CONVEX HULL IN 3 DIMENSIONS

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Preparata], [Rourke] and [Boissonnat]

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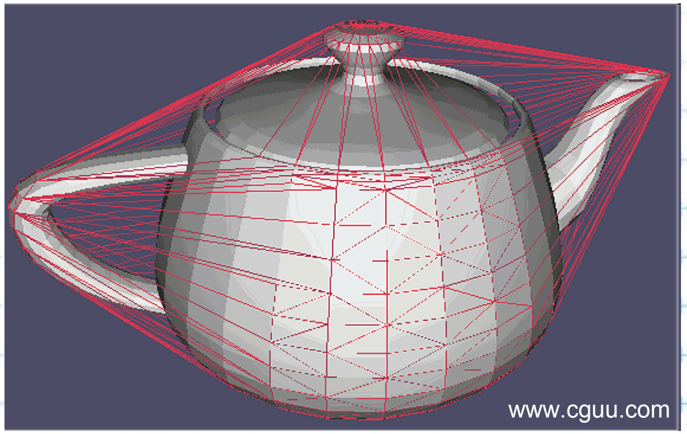
Talk overview

- Upper bounds for convex hull in 2D and 3D
- Other criteria for CH algorithm classification
- Recapitulation of CH algorithms
- Terminology refresh
- Convex hull in 3D
  - Terminology
  - Algorithms
    - Gift wrapping
    - D&C Merge
    - Randomized Incremental



Upper bounds for Convex hull algorithms

- $O(n)$ for sorted points and for simple polygon
- $O(n \log n)$ in E^2, E^3 with sorting
  – insensitive about output
- $O(n\,h)$, $O(n \log h)$, h is number of CH facets
  – output sensitive
  – $O(n^2)$ or $O(n \log n)$ for $n \sim h$
- $O(\log n)$ for new point insertion in realtime algs.
  => $O(n \log n)$ for n-points
  $O(\log n)$ search where to insert

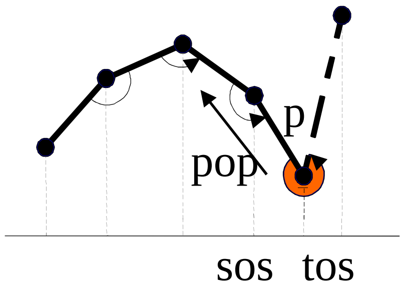
Other criteria for CH algorithm classification

- Optimality – depends on data order (or distribution)

  In the worst case x In the expected case
- Output sensitivity – depends on the result ~ O(f(h))
- Extendable to higher dimensions?
- Off-line versus on-line
  - Off-line – all points available, preprocessing for search speedup
  - On-line – stream of points, new point p_i on demand, just one new point at a time, CH valid for $\{p_1, p_2, \ldots, p_i\}$
  - Real-time – points come as they "want"

    (come not faster than optimal constant O(log n) inter-arrival delay)
- Parallelizable x serial
- Dynamic – points can be deleted
- Deterministic x approximate (lecture 13)

Graham scan

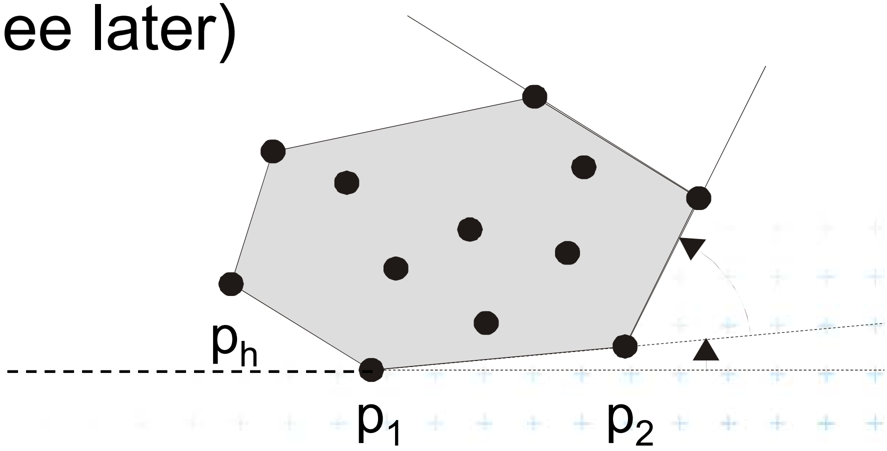


- O(n log n) time and O(n) space is
  - optimal in the worst case
  - not optimal in average case (not output sensitive)
  - only 2D
  - off-line
  - serial (not parallel)
  - not dynamic (no deleted points)

O(*n*) for polygon (discussed in seminar)

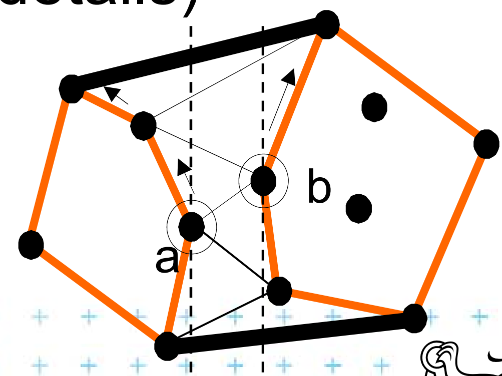
Jarvis March – Gift wrapping

- $O(hn)$ time and $O(n)$ space is
  - not optimal in worst case $O(n^2)$
  - may be optimal if $h << n$ (output sensitive)
  - 3D or higher dimensions (see later)
  - off-line
  - serial (not parallel)
  - not dynamic



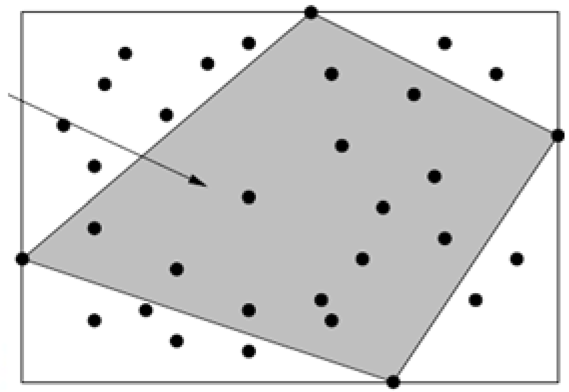
Divide & Conquer

- O(n log n) time and O(n) space is
  - optimal in worst case (in 2D or 3D)
  - not optimal in average case (not output sensitive)
  - 2D or 3D (circular ordering), in higher dims not optimal
  - off-line
  - Version with sorting (the presented one) – serial
  - Parallel for overlapping merged hulls
    (see Chapter 3.3.5 in Preparata for details)
  - not dynamic



