

Input: Point set S, its Voronoi diagram, and inserted point $y \notin S$

Output: VD after insertion of **y**

1. Find the site **x** in which cell point **y** falls, ... $O(\log n)$

2. Detect the intersections $\{a,b\}$ of bisector $L(x,y)$ with cell **x** boundary

_____ => create the first edge $e = ab$ on the border of site **x** ... $O(n)$

3. Set start intersection point $p = b$, set new intersection $c = \text{undef}$

4. site $z = \text{neighbor site}$ across the border with intersection b ... $O(1)$

5. **while**(exists(p) and $c \neq a$) // trace the bisectors from b in one direction

a. Detect intersection c of $L(y,z)$ with border of cell z

b. Report Voronoi edge pc

_____ c. $p = c$, $z = \text{neighbor site}$ across border with intersec. c

5. **if**($c \neq a$) **then** // trace the bisectors from a in other direction

a. $p = a$

b. Similarly as in steps 3,4,5 with a

} ... $O(n^2)$

$O(n^2)$ worst-case, $O(n)$ expected time for some distributions

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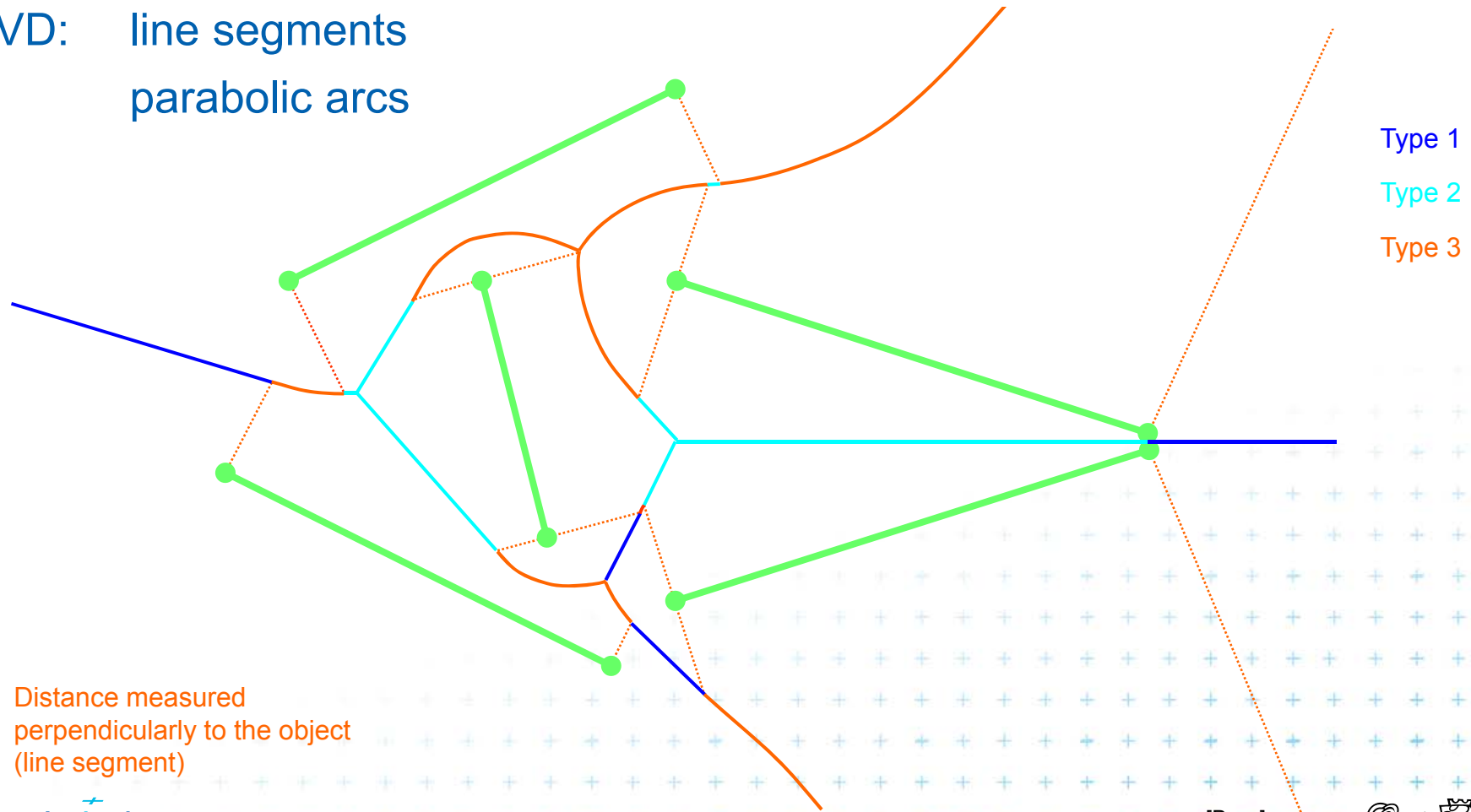
Voronoi diagram of line segments



Voronoi diagram of line segments

Input: $S = \{s_1, \dots, s_n\}$ = set of n disjoint line segments (sites)

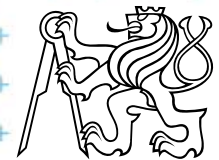
VD: line segments
parabolic arcs



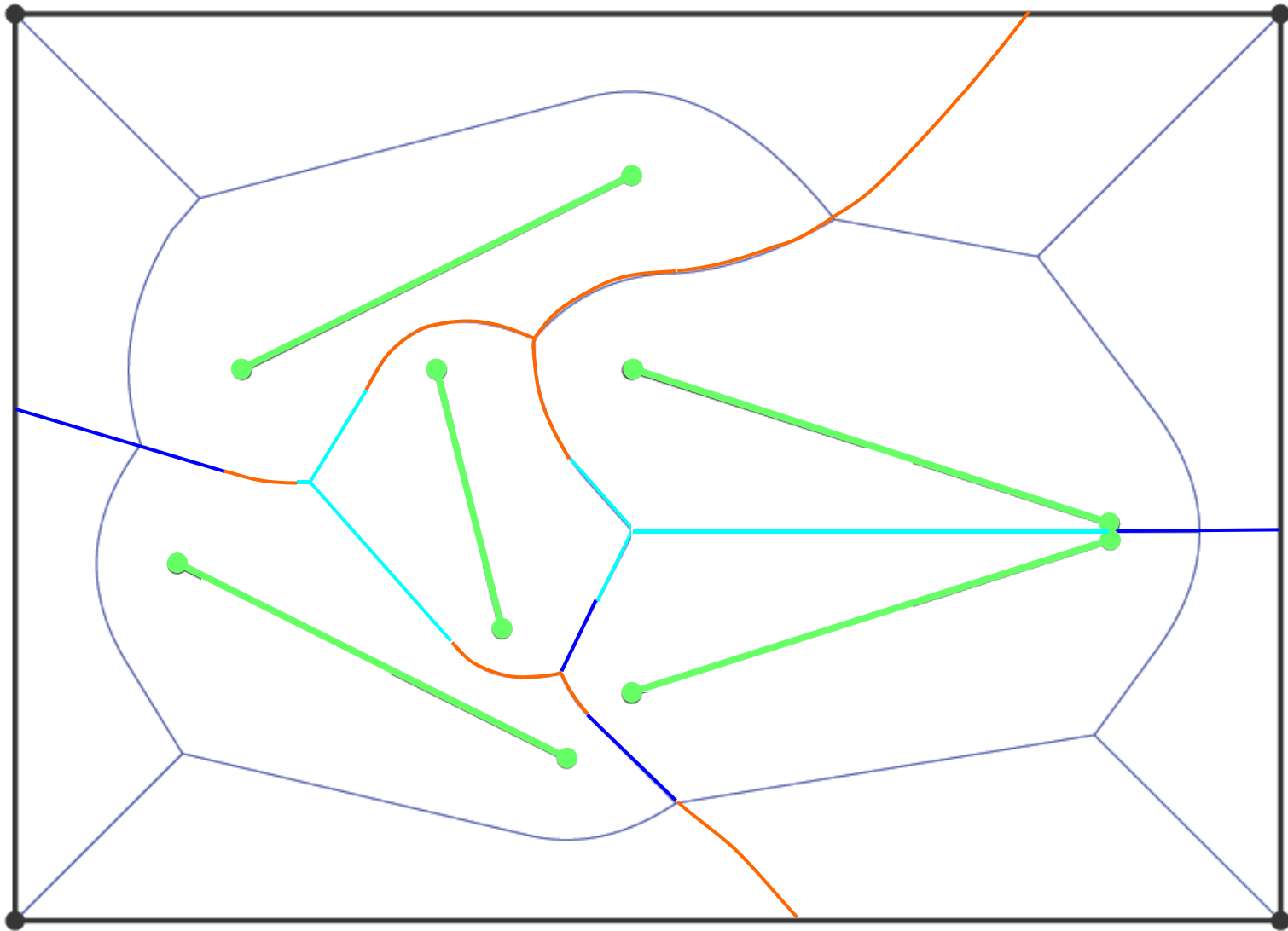
Distance measured
perpendicularly to the object
(line segment)



[Berg]



VD of line segments with bounding box



BBOX
=>
standard
DCEL

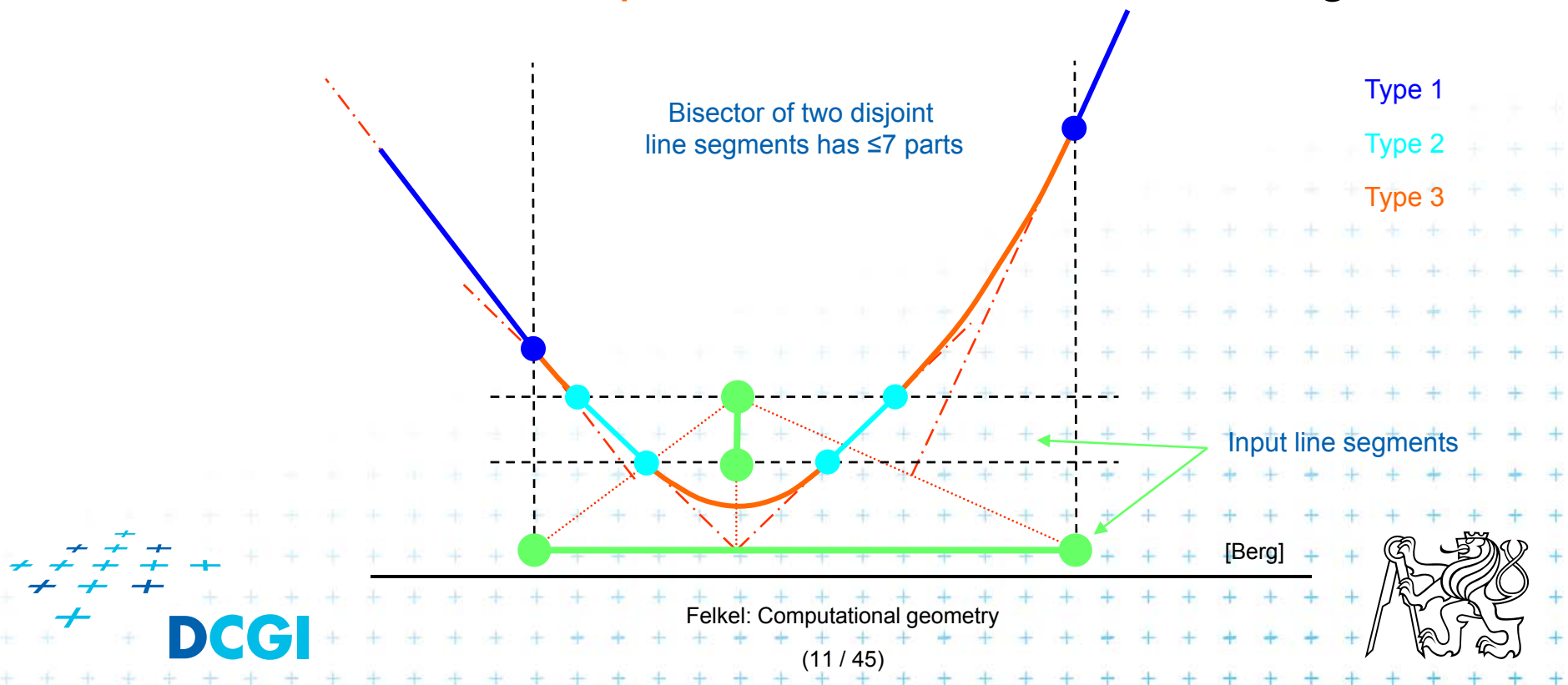


VD of 2 line-segments in detail

- Consists of line segments and parabolic arcs

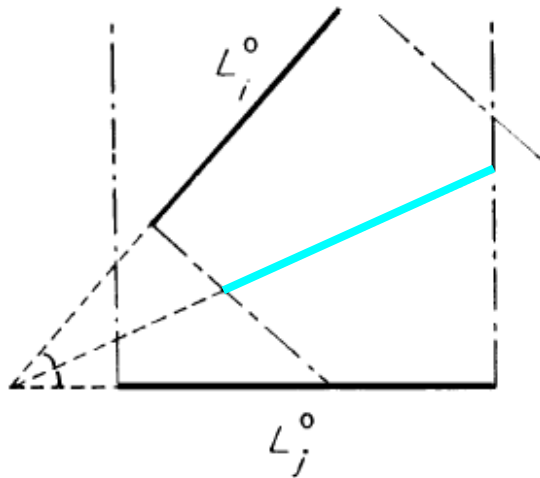
Distance from point-to-object is measured to the closest point on the object (perpendicularly to the object silhouette)

