GEOMETRIC SEARCHING PART 2: RANGE SEARCH

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg] and [Mount]

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Range search

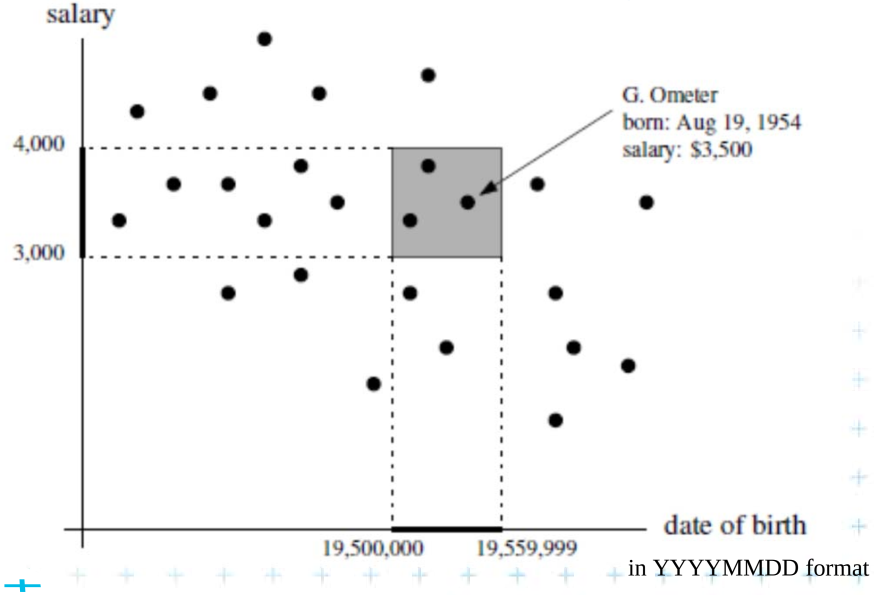
- Orthogonal range searching
- Canonical subsets
- 1D range tree
- 2D-nD Range tree
  - With fractional cascading (Layered tree)
- Kd-tree

Orthogonal range searching

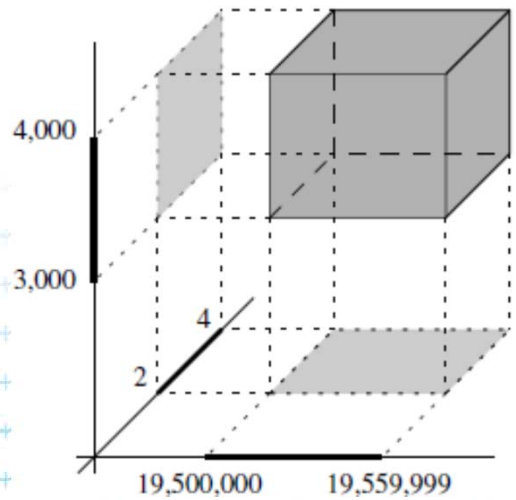
– Given a set of points P, find the points in the region Q

– Example: Databases (records->points)
  • Find the people with given range of salary, date of birth, kids, …

2D: axis parallel rectangle



3D: axis parallel box



[Berg]

Orthogonal range searching

– Given a set of points P, find the points in the region Q
  • Search space: a set of points P (somehow represented)

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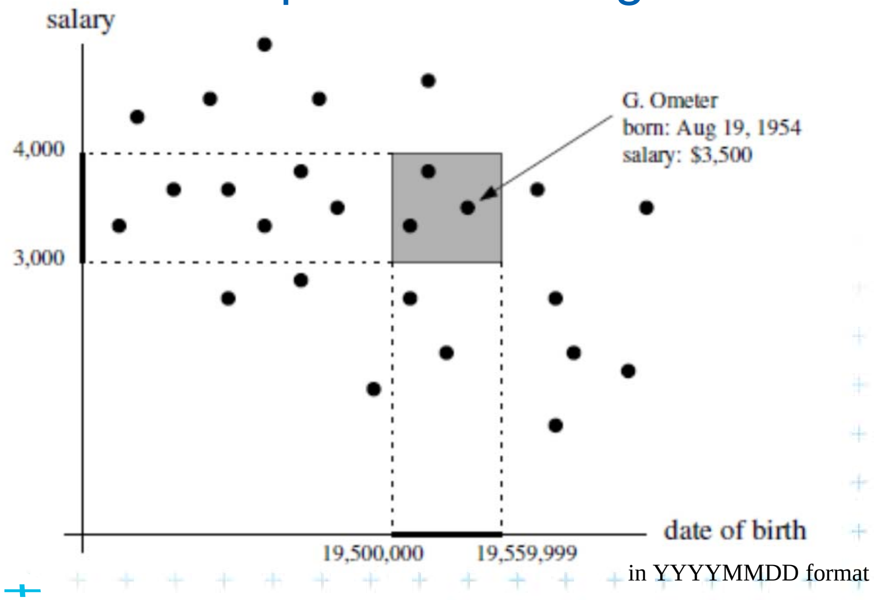