Quick hull

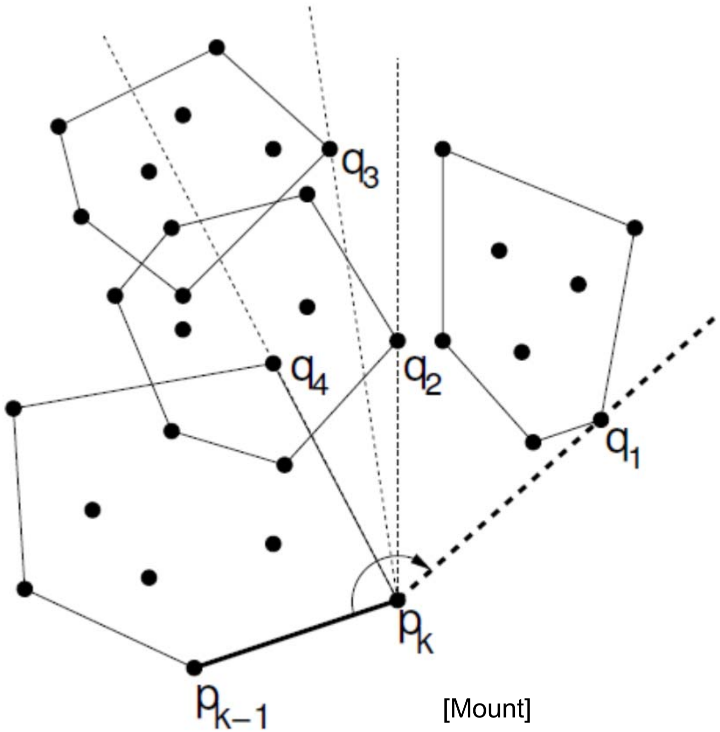
- O(n log n) expected time, O(n^2) the worst case and O(n) space *in 2D* is
  - not optimal in worst case O(n^2)
  - optimal if uniform distribution then $h << n$ (output sensitive)
  - 2D, or higher dimensions [see http://www.qhull.org/]
  - off-line
  - parallelizable
  - not dynamic



[Mount]

Chan

- *O*(*n* log *h*) time and O(*n*) space is
  - optimal for *h* points on convex hull (output sensitive)
  - 2D and 3D --- gift wrapping
  - off-line
  - Serial (not parallel)
  - not dynamic



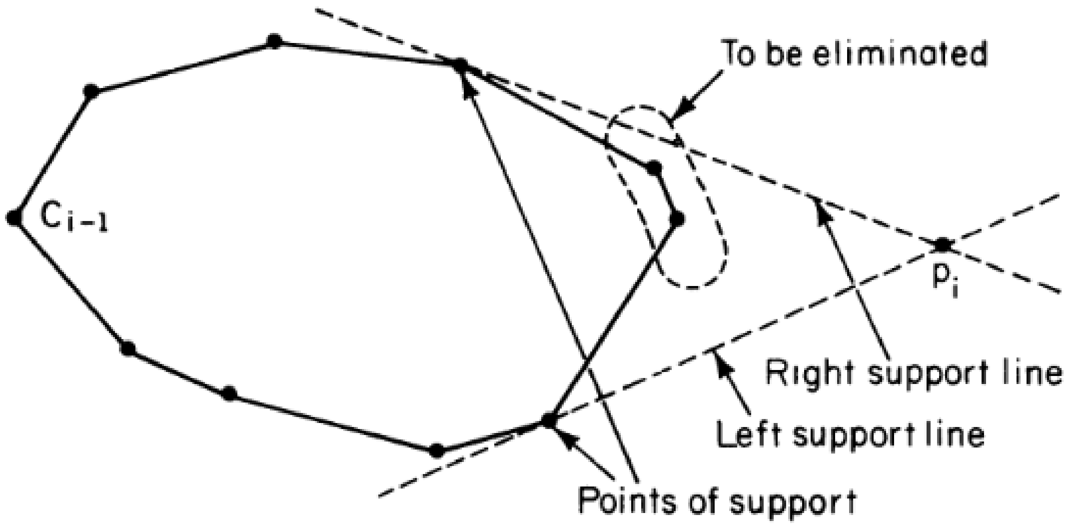
[Mount]

On-line algorithms

- Preparata's on-line algorithm
- Overmars and van Leeuven

Preparata's 2D on-line algorithm

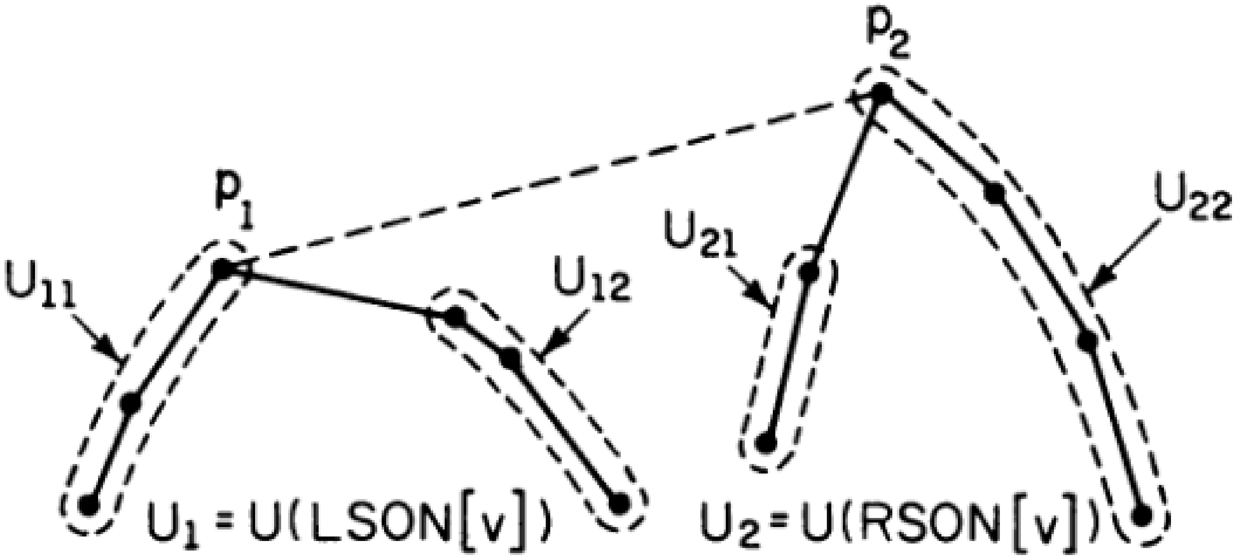
- New point *p* is tested
  - Inside –> ignored
  - Outside –> added to hull
    - Find left and right supporting lines (touch at supporting points)
    - Remove points between supporting points
    - Add *p* to CH between supporting lines



[Preparata]

Overmars and van Leeuwen

- Allow dynamic 2D CH
  (on-line insert & delete)
- Manage special tree with all intermediate CHs
- Will be discussed on seminar [7]