- **Line segment** – bisector of **end-points**₍₁₎ or of **interiors**₍₂₎
- **Parabolic arc** – of **point and interior**₍₃₎ of a line segment



VD in greater details

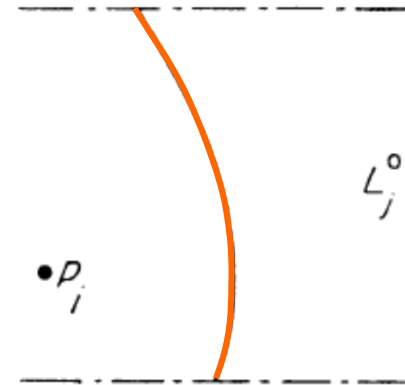
Type 2



Bisector of two
line segment interiors

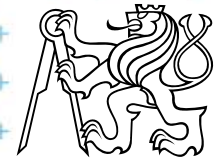
(in intersection of perpendicular slabs only)

Type 3



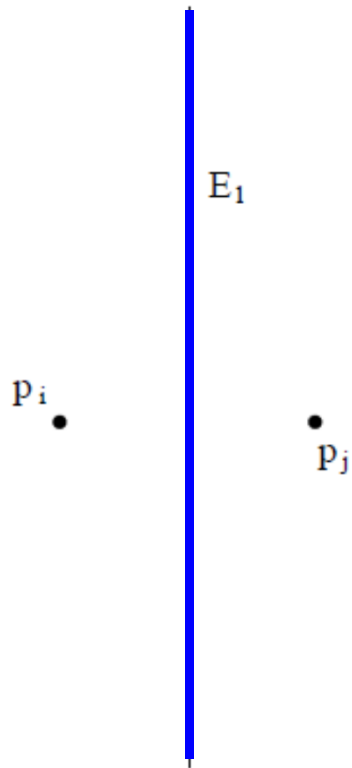
[Reiberg]

Bisector of (end-)point and
line segment interior



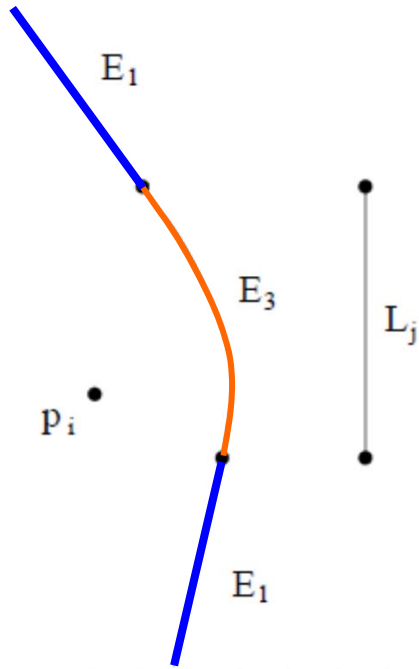
VD of points and line segments examples

2 points



Type 1

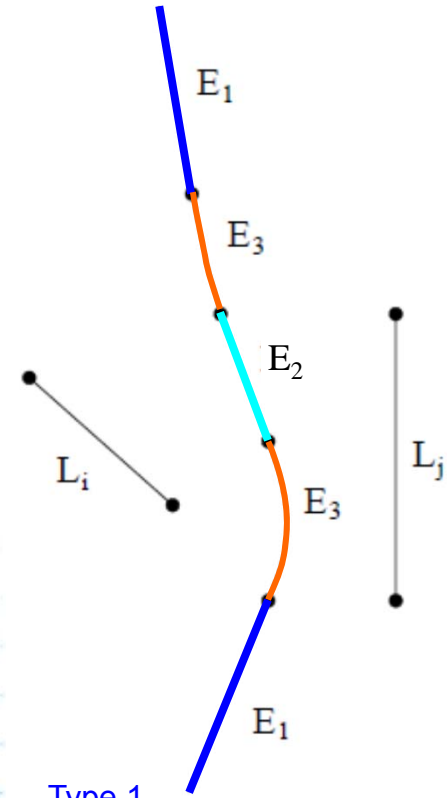
Point & segment



Type 1

Type 3

2 line segments

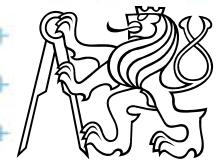


Type 1

Type 2

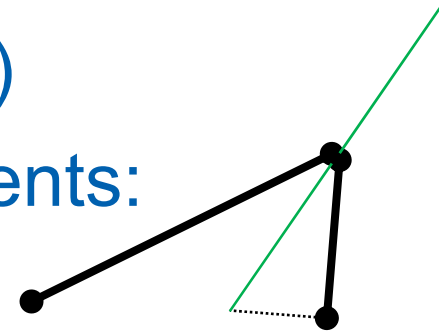
Type 3

[Reiberg]



Voronoi diagram of line segments

- More complex bisectors of line segments
 - VD contains line segments and parabolic arcs
- Still combinatorial complexity of $O(n)$
- Assumptions on the input line segments:
 - non-crossing
 - strictly disjoint end-points (slightly shorten the segm.)



if (we allow touching segments)

Shared endpoints cause complication:

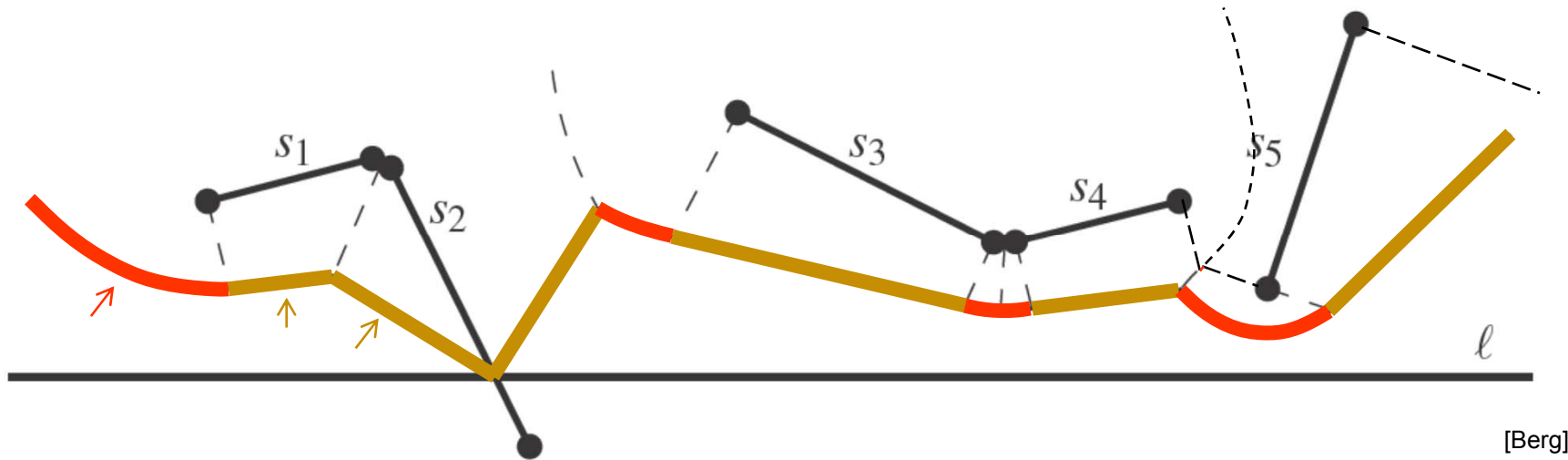
The whole region is equally close to two line segments



Fortune's algorithm for line segments



Shape of beach line for line segments

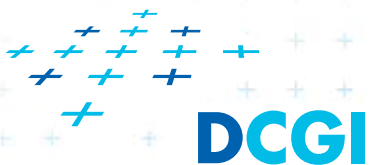


Beach line = points with distance to the closest site above sweep line l equal to the distance to l

Beach line contains

- *parabolic arcs* when closest to a site end-point
- *straight line segments* when closest to a site interior (or just the part of the site interior above l if the site s intersects l)

(This is the shape of the beach line)



Beach line breakpoints types

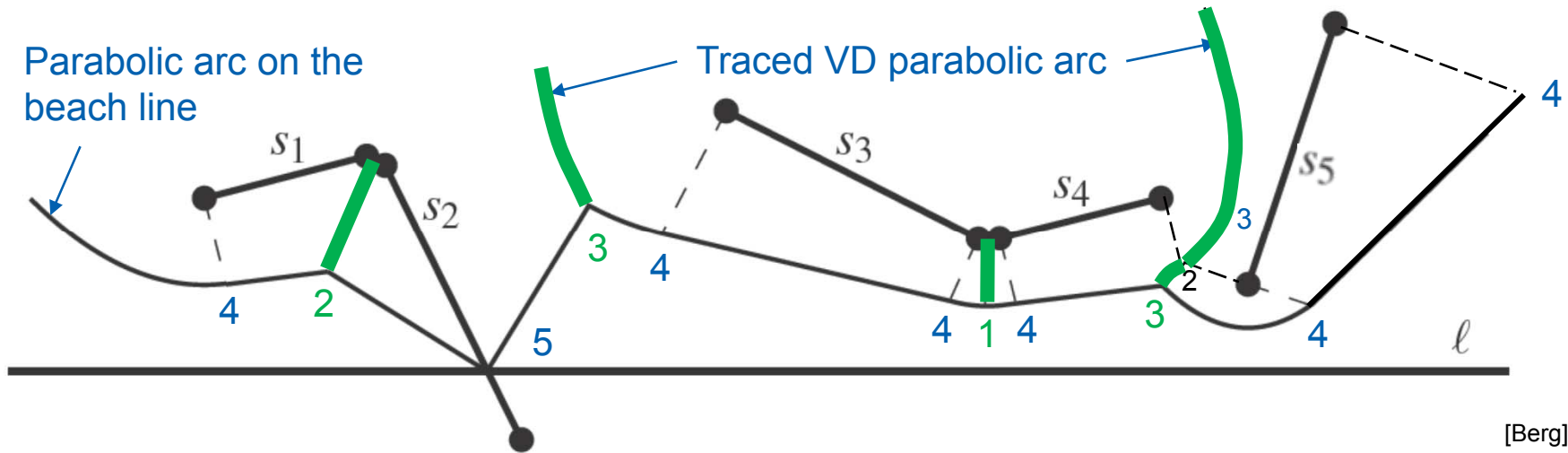
Breakpoint p is equidistant from l and equidistant and closest to:

- | | | |
|----------|--|---|
| points | 1. two site end-points | => p traces a VD line segment |
| segments | 2. two site interiors | => p traces a VD line segment |
| | 3. end-point and interior | => p traces a VD parabolic arc |
| | 4. one site end-point | => p traces a line segment
(border of the slab
perpendicular to the site) |
| | 5. site interior intersects
the scan line l | => $p =$ intersection, traces
the input line segment |

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg. only)



Breakpoints types - what they trace on VD




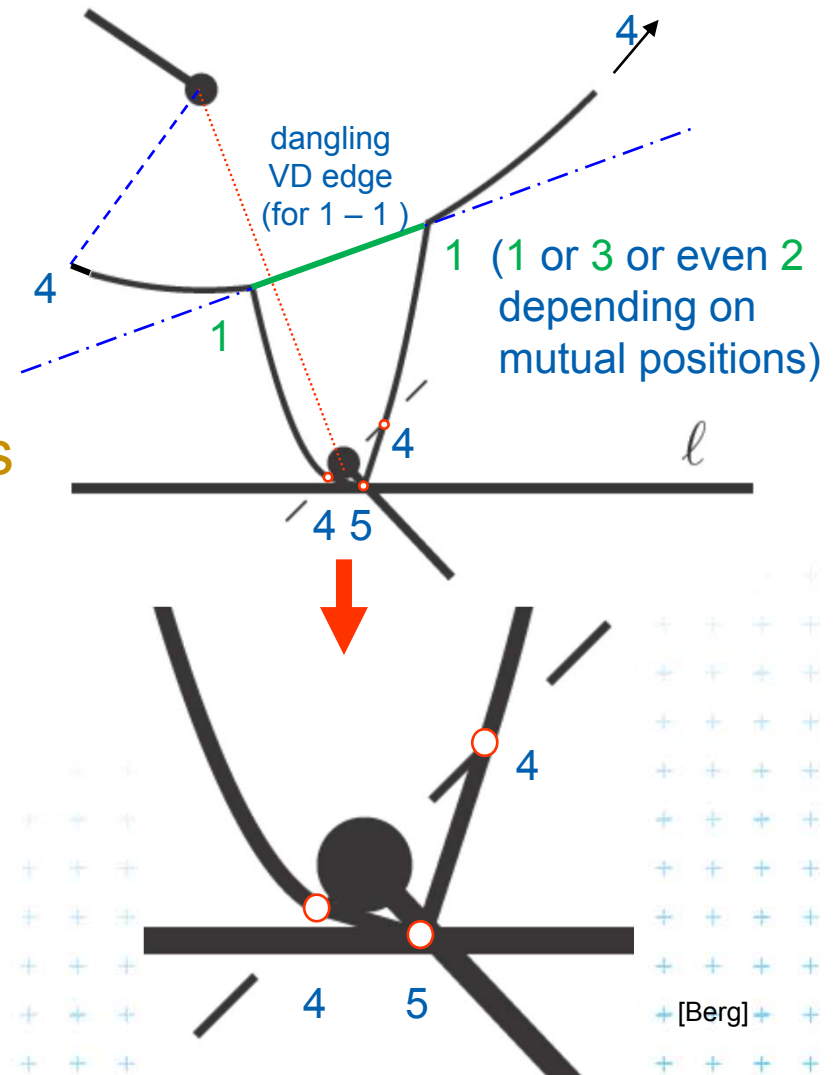
- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
 - 4 limits the slab perpendicular to the line segment
 - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)