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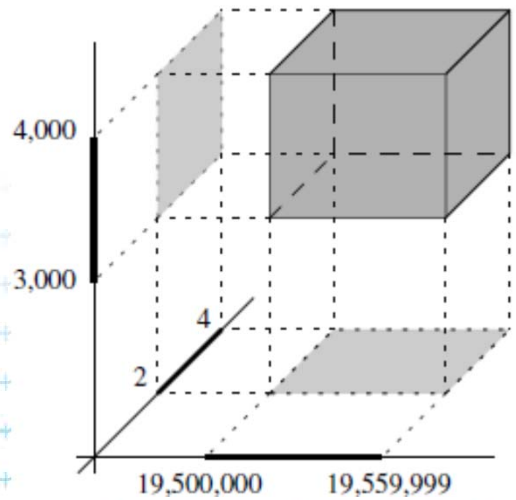

[Berg]

Orthogonal range searching

– Given a set of points P, find the points in the region Q
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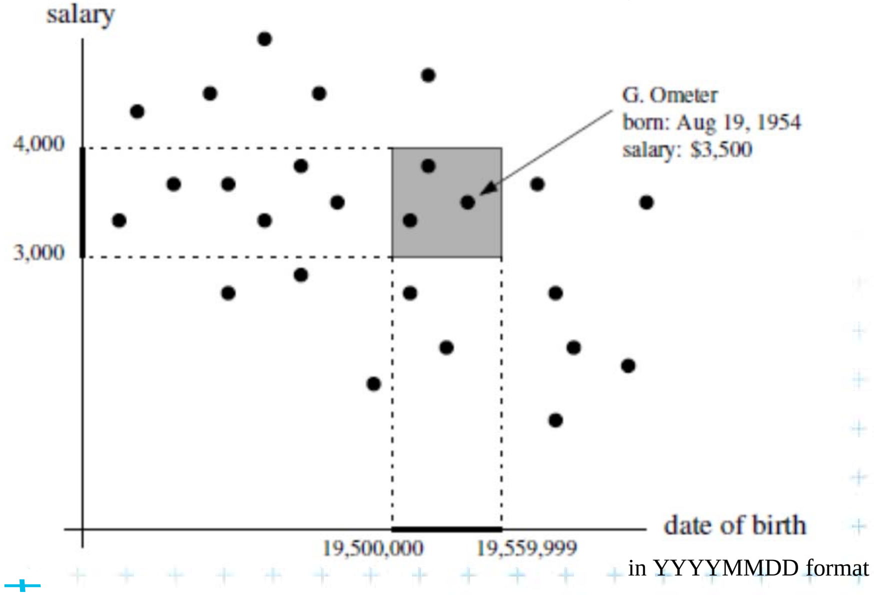