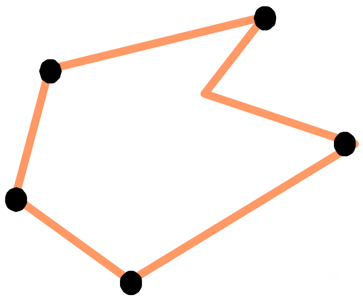
[Preparata]

Convex hull in 3D

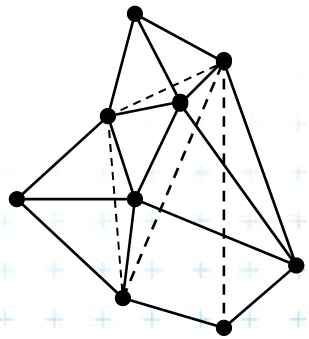
- Terminology
- Algorithms
  1. Gift wrapping
  2. D&C Merge
  3. Randomized Incremental
  4. Quick hull … minule

Terminology

- Polytope (d-polytope)
  = a geometric object with "flat" sides E^d
  (may be or may not be convex)
- Flat sides mean that
  the sides of a (*k*)-polytope
  consist of (*k-1*)-polytopes that
  may have (*k*-2)-polytopes in common.



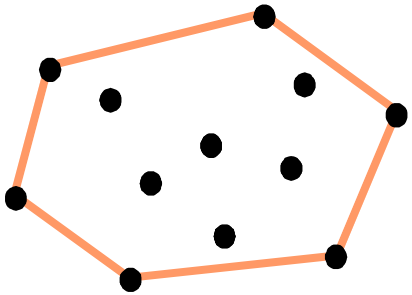
2-polytop
= polygon

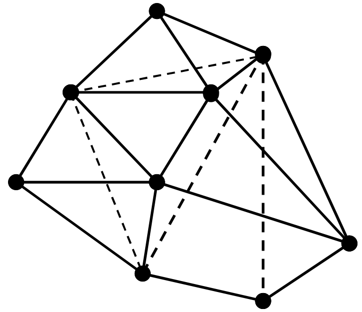


3-polytop
= polyhedron

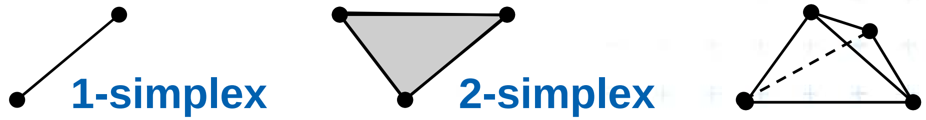
Terminology

- Convex Polytope (convex d-polytope)
  = convex hull of finite set of points in E^d





- Simplex (k-simplex, d-simplex)
  = CH of k + 1 *affine independent points* (vectors $(p_k - p_0)$ are linearly independent)

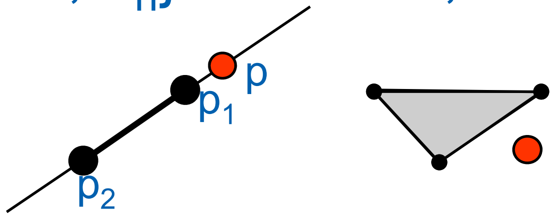


= "Special" Convex Polytope with all the points on the CH

Terminology (2)

- Affine combination
  = linear combination of the points $\{p_1, p_2, \ldots, p_n\}$
  whose coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ sum to 1, and $\lambda_i \in R$

$$\sum_{i=1}^{n} \lambda_i p_i$$



- Affine independent points
  = no one point can be expressed as affine combination of the others

- Convex combination
  = linear combination of the points $\{p_1, p_2, \ldots, p_n\}$
  whose coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ sum to 1, and $\lambda_i \in R^+_0$
  (i.e., $\forall i \in \{1, \ldots, n\}$, $\lambda_i \geq 0$)

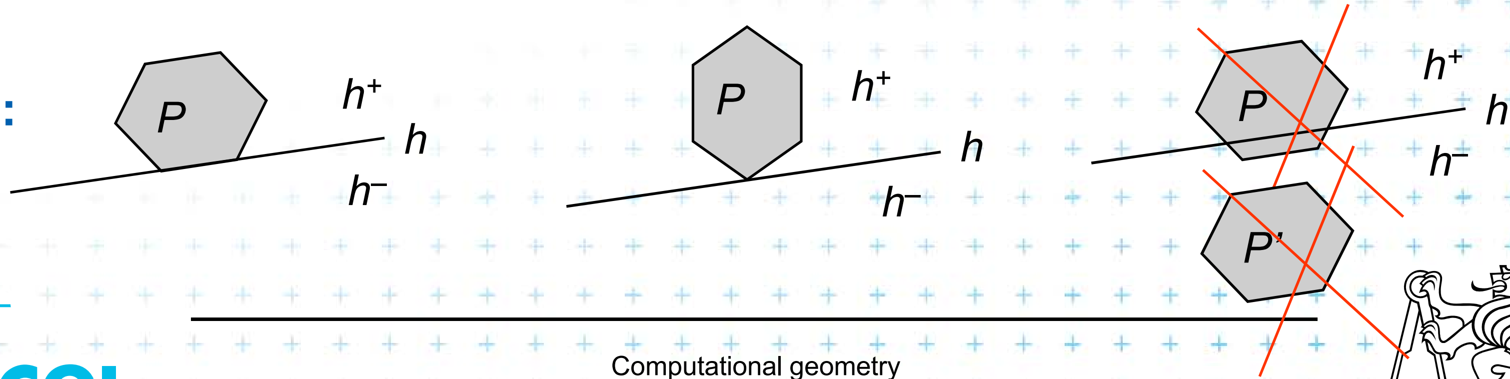
$$\Rightarrow \lambda_i \in \langle 0, 1 \rangle$$

Terminology (3)

- Any (d-1)-dimensional hyperplane *h* divides the space into (open) halfspaces h^+ *and* h^-, *so that* $E^n = h^+ \cup h \cup h^-$
- *Def:* $\overline{h^+} = h^+ \cup h$, $\overline{h^-} = h^- \cup h$ (closed halfspaces)

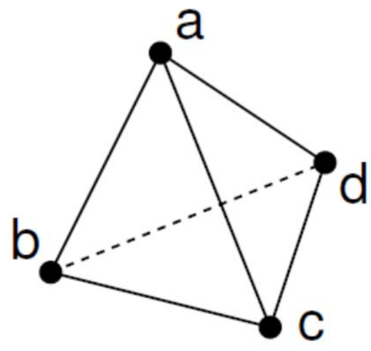
- Hyperplane supports a convex polytope *P* (Supporting hyperplane – *opěrná nadrovina*)
  – if $h \cap P$ is not empty and
  – if *P* is entirely contained within either $\overline{h^+}$ or $\overline{h^-}$