Site event – sweep line reaches an endpoint

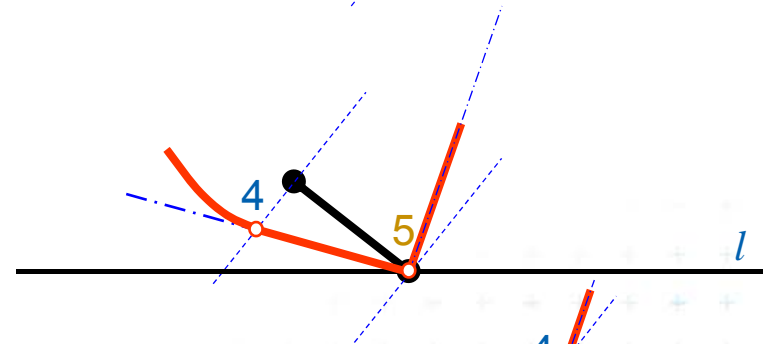
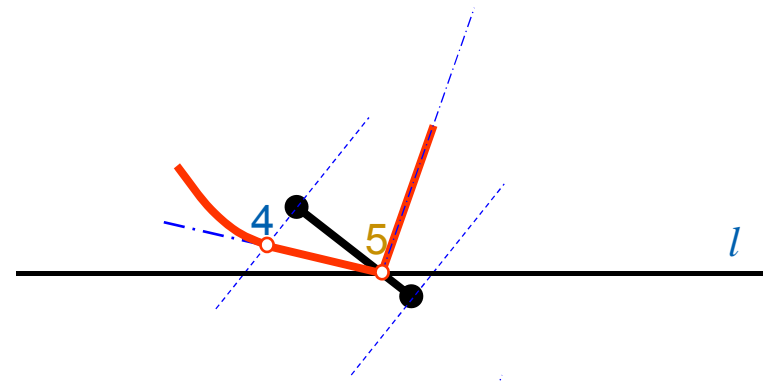
- I. At **upper endpoint** of 
- Arc above is split into two
 - four new arcs are created (2 segments + 2 parabolas)
 - Breakpoints for two **segments** are of type 4-5-4
 - Breakpoints for **parabolas** depend on the surrounding sites
 - Type **1** for two end-points
 - Type **3** for endpoint and interior
 - etc...



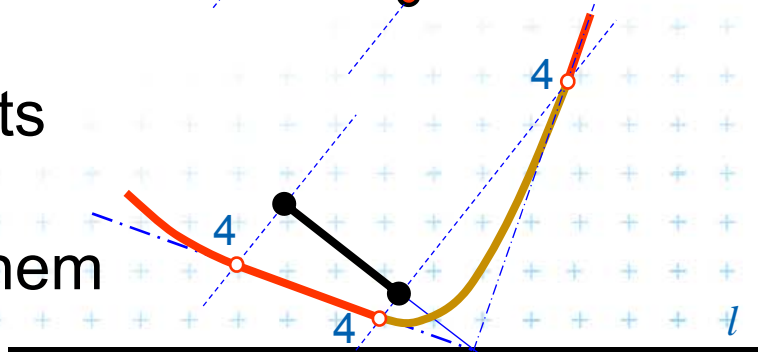
Site event – sweep line reaches an endpoint

II. At **lower endpoint** of 

- Intersection with interior
(**breakpoint of type 5**)



- is replaced by two breakpoints
(of type **4**)
with **parabolic arc** between them

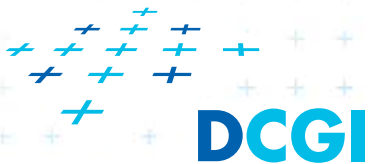


Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
 - Any of first three types (1,2,or 3) meet (circle event)
 - 3 sites involved – Voronoi vertex created
 - Type 4 (*segment interiors*) with something else
 - two sites involved – breakpoint changes its type
 - Voronoi vertex not created
(Voronoi edge may change its shape)
 - Type 5 (*on segment*) with something else
 - never happens for disjoint segments
(meet with type 4 happens before)



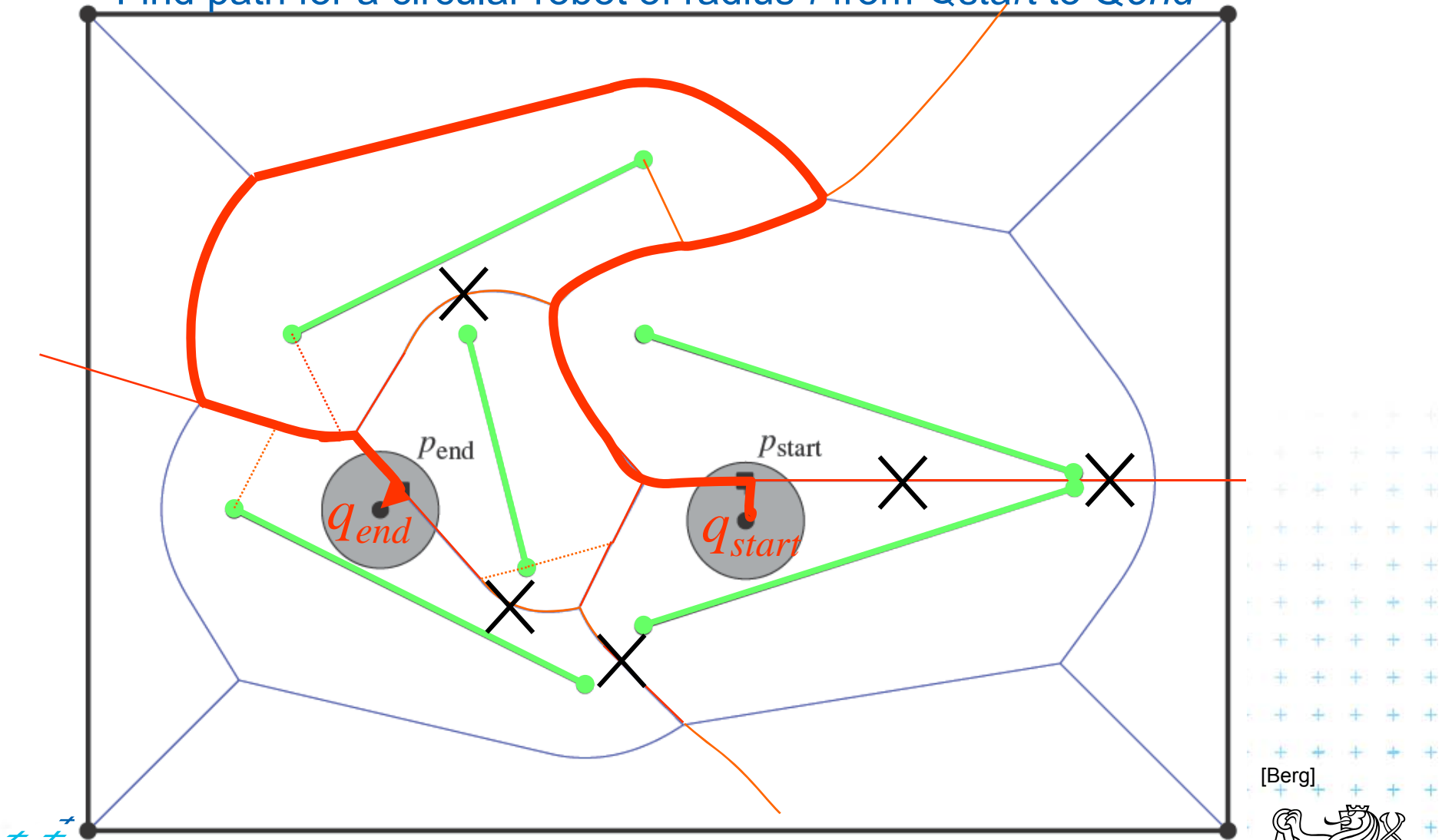
Motion planning example



Motion planning example - retraction

Rušení hran

Find path for a circular robot of radius r from Q_{start} to Q_{end}



[Berg]



Motion planning example - retraction Rušení hran

Find path for a circular robot of radius r from Q_{start} to Q_{end}

- Create Voronoi diagram of line segments, take it as a graph
- Project Q_{start} to P_{start} on VD and Q_{end} to P_{end}
- Remove segments with distance to sites smaller than radius r of a robot
- Depth first search if path from P_{start} to P_{end} exists
- Report path $Q_{start} P_{start} \dots path \dots P_{end} Q_{end}$

- $O(n \log n)$ time using $O(n)$ storage



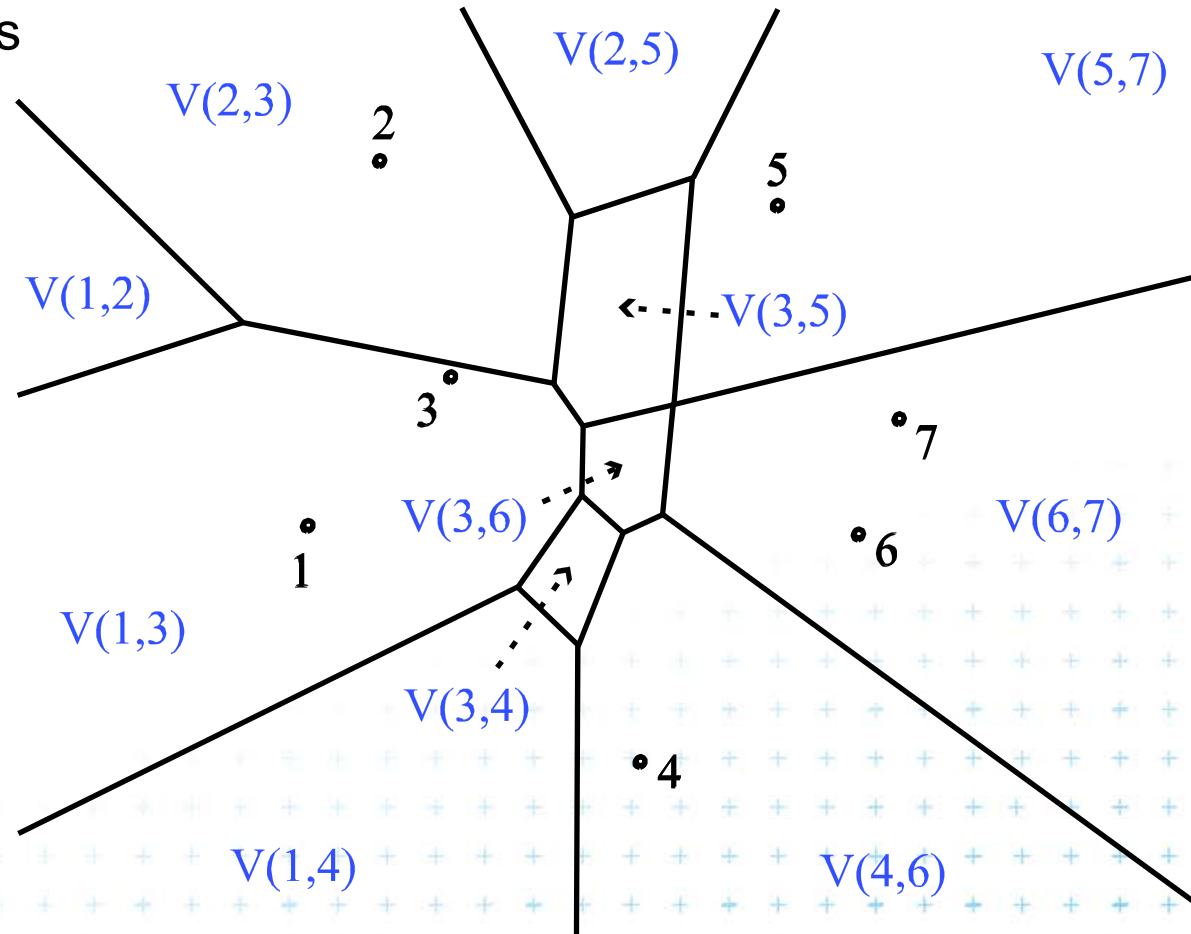
Higher order VD



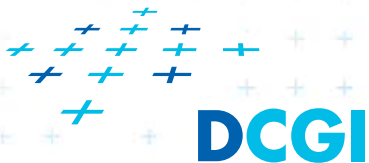
Order-2 Voronoi diagram (nearest to two sites)