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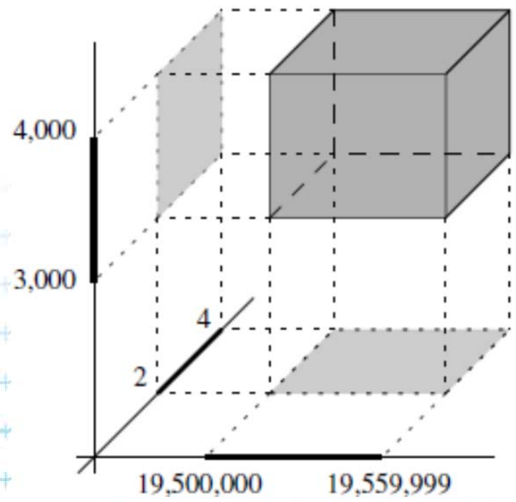
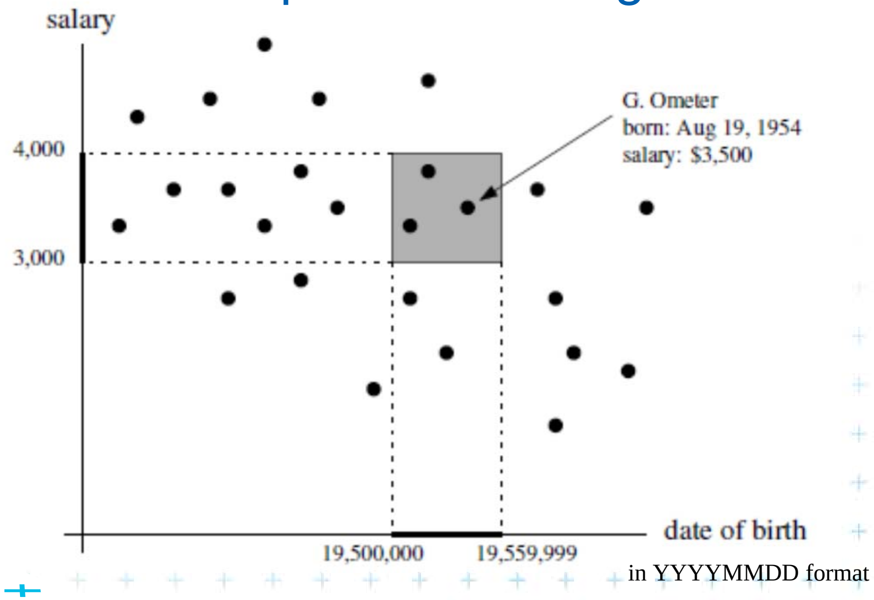


[Berg]

Orthogonal range searching

– Given a set of points P, find the points in the region Q
  - Search space: a set of points P (somehow represented)
  - Query: intervals Q (axis parallel rectangle)
  - Answer: points contained in Q
– Example: Databases (records->points)
  - Find the people with given range of salary, date of birth, kids, …

2D: axis parallel rectangle



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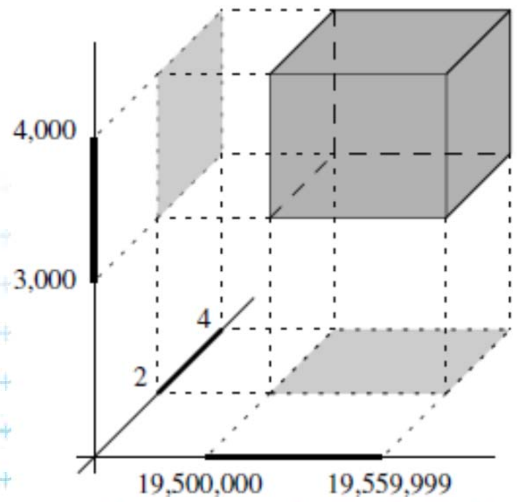

[Berg]

Orthogonal range searching

- Query region = axis parallel rectangle
  - nDimensional search can be decomposed into set of 1D searches (separable)

Other range searching variants

- Search space S: set of
  - line segments,
  - rectangles, …
- Query region Q: any other searching region
  - disc,
  - polygon,
  - halfspace, …
- Answer: subset of S laying in Q

- We concentrate on points in orthogonal ranges

How to represent the search space?

Basic idea:

- Not all possible combination can be in the output (not the whole power set)
- => Represent only the “selectable” things (a well selected subset –> one of the canonical subsets)

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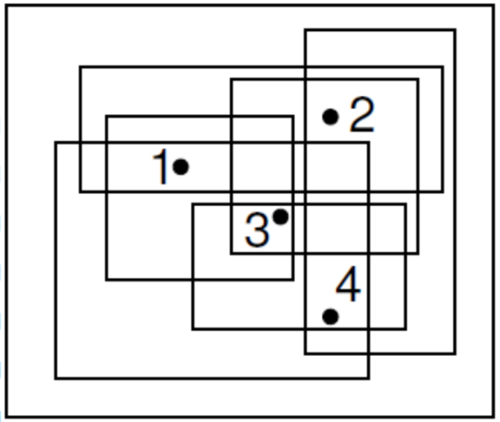
Example?