In 2D:



Faces and facets

- Face of the convex polytope
  = Intersection of convex polytope *P*
  with a supporting hyperplane *h*
  - Faces are convex polytopes of dimension *d* ranging from 0 to d – 1
  - 0-face = vertex
  - 1-face = edge
  - (d – 1)-face = facet



Proper faces in 3D:
Vertices: a,b,c,d
Edges: ab, ac, ad, bc, bd, cd
Facets: abc, abd, acd, bcd

In 3D we often say *face*, but more precisely a *facet*

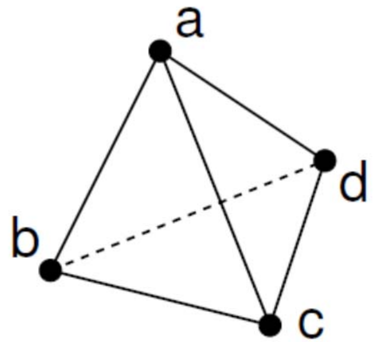
(In 3D a 2-face = facet)
(In 2D a 1-face = facet)

Proper faces

- Proper faces

  = Faces of dimension *d* ranging from 0 to $d - 1$
- Improper faces

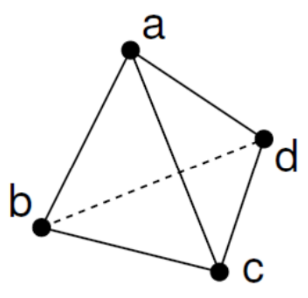
  = proper faces + two additional faces:
  - {} = Empty set = face of dimension -1
  - Entire convex polytope = face of dimension *d*

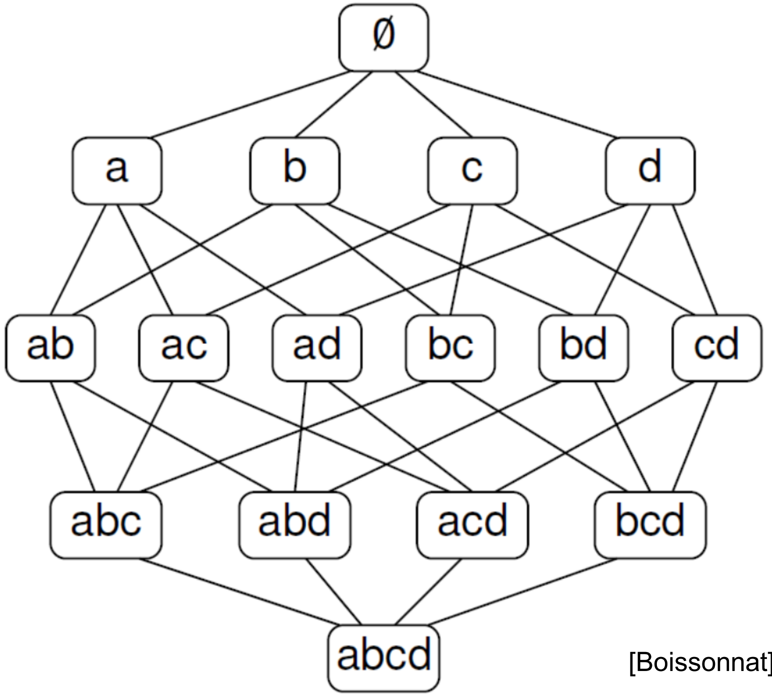


Improper faces in 3D:
- Empty set {}
- Vertices: a,b,c,d
- Edges: ab, ac, ad, bc, bd, cd
- Facets: abc, abd, bcd
- Entire polytope: abcd

Incident graph

- Stores topology of the polytope
- Ex: 3-simplex:





Dimension: -1, 0, 1, 2, 3

[Boissonnat]

- d-simplex is a very regular face structure:
  – 1-face for each pair of vertices
  – 2-face for each triple of vertices

Facts about polytopes

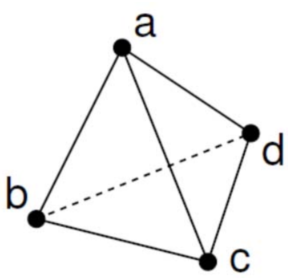
- Boundary o polytope is *union of its proper faces*
- Polytope has *finite number of faces (next slide)*. Each face is a polytope
- Convex polytope is *convex hull of its vertices (the def),* its bounded
- Convex polytope is the *intersection of finite number of closed halfspaces* $\overline{h^{+}}$ (conversely not: intersection of closed halfspaces may be unbounded => called *unbounded polytope*)

Number of faces on a d-simplex

- Number of *j*-dimensional faces on a *d*-simplex

$$\binom{d+1}{j+1} = \frac{(d+1)!}{(j+1)!(d-j)!}$$

- Ex.: Tetrahedron = 3-simplex:



  – facets (2-dim. faces) $\binom{3+1}{2+1} = \frac{4!}{3!1!} = 4$
  – edges (1-dim. faces) $\binom{3+1}{1+1} = \frac{4!}{2!2!} = 6$
  – vertices (0-dim faces) $\binom{3+1}{0+1} = \frac{4!}{1!3!} = 4$

Complexity of 3D convex hull is O(n)

- 3-polytope - has polygonal faces
- convex 3-polytope (CH of a point set in 3D)
- simplical 3-polytope
  – has triangular faces (=> more edges and faces)
- simplical convex 3-polytope with all *n* points on CH
  – the worst case complexity
  – => maximum # of edges and faces for given points
  – has triangular facets, each generates 3 edges, shared by 2 triangles => 3F = 2E   2-manifold

F = 2V – 4   => F ≤ 2V – 4   F = O(n)

E = 3V – 6   => E ≤ 3V – 6   E = O(n)

Complexity of 3D convex hull is O(n)

- The worst case complexity → if all *n* points on CH

=> use simplical convex 3-polytop for complexity derivation

1. has all points on its surface – on the Convex Hull
2. has triangular facets, each generates 3 edges, shared by 2 triangles => $3F = 2E$