$V(p_i, p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site

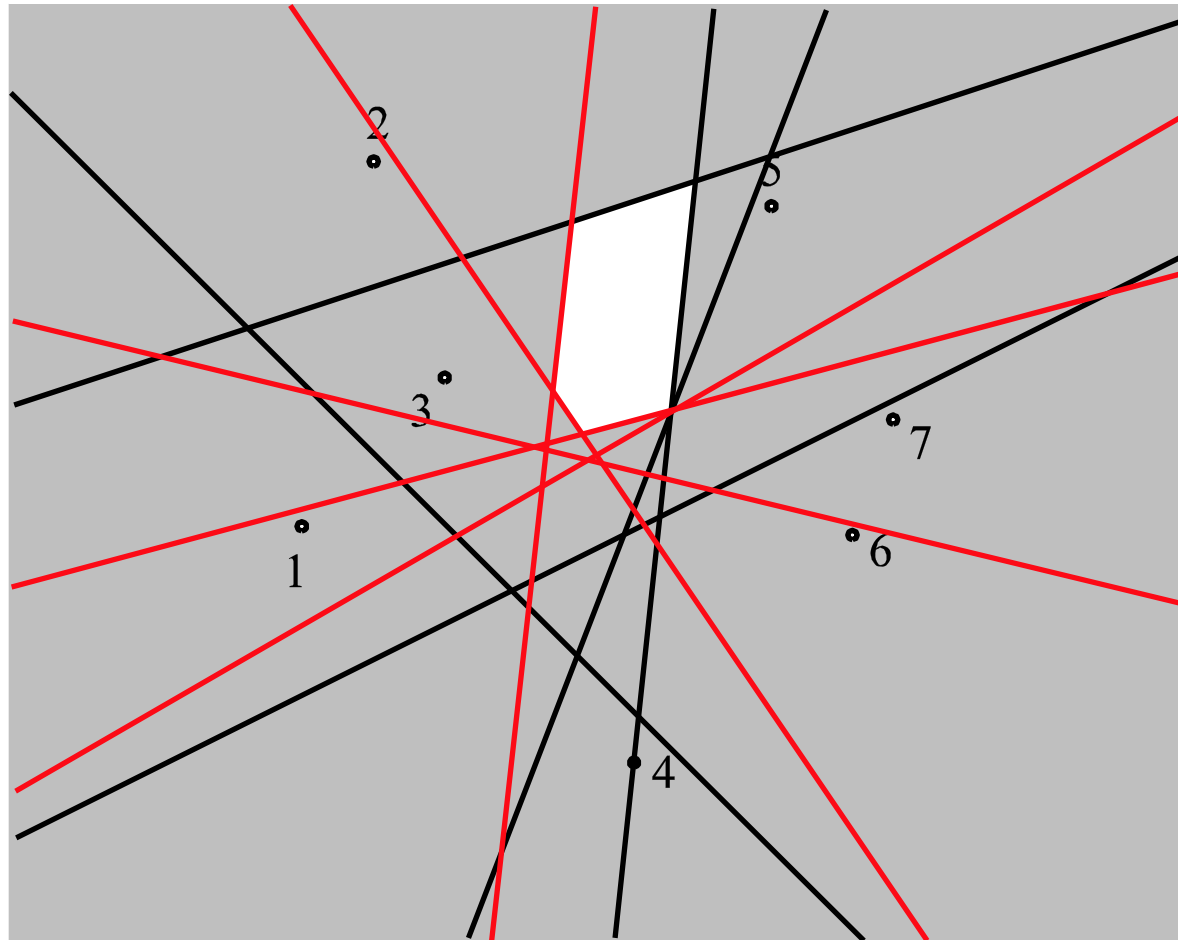
Property
The order-2 Voronoi regions are convex



[Nandy]



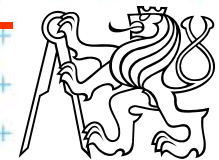
Construction of $V(3,5) = V(5,3)$



[Nandy]

Intersection of all halfplanes
except $h(3,5)$ and $h(5,3)$

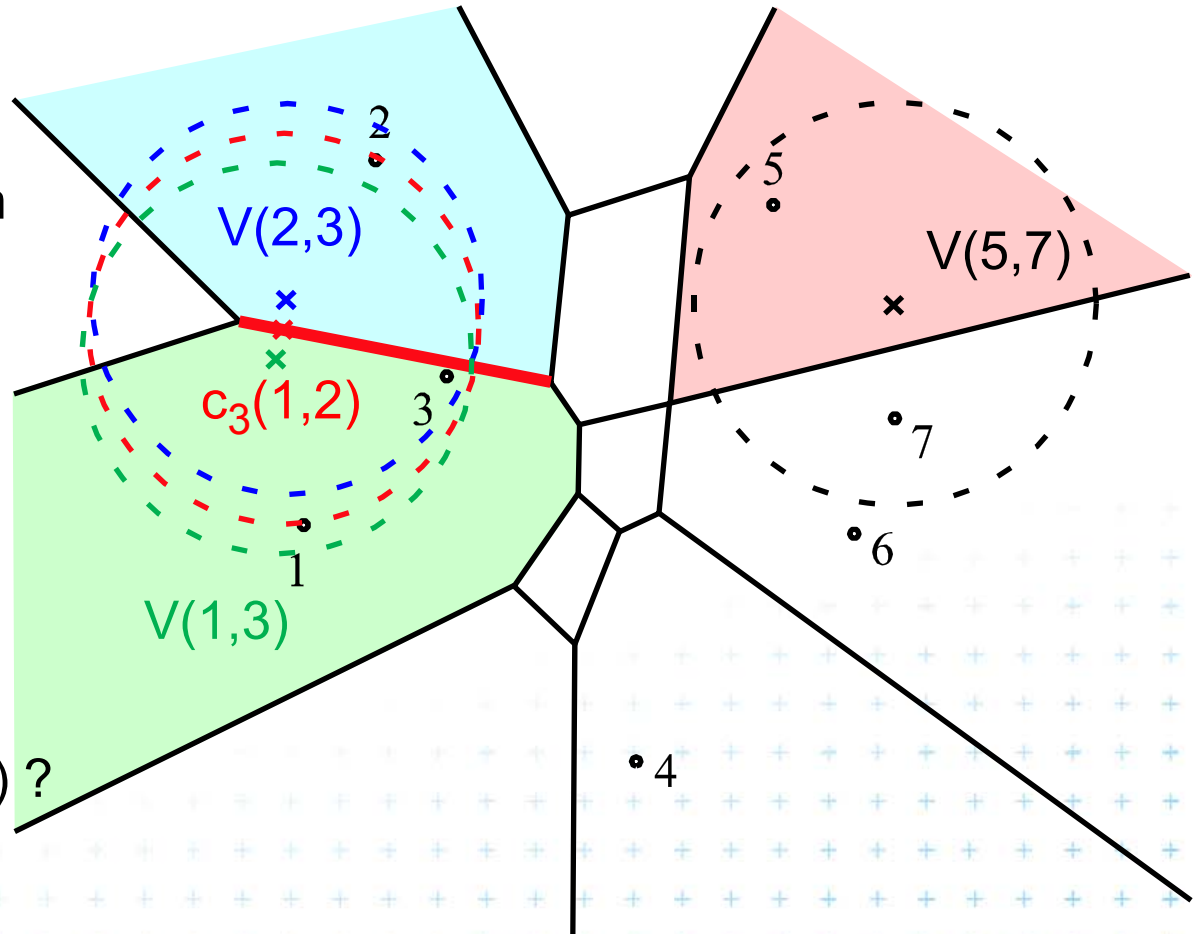
$$\bigcap_{x \neq 5} h(3, x) \cap \bigcap_{x \neq 3} h(5, x)$$



Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites s and t and containing one site p
 $\Rightarrow c_p(s,t)$

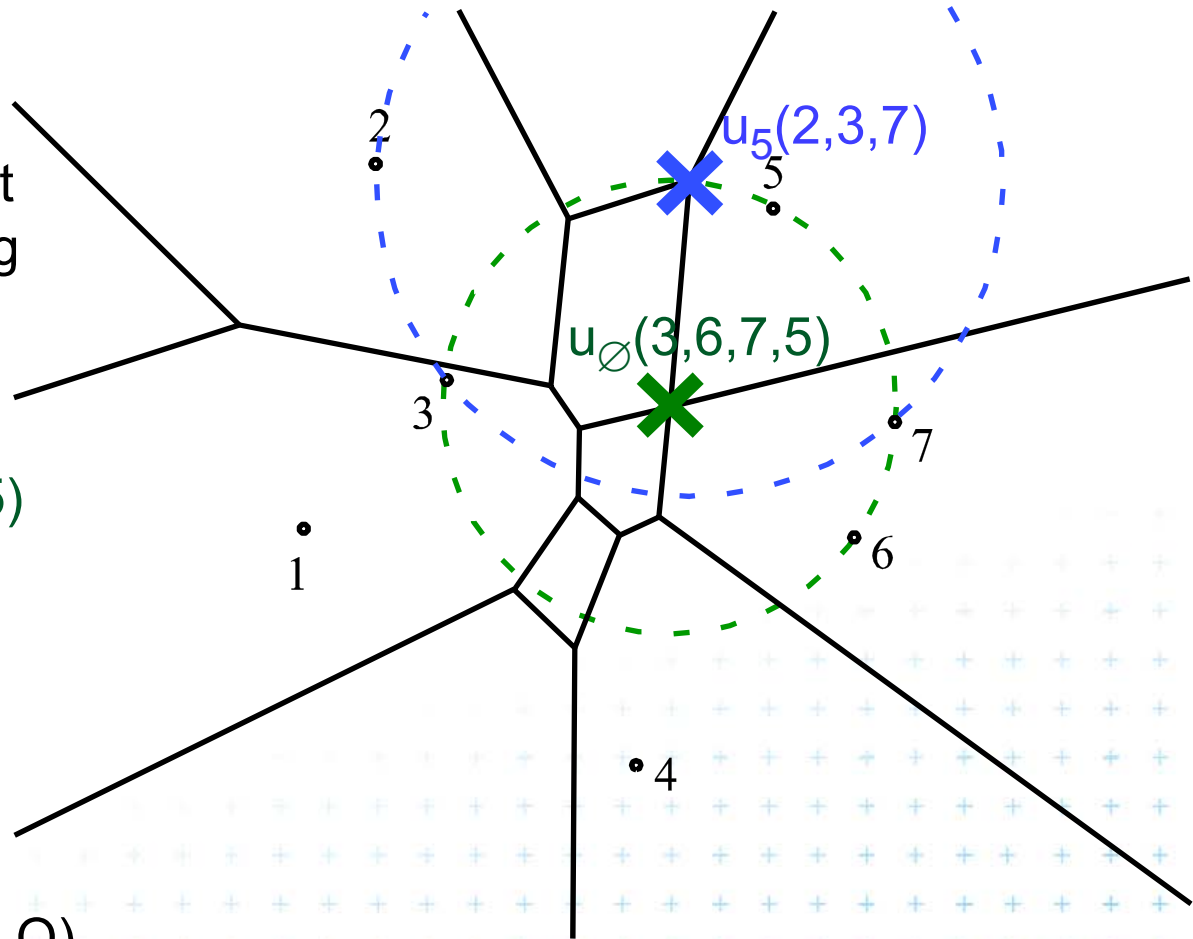
Question
Which are the regions on both sides of $c_p(s,t)$?
 $\Rightarrow V(p,s)$ and $V(p,t)$



Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

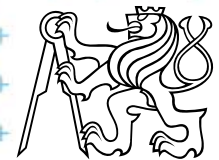
$\Rightarrow u_p(Q)$ or $u_\emptyset(Q \cup p)$
 $u_5(2,3,7), u_\emptyset(3,6,7,5)$



(circle circumscribed to Q)



[Nandy]