Subsets selectable by given range class

- The number of subsets that can be selected by simple ranges Q is limited
- It is usually much smaller than the power set of P
  - Power set of P where P = {1,2,3,4} (potenční množina) is {{ }, {1},{2},{3},{4}, {1,2},{1,3},{1,4}, {2,3},…,{2,3,4}, {1,2,3,4} } … $O(2^n)$
    i.e. set of all possible subsets
  - Simple rectangular queries are limited
    - Defined by max 4 points along 4 sides => $O(n^4)$ of $O(2^n)$ power set
    - Moreover – not all sets can be formed by □ query Q

      e.g. sets {1,4} and {1,2,4} cannot be formed



[Mount]

Canonical subsets S_i

Search space $S = (P, Q)$ represented as a collection of canonical subsets $\{S_1, S_2, \dots, S_k\}$, each $S_i \quad S$,

- S_i may overlap each other (elements can be multiple times there)
- Any set can be represented as disjoint union disjunktní sjednocení of canonical subsets S_i each element knows from which subset it came
- Elements of disjoint union are ordered pairs (x, i) (every element x with index i of the subset S_i)

S_i may be selected in many ways

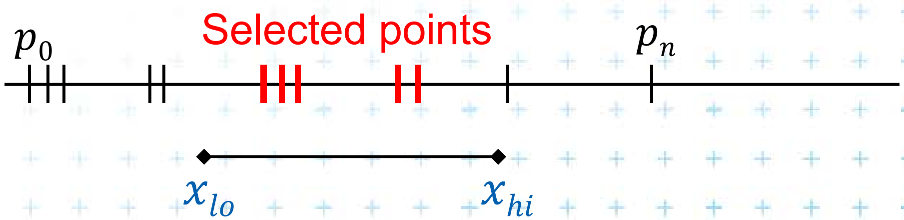
- from *n* singletons $\{pi\}$ … $O(n)$
- to power set of P … $O(2^n)$

- Good DS balances between total number of canonical subsets and number of CS needed to answer the query

1D range queries (interval queries)

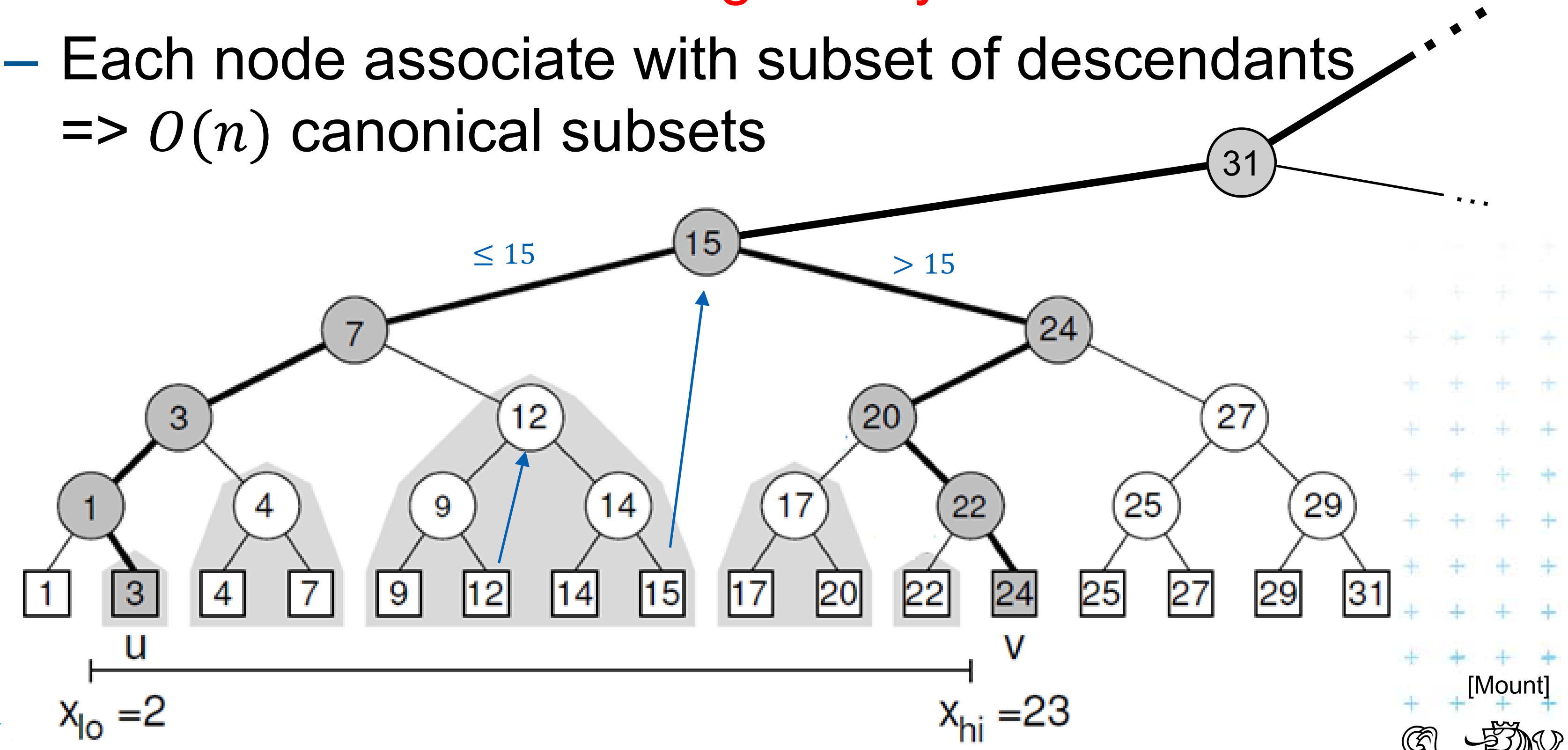
- Query: Search the interval $[x_{lo}, x_{hi}]$
- Search space: Points $P = \{p_1, p_2, \dots, p_n\}$ on the line
  a) Binary search in an array
    - Simple, but
    - not generalize to any higher dimensions