2-manifold

$F = 2E / 3$

- $V - E + F = 2$ … Euler formula for V = n points

$V - E + 2E/3 = 2$

$V - 2 = E / 3$

$E = 3V - 6, \quad V = n$

$E = O(n)$

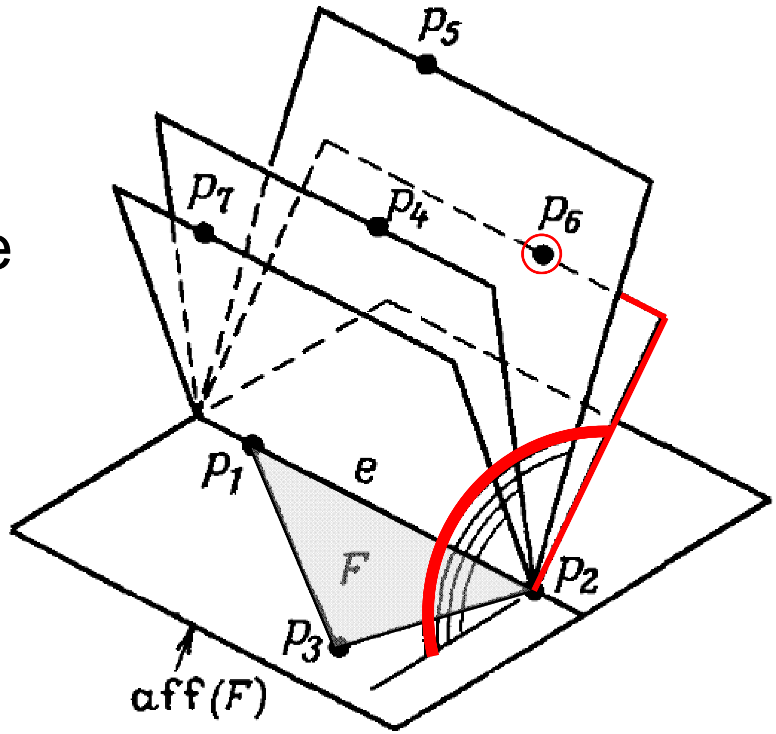
$F = 2E / 3$

$F = 2V - 4$

$F = O(n)$

1. Gift wrapping in higher dimensions

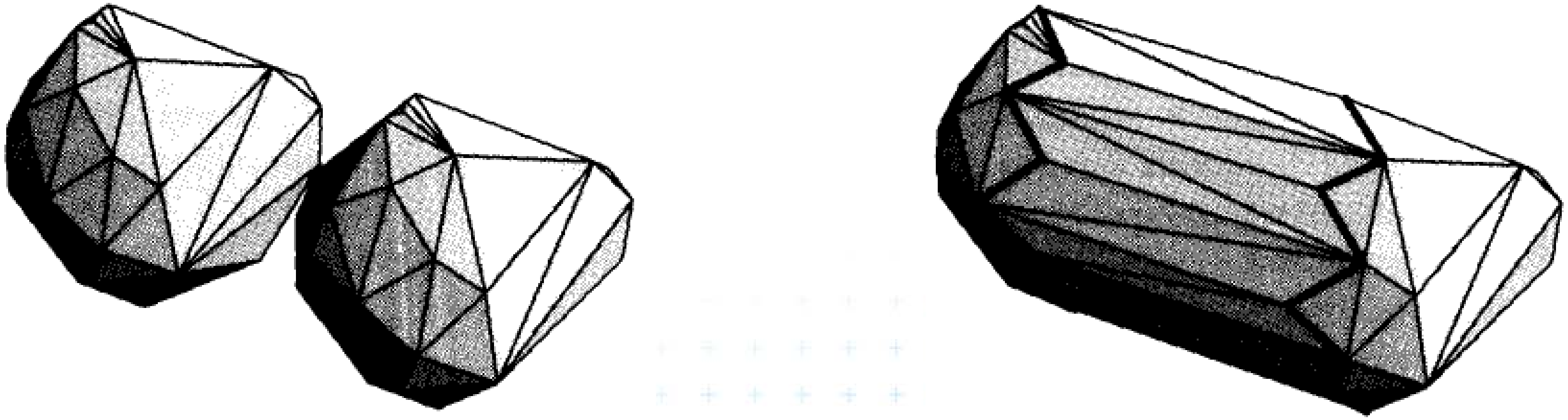
- First known algorithm for n-dimensions (1970)
- Direct extension of 2D alg.
- Complexity O(nF)
  - F is number of CH facets
  - Algorithm is output sensitive
  - Details on seminar, assignment [10]



[Preparata]

2. Divide & conquer 3D convex hull [Preparata, Hong77]

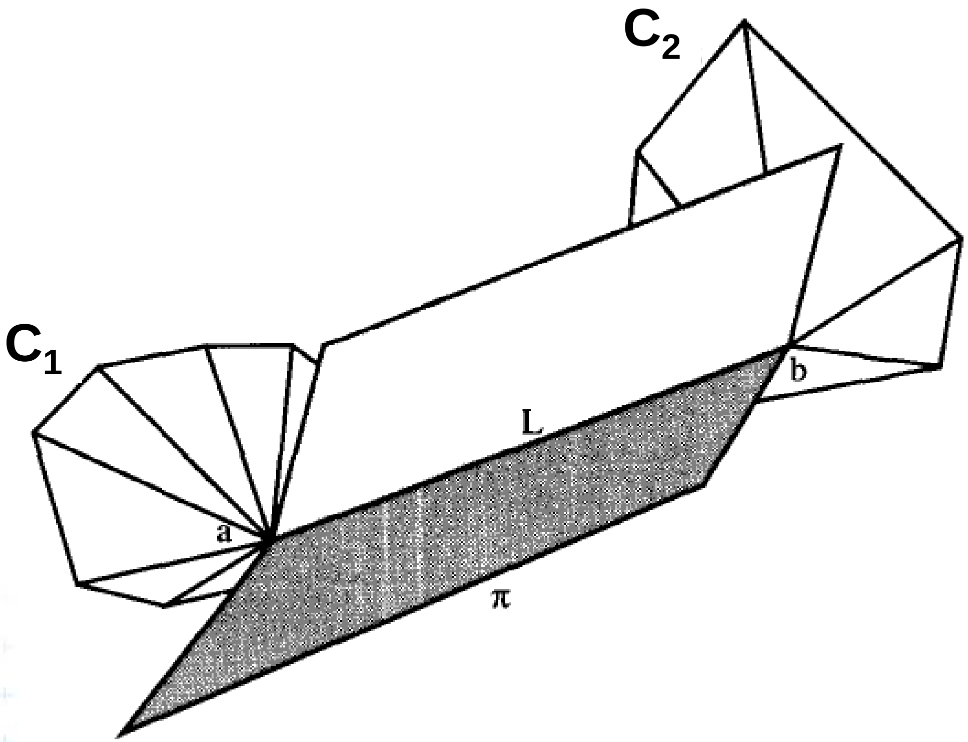
- Sort points in x-coord
- Recursively split, construct CH, merge
- Merge takes O(n) => O(n log n) total time



[Rourke]

Divide & conquer 3D convex hull [Preparata, Hong 77]