Order-2 Voronoi vertex $u_p(Q)$

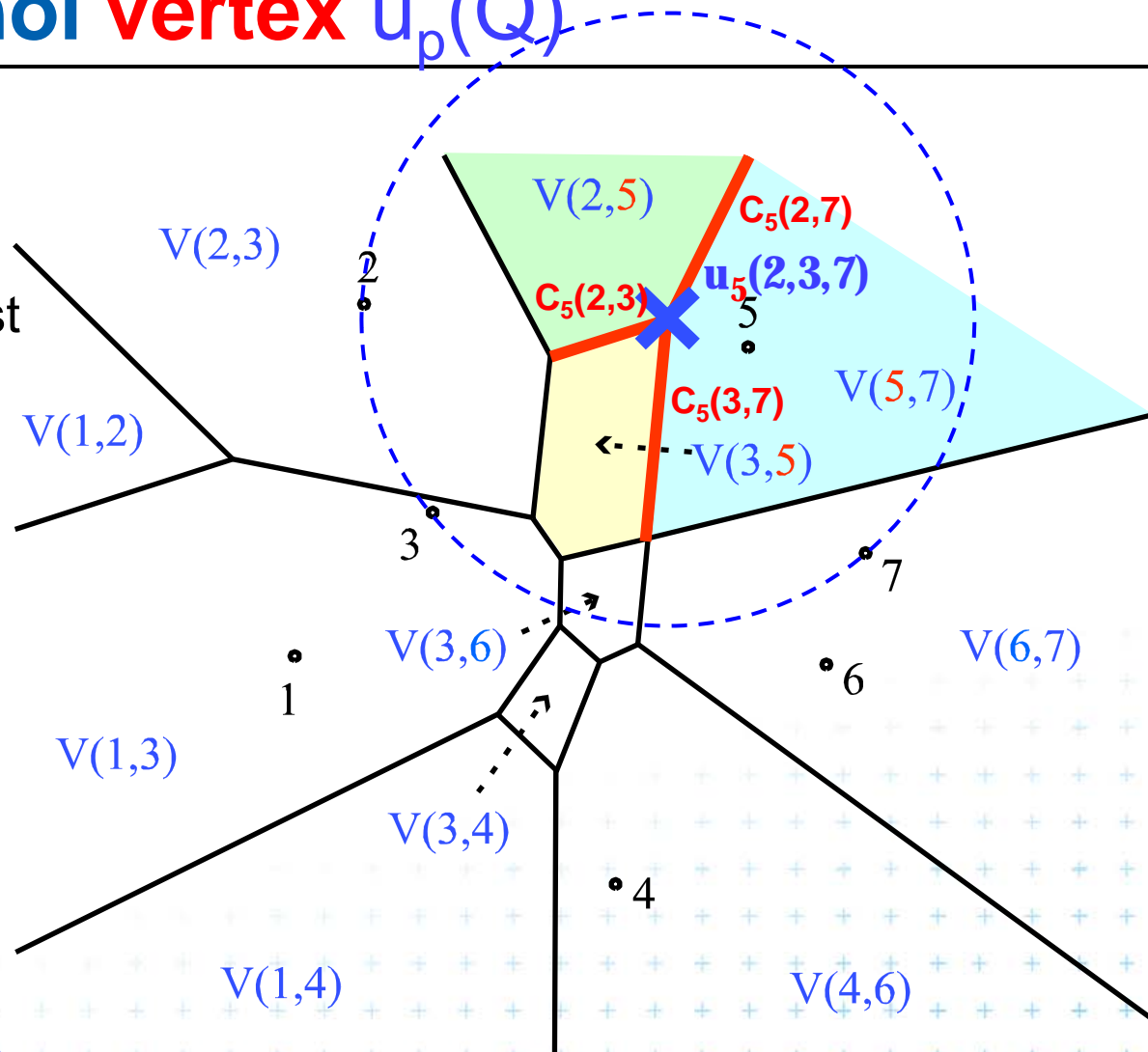
vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_p(Q)$
 $u_5(2,3,7)$

Cell 5 is inside for all incident edges:

- $C_5(2,3)$
- $C_5(2,7)$
- $C_5(3,7)$

\Rightarrow 5 is inside for circle with center in vertex



[Nandy]



Order-2 Voronoi vertex $u_{\emptyset}(Q \cup p)$

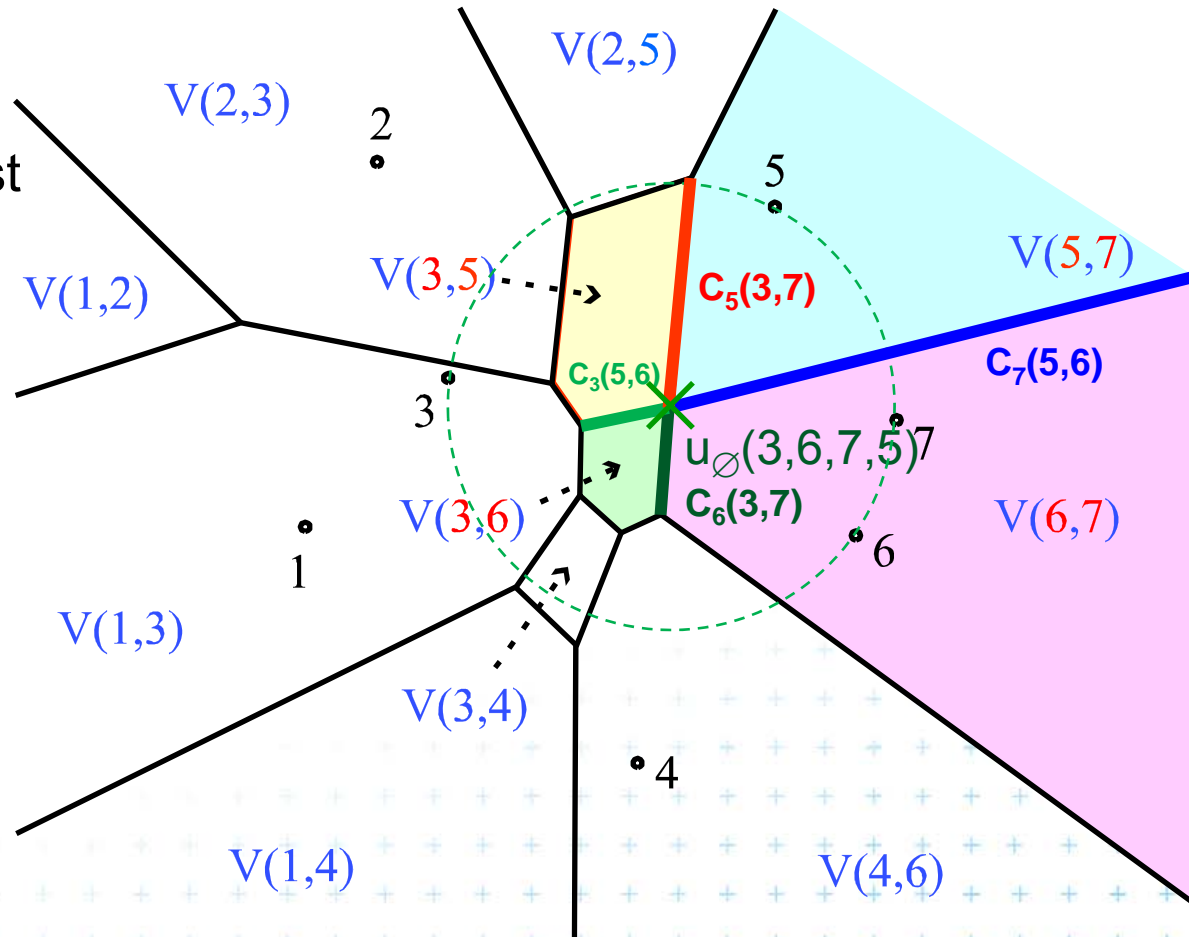
vertex : center of a circle passing through at least 3 sites and containing either site p or **nothing**

Case $u_{\emptyset}(Q \cup p)$
 $u_{\emptyset}(3,6,7,5)$

Cell 5 is not inside for all incident edges:

- $C_5(3,7)$
- $C_6(3,7)$
- $C_3(5,6)$
- $C_7(5,6)$

=> 5 on circle with center in vertex



[Nandy]



Order-k Voronoi Diagram

Theorem věta

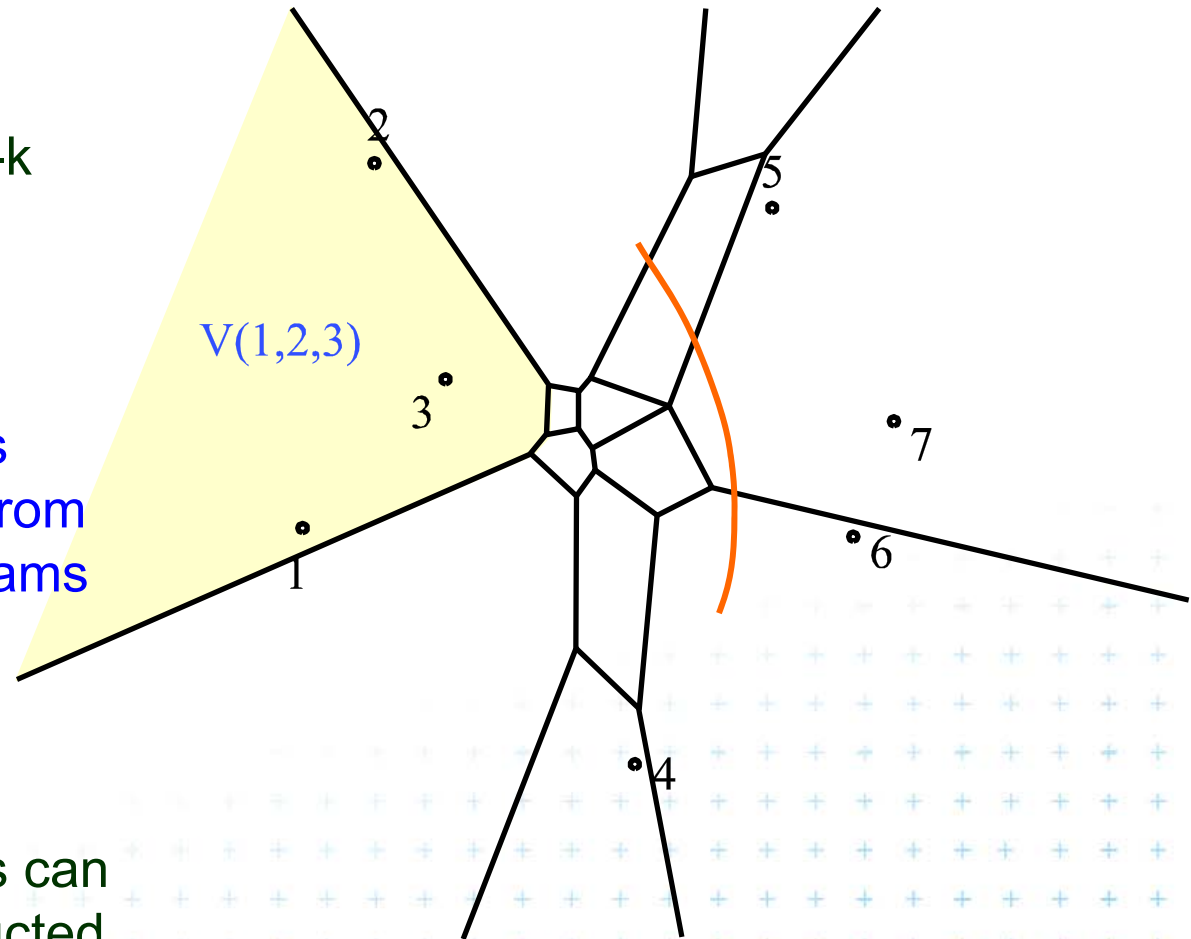
The size of the order-k diagrams is $O(k(n-k))$

Theorem věta

The order-k diagrams can be constructed from the order-(k-1) diagrams in $O(k(n-k))$ time

Corollary důsledek

The order-k diagrams can be iteratively constructed in $O(n \log n + k^2(n-k))$ time



[Nandy]