  b) Balanced binary search tree
    - 1D range tree
    - maintains canonical subsets
    - generalize to higher dimensions



1D range tree definition

- Balanced binary search tree (with repeated keys)
  - leaves – sorted points
  - inner node label – the largest key in its left child
  - Each node associate with subset of descendants => $O(n)$ canonical subsets



Canonical subsets and <2,23> search

- Canonical subsets for this subtree are #

{ {1}, {3}, …, {31}, 16

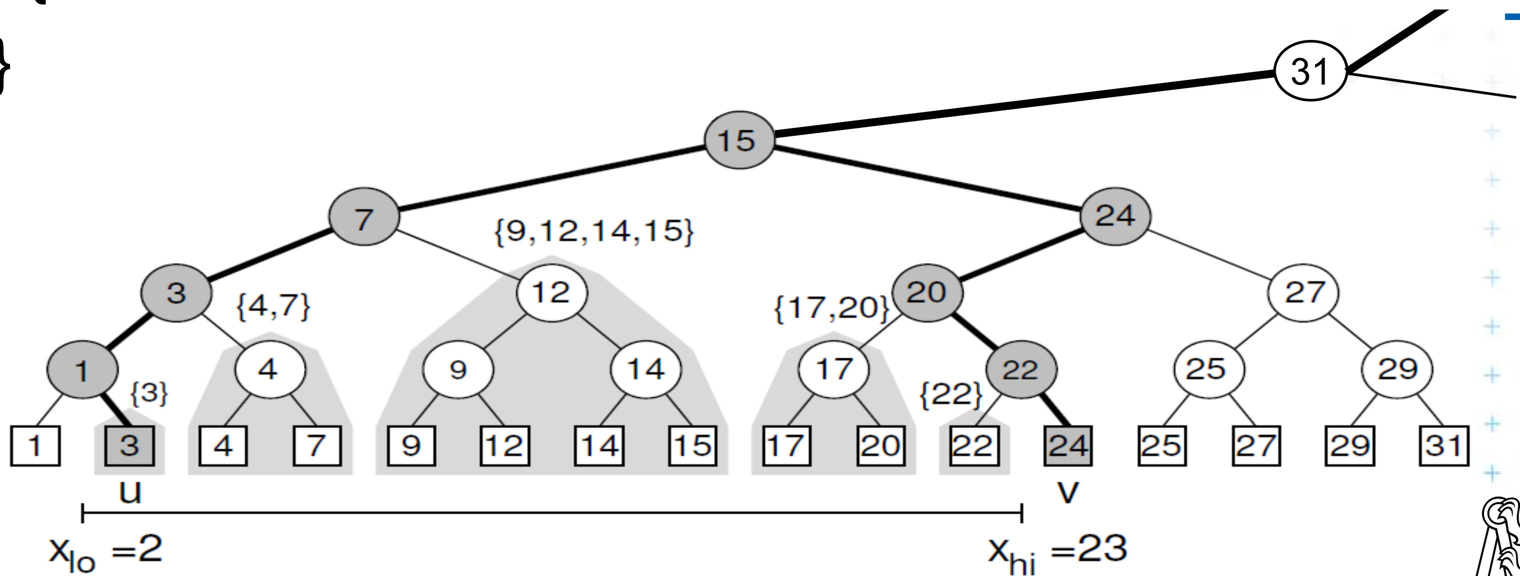
{1, 3}, {4, 7}, …, {29, 31} 8

{1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31} 4

{1, 3, 4, 7, 9, 12, 14, 15}, {17, 20, 22, 24, 25, 27, 29, 31} 2

{1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31} 1

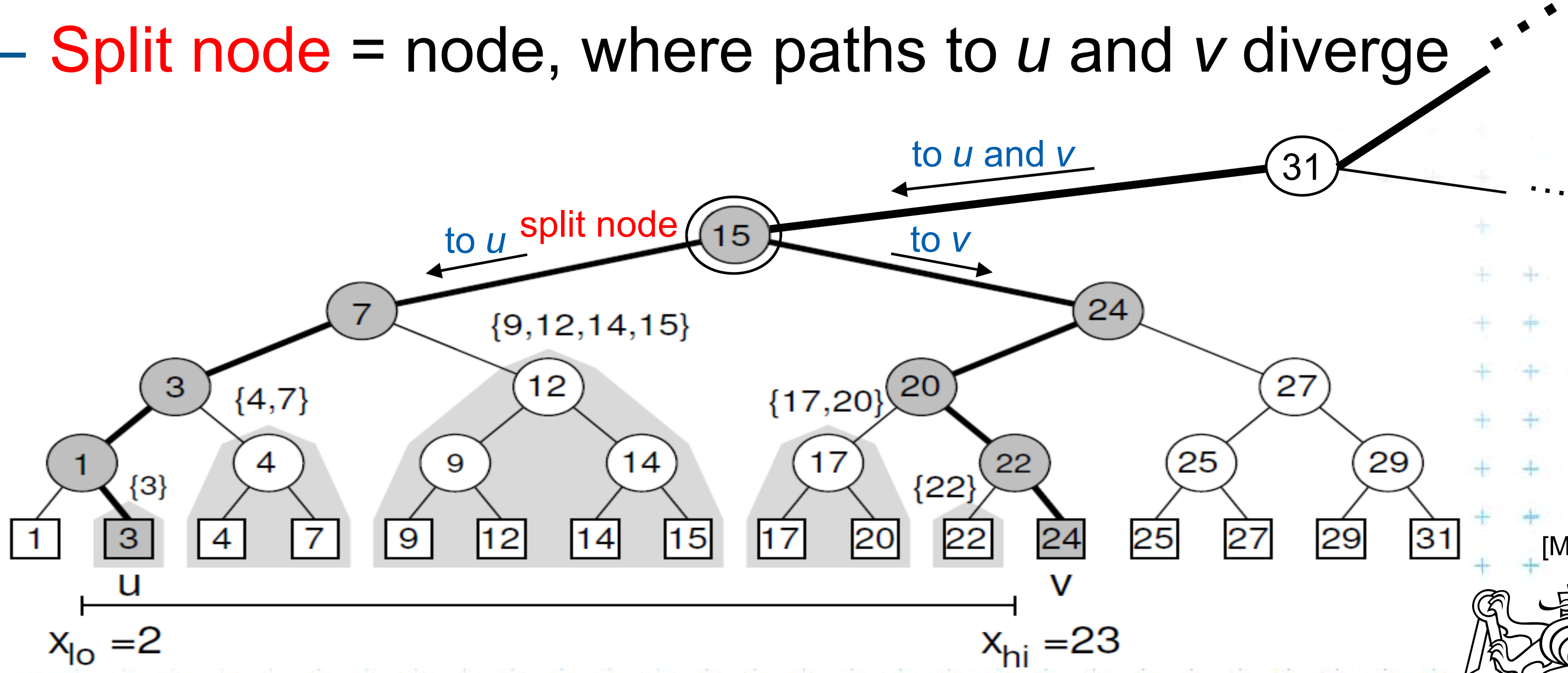
} O(n)



[Mount]