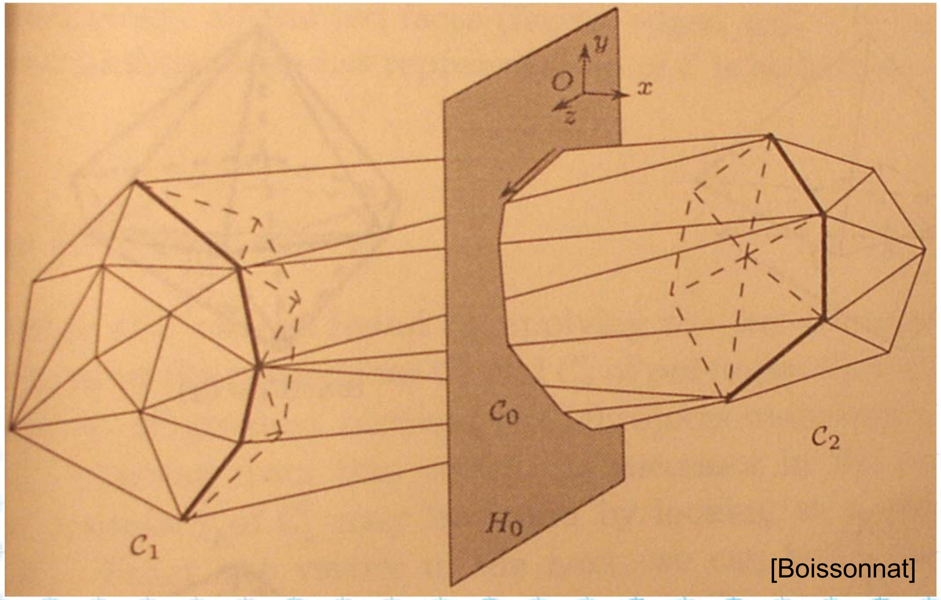
- Merge(C_1 with C_2) uses gift wrapping
  - Gift wrap plane around edge *e* – find new point *p* on C_1 or on C_2 (neighbor of *a* or *b*)
  - Search just the CW or CCW neighbors around *a, b*



[Rourke]

Divide & conquer 3D convex hull [Preparata, Hong 77]

- Performance O(*n* log *n*) rely on circular ordering
  - In 2D: Ordering of points around CH
  - In 3D: Ordering of vertices around 2-polytop C_0 (vertices on intersection of new CH edges with separating plane H_0) [ordering around horizon of C_1 and C_2 does not exist, as both horizons may be non-convex and even not simple polygons]



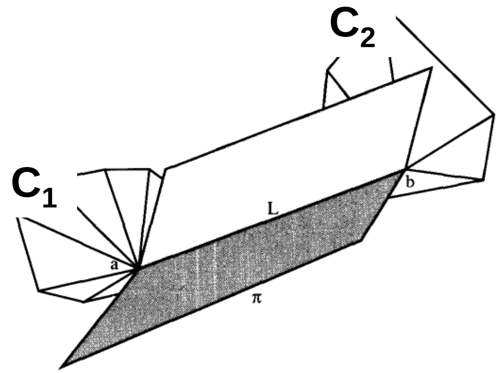
[Boissonnat]

  - In ≥ 4D: Such ordering does not exist

Divide & conquer 3D convex hull [Preparata, Hong 77]

Merge(C_1 with C_2)

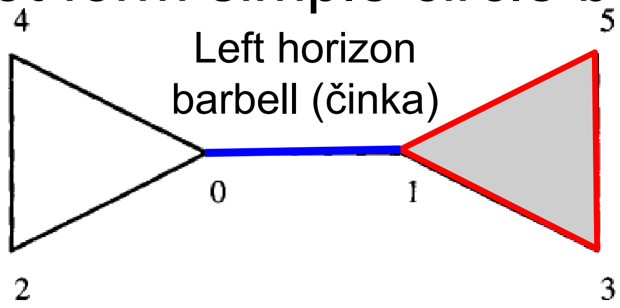
- Find the first CH edge L connecting C_1 with C_2
- $e = L$
- While not back at L *do* **CHYBA**
  - store e to C
  - Gift wrap plane around edge e – find new point P on C_1 or on C_2 (neighbor of a or b)
  - e = new edge to just found end-point P
  - Store new triangle eP to C
- Discard hidden faces inside CH from C
- Report merged convex hull C

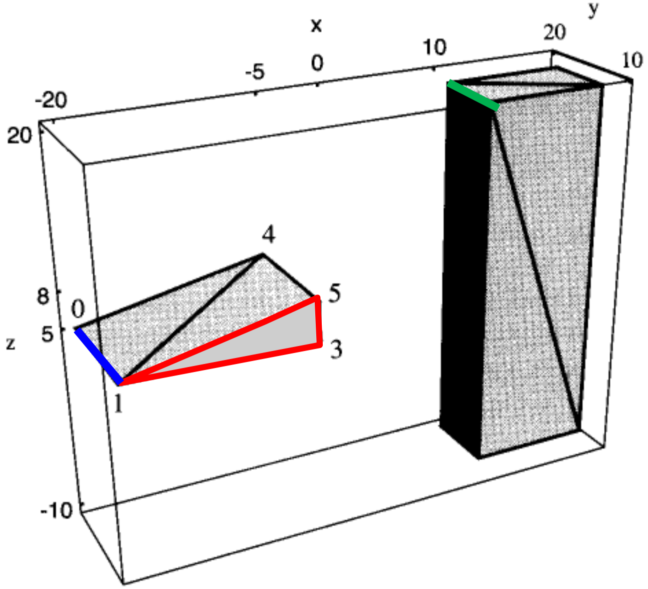


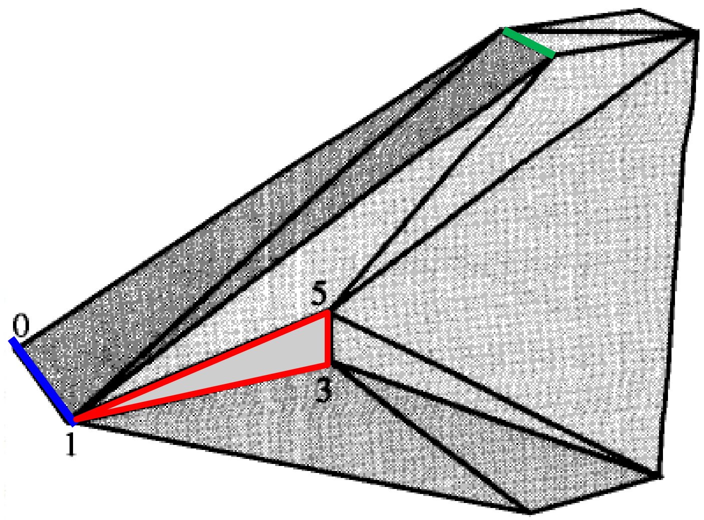
Divide & conquer 3D convex hull [Preparata, Hong 77]

- Problem of the wrapping phase [Edelsbrunner 88]
  - The edges on horizon do not form simple circle but a "barbell" 0,2,4,0,1,3,5,1

Do not stop here!