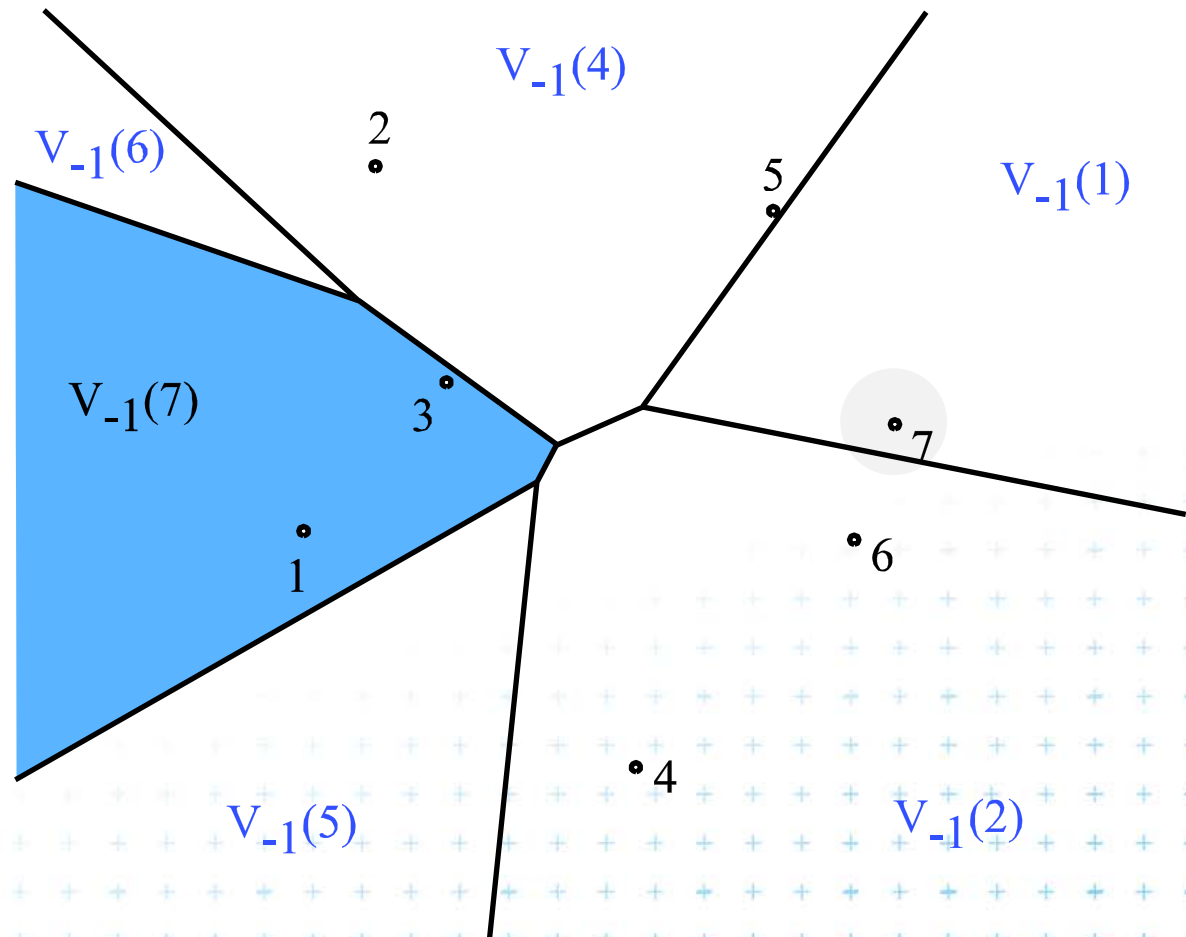
Order $n-1$ VD (Farthest-point Voronoi diagram)



Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from $p_i=7$ than from any other site

$\text{Vor}_{-1}(P) = \text{Vor}_{n-1}(P)$
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



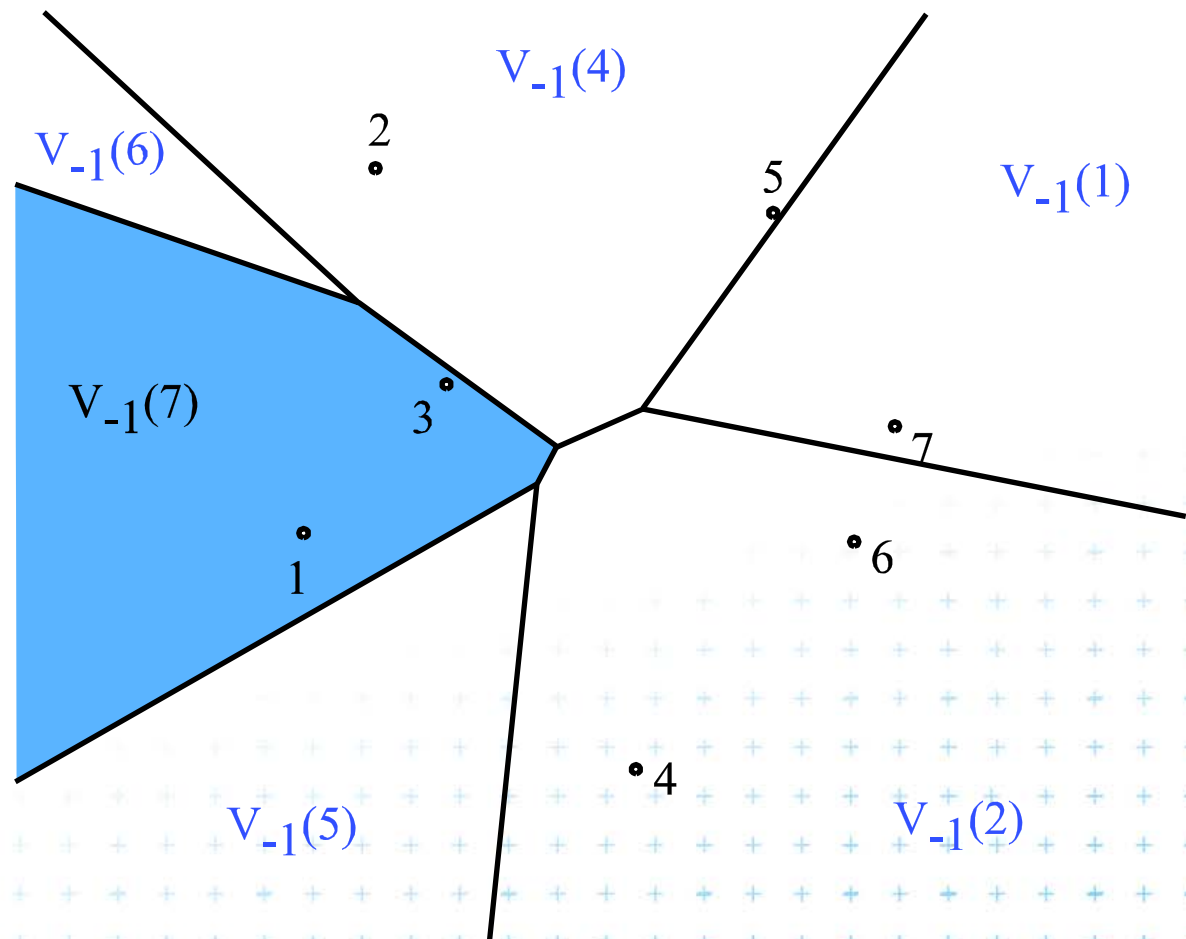
Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell

= set of points in the plane farther from p_i than from any other site

$\text{Vor}_{-1}(P)$ diagram

= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



[Nandy]



Farthest-point Voronoi region (cell)

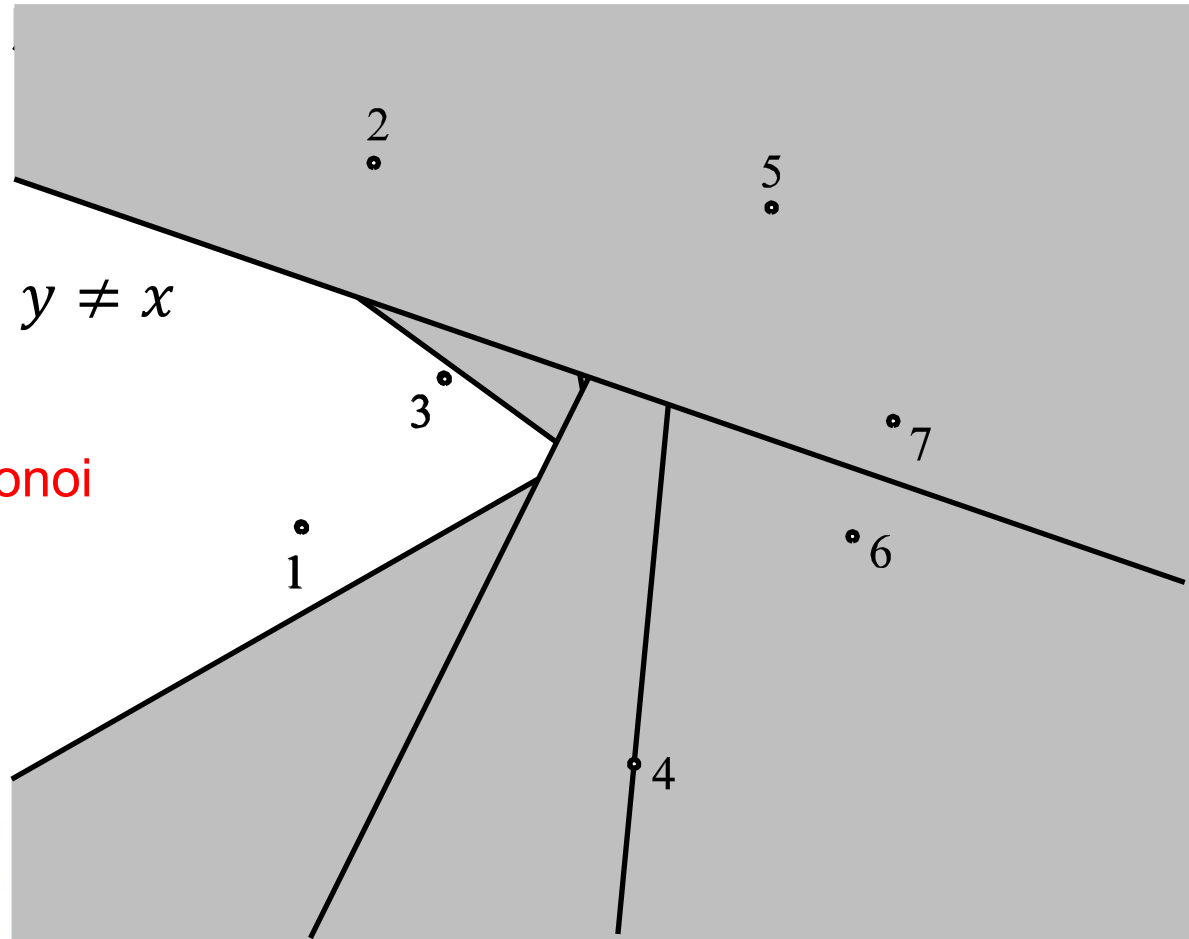
Computed as intersection of halfplanes, but we take “other sides” of bisectors

Construction of $V_{-1}(7)$

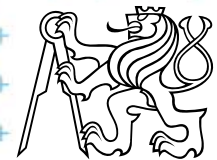
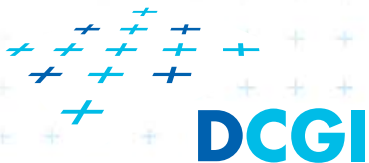
$$V_{-1}(y) = \bigcap_{x=1}^n h(y, x), \quad y \neq x$$

Property

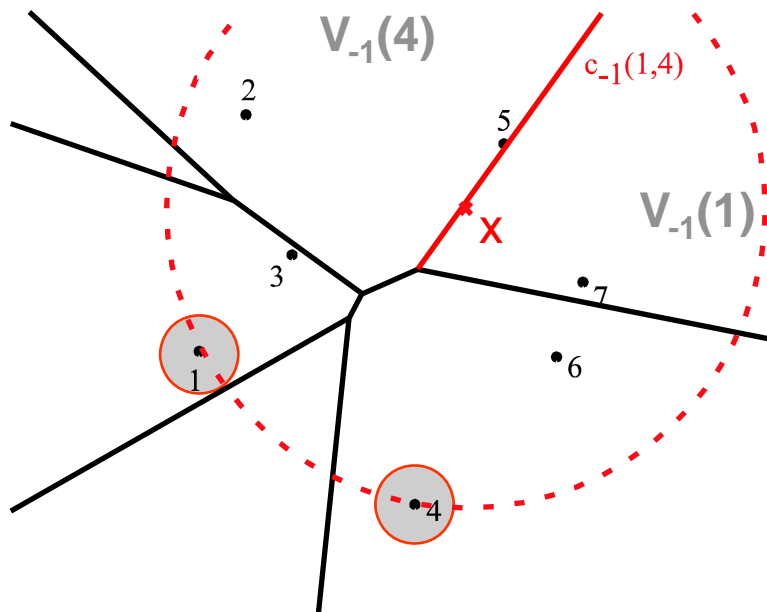
The farthest point Voronoi regions are convex and unbounded



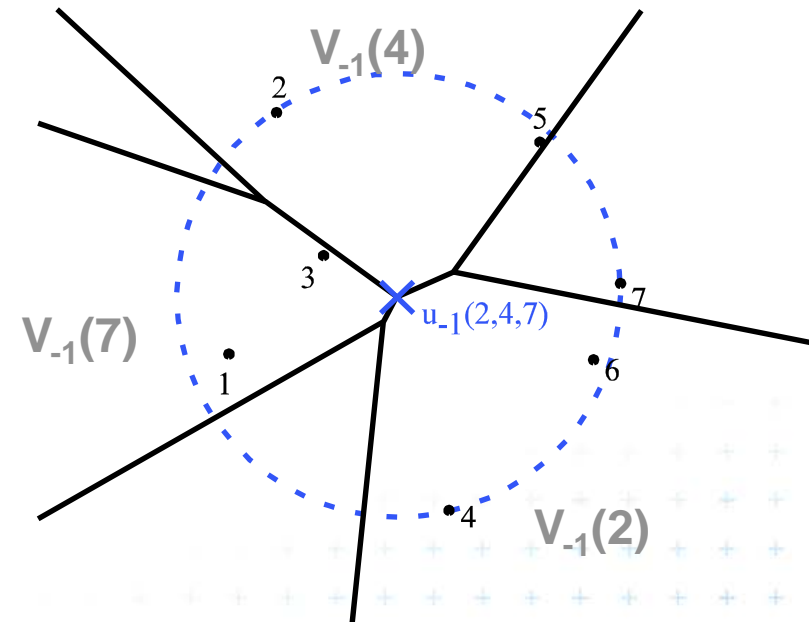
[Nandy]