1D range tree search interval <2,23>

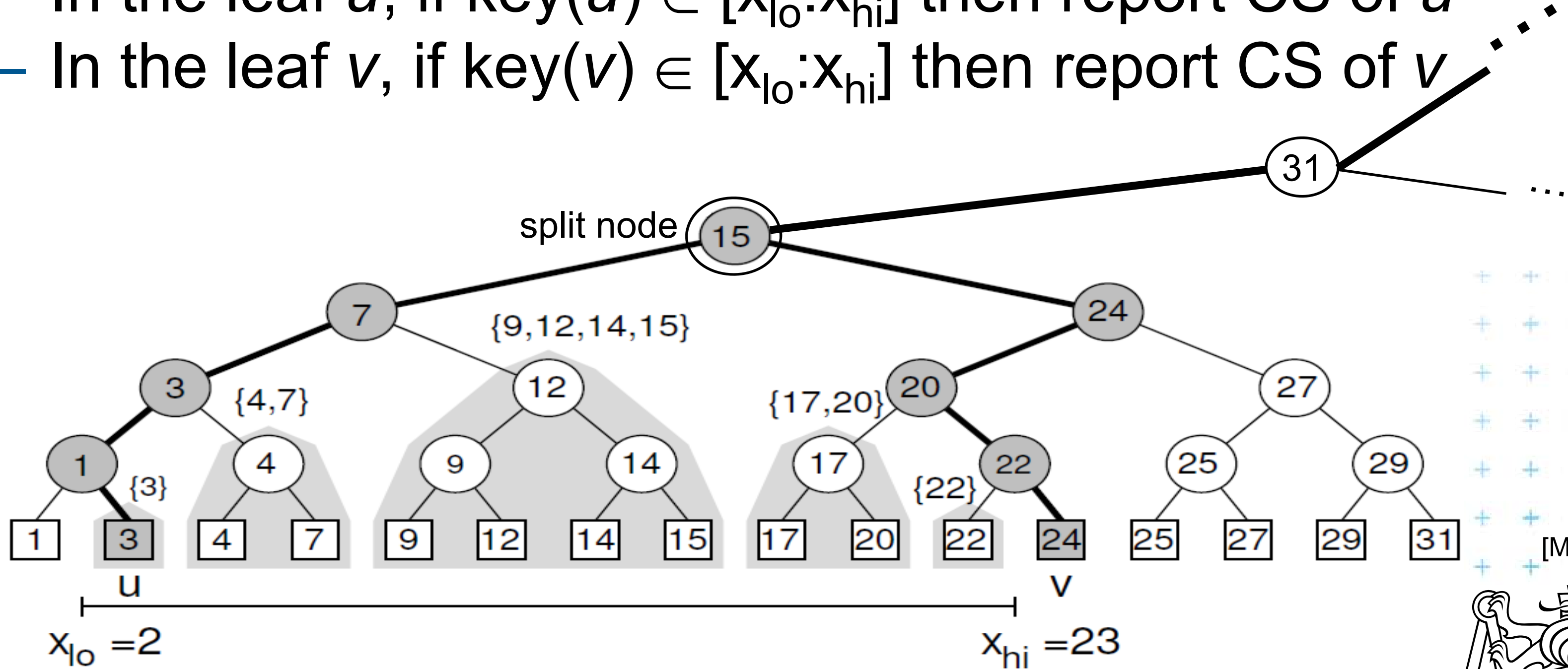
- Canonical subsets for any range found in O(log *n*)
  - Search x_{lo}: Find leftmost leaf *u* with key(*u*) $\geq x_{lo}$ 2 -> $\boxed{3}$
  - Search x_{hi}: Find leftmost leaf *v* with key(*v*) $\geq x_{hi}$ 23 -> $\boxed{24}$
  - Points between *u* and *v* lie within the range => report canon. subsets of maximal subtrees between *u* and *v*
  - Split node = node, where paths to *u* and *v* diverge



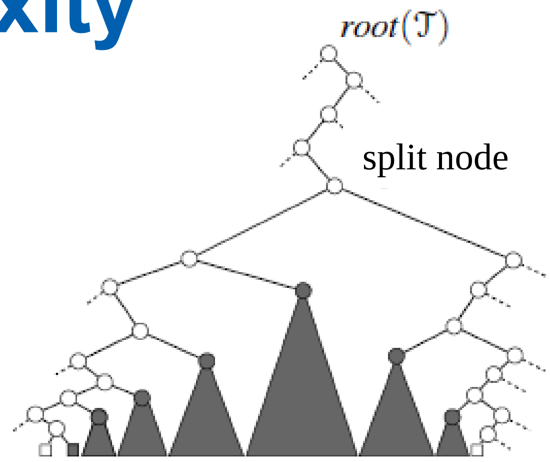
[Mount]