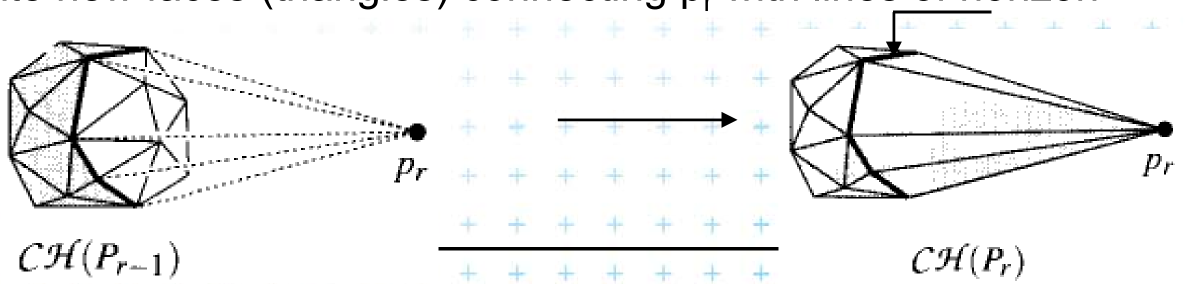


[Berg]

3. Randomized incremental alg. principle

1. Create tetrahedron (smallest CH in 3D) CH(P_4)
   - Take 2 points p_1 and p_2
   - Search the 3rd point not lying on line p_1p_2
   - Search the 4th point not lying in plane $p_1p_2p_3$ …if not found, use 2D CH
2. Perform random permutation of remaining points $\{p_5, \dots, p_n\}$
3. For p_r in $\{p_5, \dots, p_n\}$ do add point p_r to CH(P_{r-1})

   Notation: for $r \geq 1$ let $P_r = \{p_1, \dots, p_r\}$ is set of already processed pts
   - If p_r lies inside or on the boundary of CH(P_{r-1}) then do nothing
   - If p_r lies outside of CH(P_{r-1}) then
     - find and remove visible faces
     - create new faces (triangles) connecting p_r with lines of horizon



[Berg]

DCGI

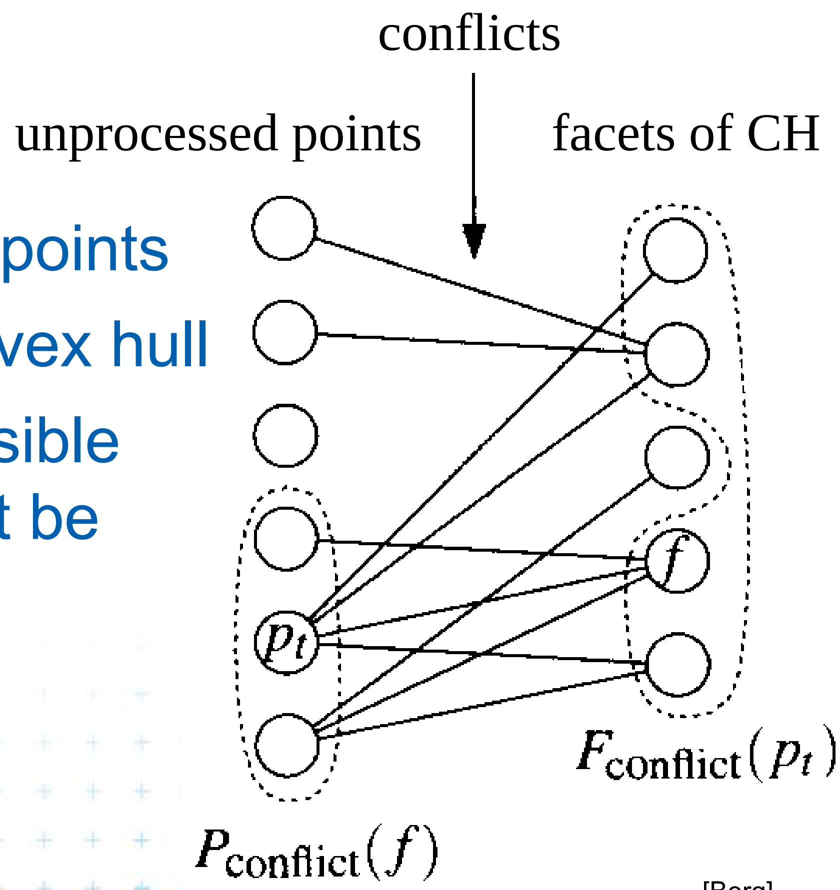
Conflict graph

- Stores unprocessed points with facets of CH they see
- Bipartite graph

  points p_t, $t > r$ … unprocessed points

  facets of CH(P_r)… facets of convex hull

  conflict arcs … conflict, as visible facets cannot be in CH



[Berg]

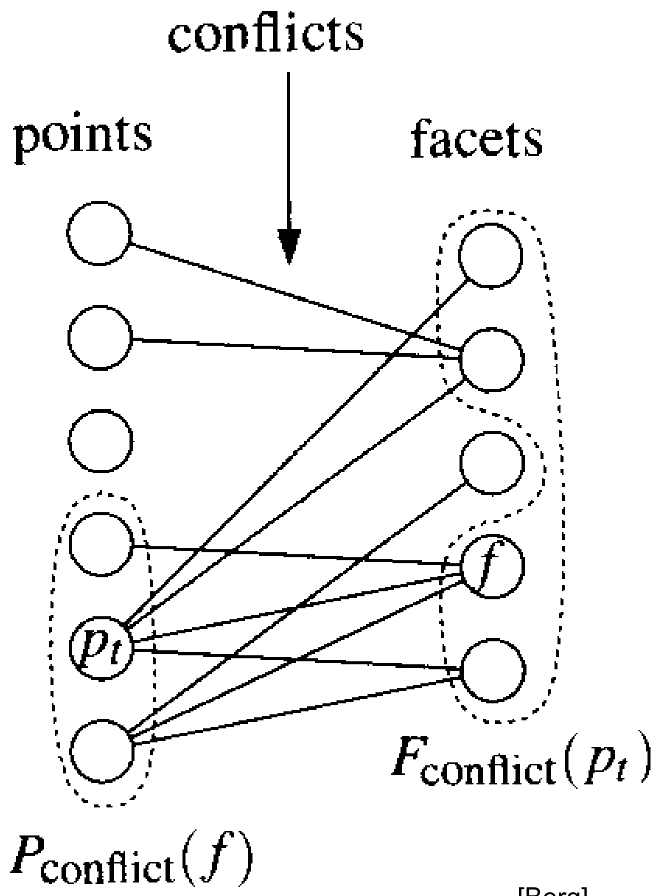
- Maintains sets:

  $P_{conflict}(f)$ … points, that see f

  $F_{conflict}(p_r)$… facets visible from p_r (visible region – deleted after insertion of p_r)