Farthest point Voronoi edges and vertices



edge : set of points equidistant from 2 sites and closer to all the other sites



vertex : point equidistant from at least 3 sites and closer to all the other sites

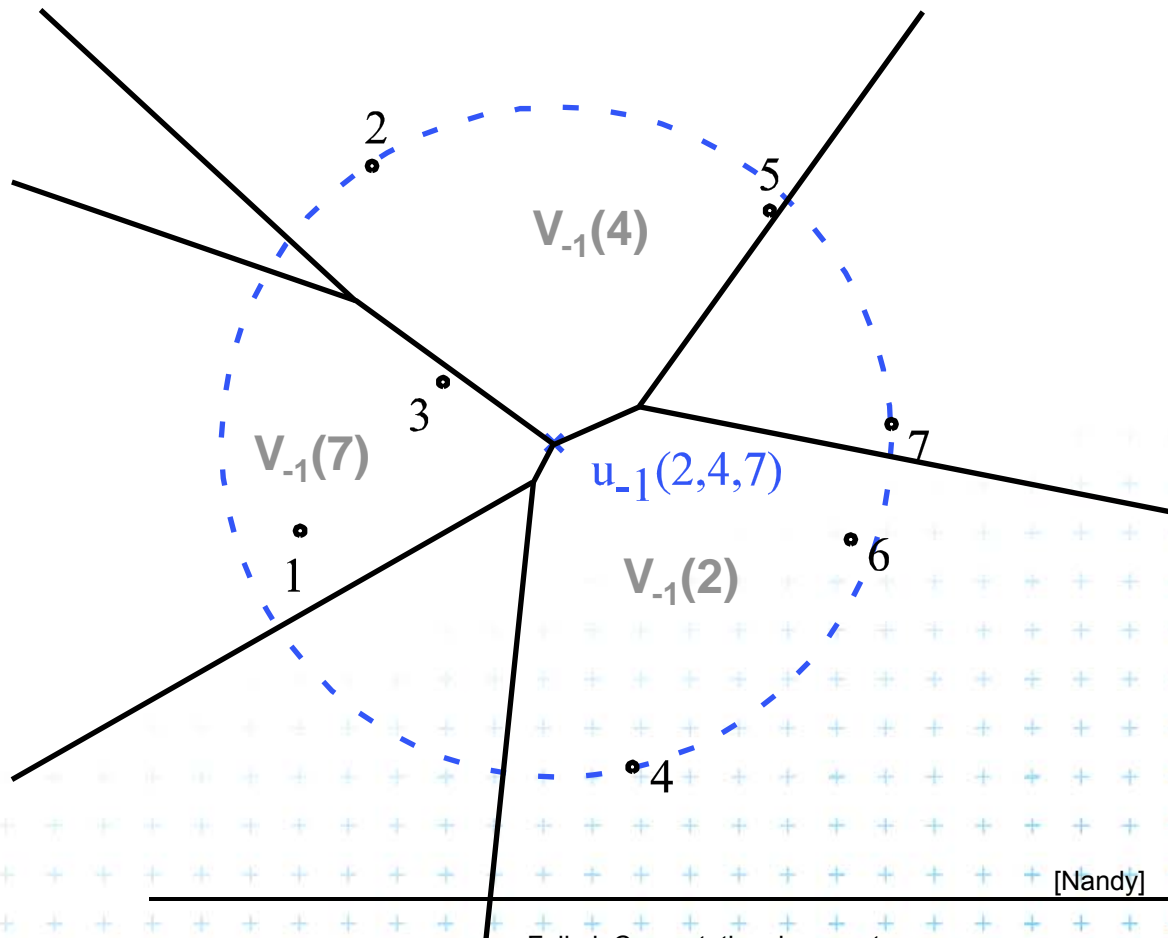


[Nandy]



Application of $\text{Vor}_{-1}(P)$: Smallest enclosing circle

- Construct $\text{Vor}_{-1}(P)$ and find minimal circle with center in $\text{Vor}_{-1}(P)$ vertices or on edges



[Nandy]

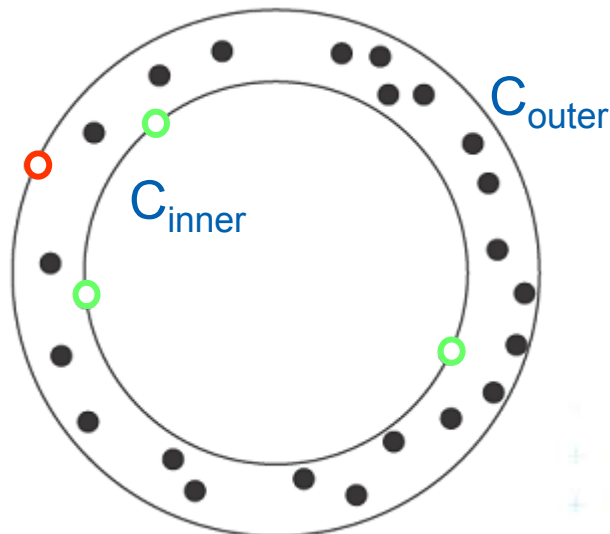


Farthest-point Voronoi diagrams example

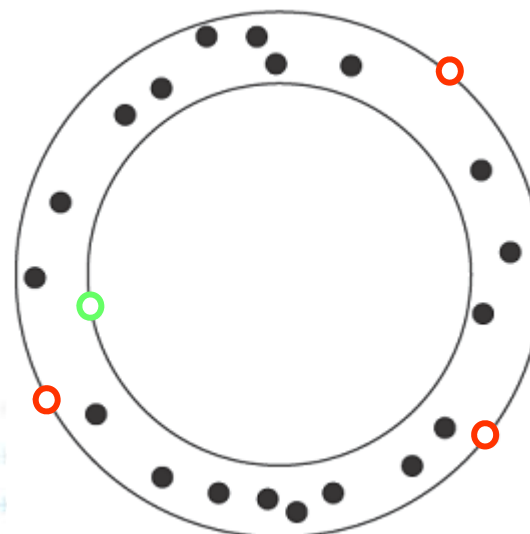
Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles C_{inner} and C_{outer})

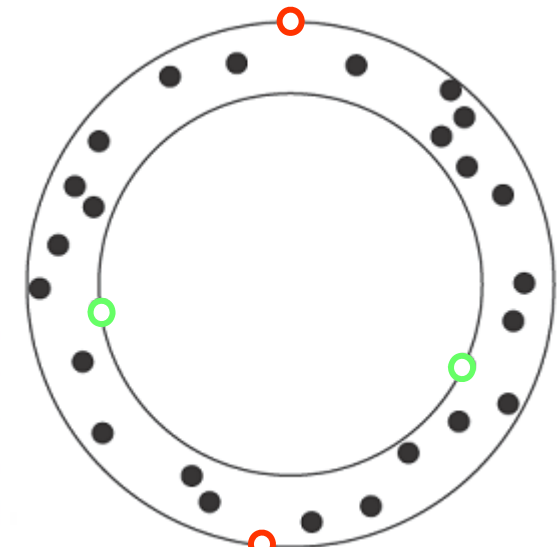
Three cases to test – one will win:



a) 3 in – 1 out



b) 1 point in – 3 out



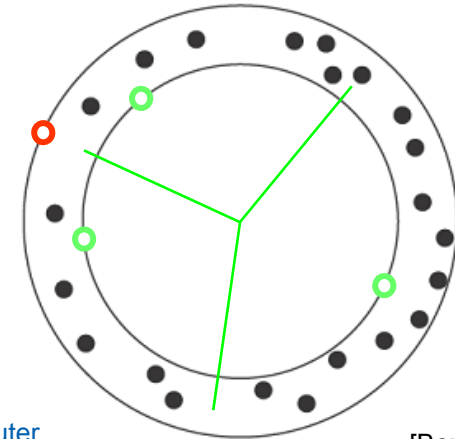
c) 2 in – 2 out



Smallest width annulus – cases with 3 pts

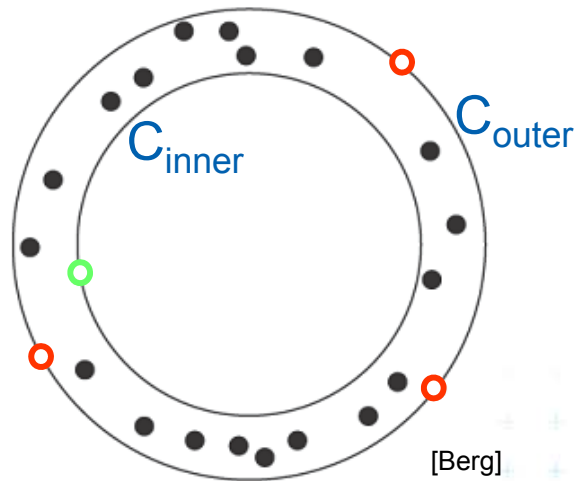
a) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram (1st order VD)*
- The **remaining point** on C_{outer} in $O(n)$ for each vertex
 - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C_{outer}
 - ⇒ $O(n^2)$



[Berg]

3 in – 1 out



[Berg]

1 point in – 3 out

b) C_{outer} contains at least 3 points

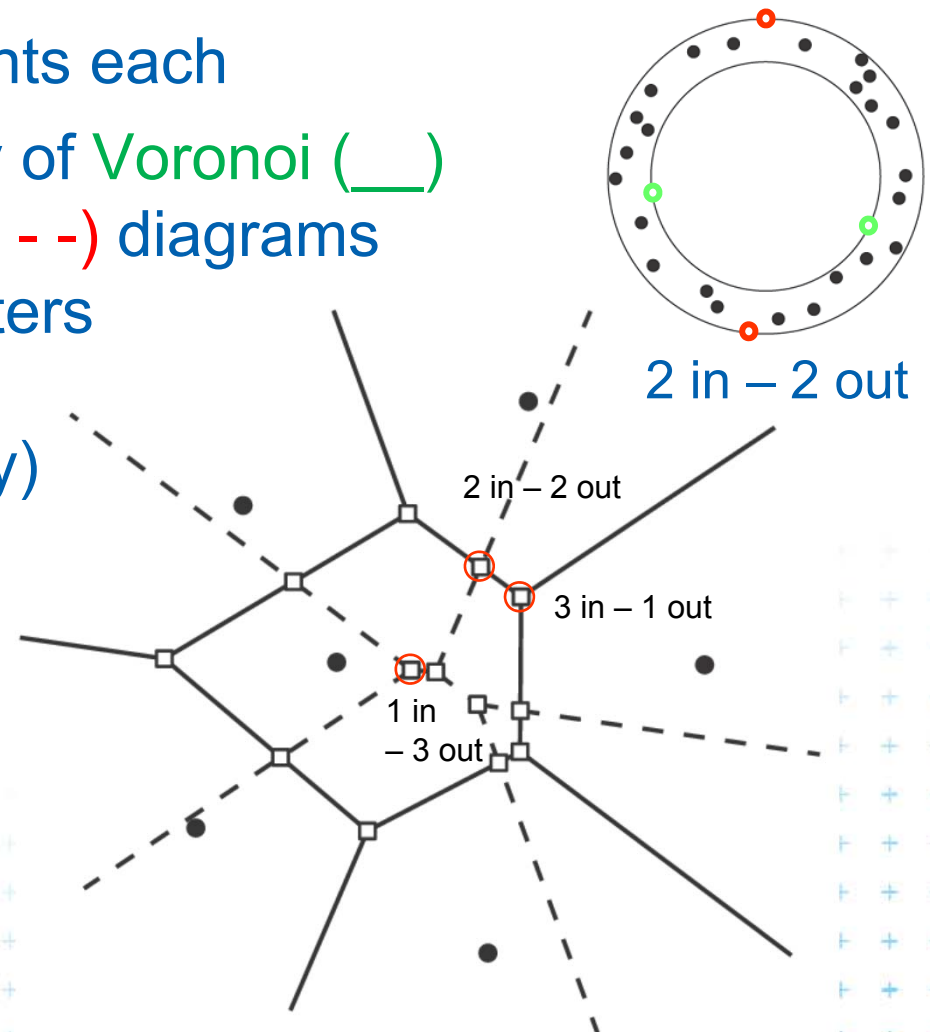
- Center is the *vertex of the farthest Voronoi diagram*
- The **remaining point** on C_{inner} in $O(n)$
 - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C_{inner}
 - ⇒ $O(n^2)$



Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (—) and farthest-point Voronoi (---) diagrams
 $\Rightarrow O(n^2)$ candidates for centers
 (we need only vertices, not the complete overlay)
- annulus computed in $O(1)$ from center and 4 points (same for all 3 cases)
- $O(n^2)$



[Berg]



Smallest width annulus

Smallest-Width-Annulus

Input: Set P of n points in the plane

Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram $Vor(P)$ and farthest-point Voronoi diagram $Vor_{-1}(P)$ of P
2. For each vertex of $Vor(P)$ (r) determine the *farthest point* (R) from P
 $\Rightarrow O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $Vor_{-1}(P)$ (R) determine the *closest point* (r) from P
 $\Rightarrow O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges $Vor(P)$ and $Vor_{-1}(P)$ test if they intersect
 \Rightarrow another set of four points defining candidate annulus – c)
5. For all candidates of all three types chose the smallest-width annulus

1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

$O(n^2)$ time using $O(n)$ storage



DCGI

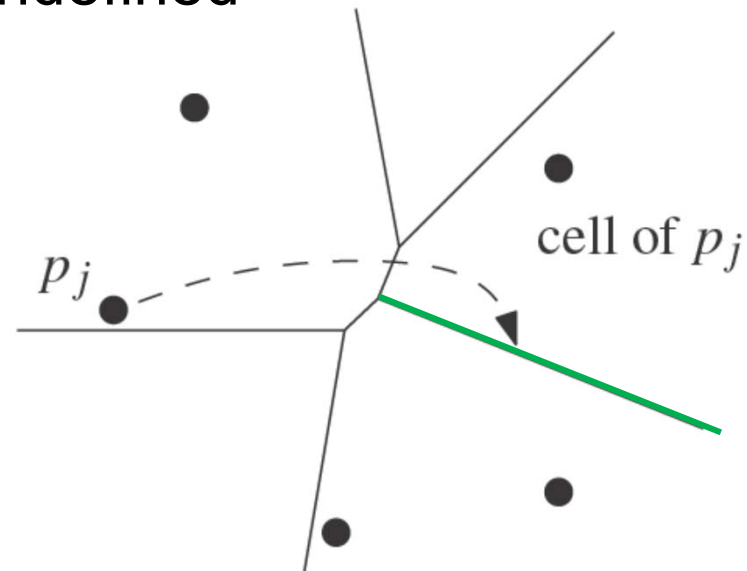


Order $n-1$ VD construction



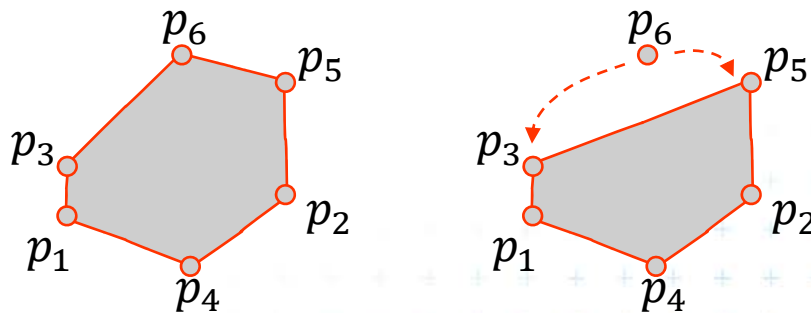
Modified DCEL for farthest-point Voronoi d

- Half-infinite edges \rightarrow we adapt DCEL
- Half-edges with origin in infinity
 - Special vertex-like record for origin in infinity
 - Store **direction** instead of coordinates
 - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
 - store a **pointer to the most CCW half-infinite half-edge** of its cell in DCEL

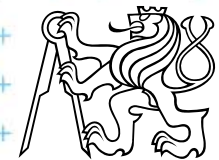


Idea of the algorithm

1. Create the convex hull and number the CH points randomly
2. Remove the points starting in the last of this random order and store $cw(p_i)$ and $ccw(p_i)$ points at the time of removal.
3. Include the points back and compute V_{-1}



p_i	$ccw(p_i)$	$cw(p_i)$
p_6	p_3	p_5
p_5	p_3	p_2
...		



Farthest-point Voronoi d. construction

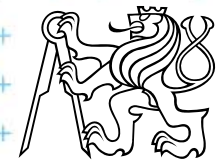
Farthest-point Voronoi

$O(n \log n)$ time in $O(n)$ storage

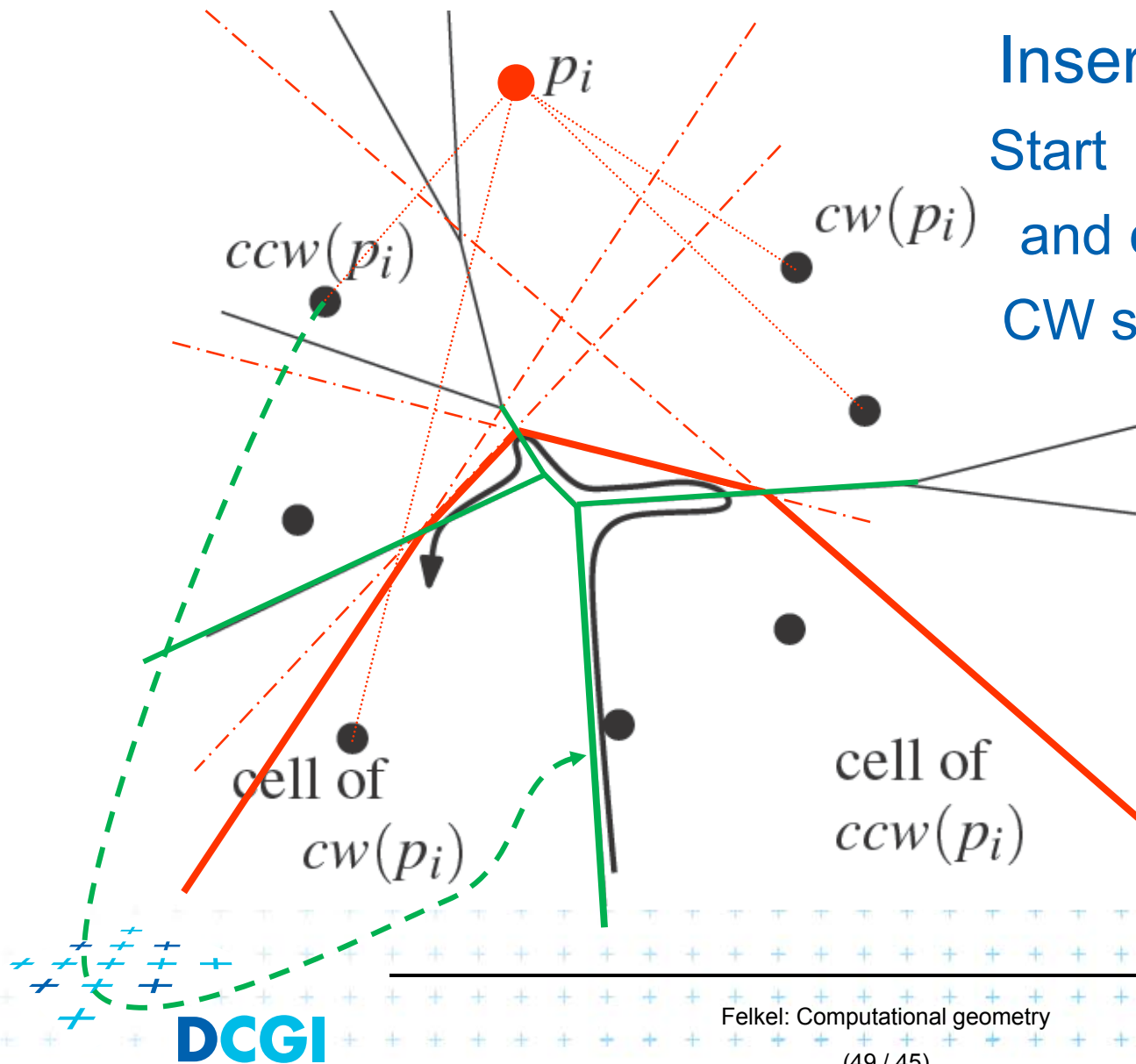
Input: Set of points P in plane

Output: Farthest-point VD $\text{Vor}_{-1}(P)$

1. Compute convex hull of P
2. Put points in CH(P) of P in random order p_1, \dots, p_h
3. Remove p_h, \dots, p_4 from the cyclic order (around the CH).
When removing p_i , store the neighbors: $cw(p_i)$ and $ccw(p_i)$ at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $\text{Vor}_{-1}(\{p_1, p_2, p_3\})$ as init
5. **for** $i = 4$ **to** h **do**
6. Add site p_i to $\text{Vor}_{-1}(\{p_1, p_2, \dots, p_{i-1}\})$ between site $cw(p_i)$ and $ccw(p_i)$
7. - start at most CCW edge of the cell $ccw(p_i)$
8. - continue CW to find intersection with bisector($ccw(p_i), p_i$)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i



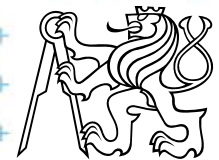
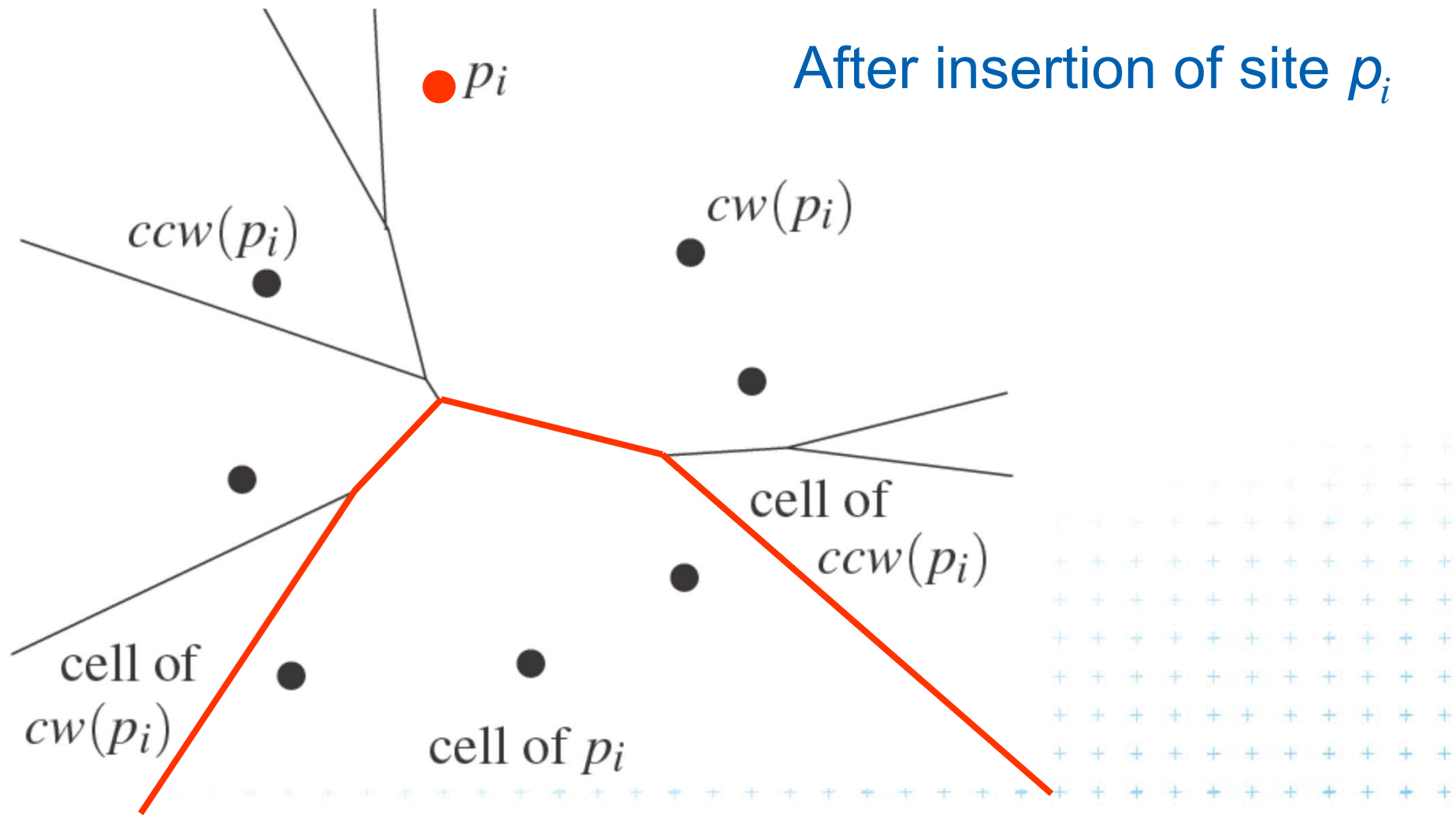
Farthest-point Voronoi d. construction



Insertion of **site p_i**
Start with site $ccw(p_i)$ and ccw edge of its cell
CW search of intersection



Farthest-point Voronoi d. construction



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