Conflict graph – init and final state

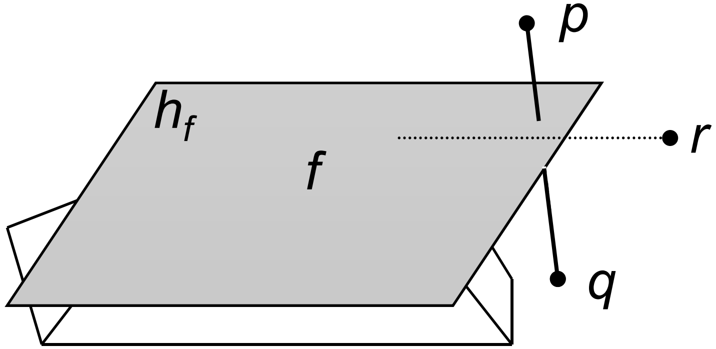
- Initialization
  - Points $\{p_5, \ldots, p_n\}$ (not in tetrahedron)
  - Facets of the tetrahedron (four)
  - Arcs – connect each tetrahedron facet with points visible from it
- Final state
  - Points – {} = empty set
  - Facets of the convex hull
  - Arcs - none



[Berg]

Visibility between point and face

- Face *f* is visible from a point *p* if that point lies in the open half-space on the other side of h_f than the polytope



f is visible from *p* (*p* is *above* the plane)

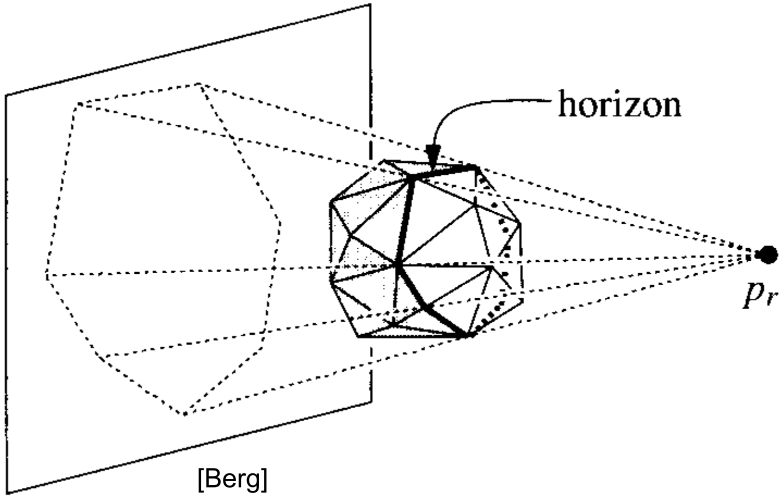
f is not visible from *r* lying *in the plane* of *f* (this case will be discussed next)

f is not visible from *q*

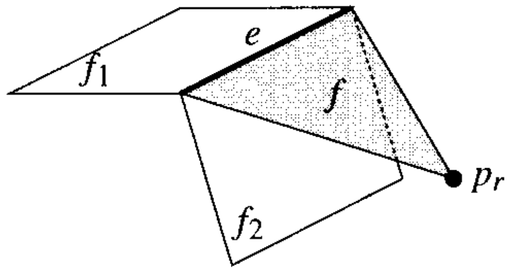
$p \in P_{conflict}(f)$, p is among the points that see the face f
$f \in F_{conflict}(p)$ f is among the faces visible from point p

New triangles to horizon

- Horizon = edges *e* incident to visible and invisible facets



[Berg]



[Berg]

- New triangle *f* connects edge *e* on horizon and point p_r and
  - creates new node for facet *f* updates the conflict graph
  - add arcs to points visible from *f* (subset from $P_{coflict}(f_1) \cup P_{coflict}(f_2)$)
- Coplanar triangles on the plane ep_r
  - are merged with new triangle.
  - Conflicts in G are copied from the deleted triangle (same plane)

Overview of new point insertion

Processing of point p_r outside

- Remove facets that p_r sees from the CH (do not delete them from the graph G)
- Find horizon edges (around the hole in CH)
- Create new facets from horizon edges to p_r
  - add them to CH
  - create face nodes f in G for them
- Compute what p_r sees – search only from $P(e) \ = P_{conflict}(f_1) \cup P_{conflict}(f_2)$)
- Delete node p_r and face $F_{conflict}(p_r)$ from G

Incremental Convex hull algorithm

IncrementalConvexHull(P)

Input: Set of n points in general position in 3D space

Output: The convex hull $C = CH(P)$ of P

1. Find four points that form an initial tetrahedron, $C = CH(\{p_1, p_2, p_3, p_4\})$
2. Compute random permutation $\{p_5, p_6, \ldots, p_n\}$ of the remaining points
3. Initialize the conflict graph G with all visible pairs (p_t, f),
   where f is facet of C and $p_t, t > 4$, are non-processed points
4. **for** r = 5 to n **do** …inserting p_r, into C
5. **if**($F_{conflict}(p_r)$ is not empty) **then** …p_r is outside, insert p_r, into C
6. Delete all facets $F_{conflict}(p_r)$ from C … only from hull C, not from G
7. Walk around visible region boundary, create list L of horizon edges
8. **for** all $e \in L$ **do**
9. connect e to p_r by a new triangular facet f
10. **if** f is coplanar with its neighbor facet f' along e
11. **then** merge f and f' in C, take conflict list from f'
12. **else** … determine conflicts for new facet f

… [continue on the next slide]

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Incremental Convex hull algorithm (cont…)

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else … not coplanar => determine conflicts for new facet f
    Insert f into hull C
    Create node for f in G          //… new face in conflict graph G
    Let f_1 and f_2 be the facets incident to e in the old CH(P_{r-1})
    P(e) = P_conflict(f_1) ∪ P_conflict(f_2)
    for all points p ∈ P(e) do
        if f is visible from p, then add(p, f) to G … new edges in G
Delete the node corresponding to p_r and the nodes corresponding
to facets in F_conflict(p_r) from G, together with their incident arcs
return C
```

Complexity: Convex hull of a set of points in E^3 can be computed incrementally in $O(n \log n)$ randomized expected time (process $O(n)$ points, but number of facets and arcs depend on the order of inserting points – up to $O(n^2)$)

For proof see: [Berg, Section11.3]

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Convex hull in higher dimensions

- Convex hull in *d* dimensions can have $\Omega(n^{\lfloor d/2 \rfloor})$
  Proved by [Klee, 1980]
- Therefore, 4D hull can have quadratic size
- No *O(n log n)* algorithm possible for d>3
- These approaches can extend to d>3
  - Gift wrapping
  - D&C
  - Randomized incremental
  - QuickHull

Conclusion

- Recapitulation of 2D algorithms
- >=3D algorithms
  - Gift wrapping
  - D&C
  - Randomized incremental
  - QuickHull

References

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