1D range tree search

- Reporting the subtrees (below the split node)
  - On the path to *u* whenever the *path goes left*, report the canonical subset (CS) associated to right child
  - On the path to *v* whenever the *path goes right*, report the canonical subset associated to left child
  - In the leaf *u*, if key(*u*) $\in [x_{lo}:x_{hi}]$ then report CS of *u*
  - In the leaf *v*, if key(*v*) $\in [x_{lo}:x_{hi}]$ then report CS of *v*



1D range tree search complexity



[Berg]

- Path lengths O(log n)

  => O(log n) canonical subsets (subtrees)

- Range counting queries
  - Return just the number of points in given range
  - Sum the total numbers of leaves stored in maximum subtree roots … O(log *n*) time

- Range reporting queries
  - Return all *k* points in given range
  - Traverse the canonical subtrees … O(log *n* + *k*) time

- O(*n*) storage, O(*n* log *n*) preprocessing (sort P)

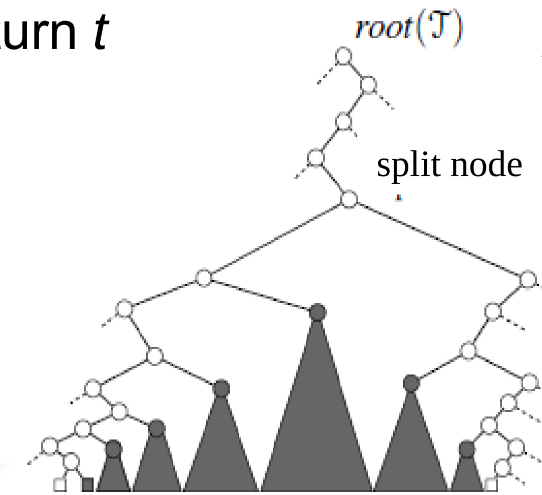
Find split node

FindSplitNode(*T*, [x:x’])

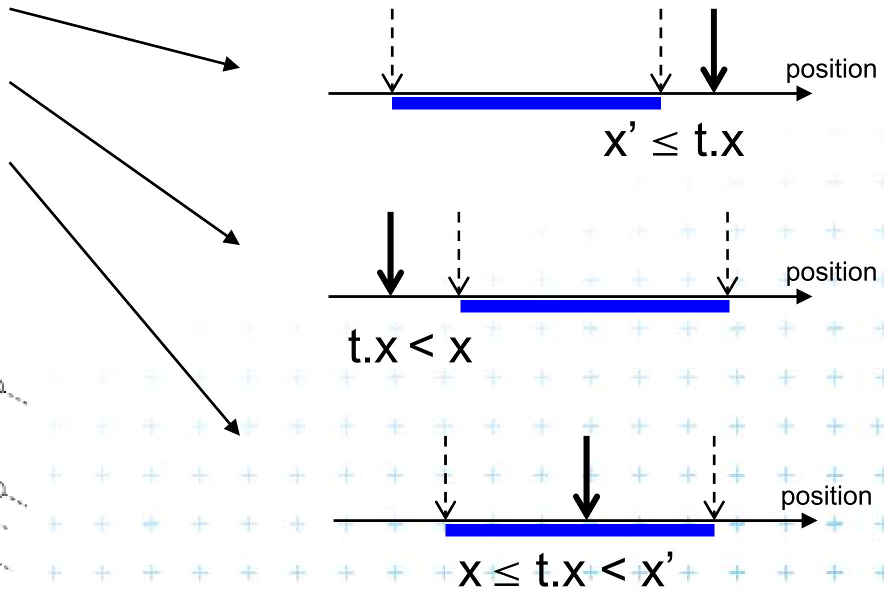
Input: Tree *T* and Query range [x:x’], x ≤ x’

Output: The node, where the paths to x and x’ split or the leaf, where both paths end

1. *t = root(T)*
2. while(*t* is not a leaf **and (**x’ ≤ t.x **or** t.x < x**)**) // *t* out of the range [x:x’]
3. if(*x’* ≤ t.x) *t = t.left*
4. else *t = t.right*
5. return *t*



[Berg]



STOP

1D range search

(2D on slide 30)

1dRangeQuery(*t*, [x:x’])

Input: 1d range tree *t* and Query range $[x : x']$

Output: All points in *t* lying in the range

t_{split} = FindSplitNode(*t, x, x’*) // find interval point t $\in$ [x:x’]

if(t_{split} is leaf) // e.g. Searching [16:17] or [16:16.5] both stops in the leaf 17 in the previous example

  check if the point in t_{split} must be reported // $t_x \in [x : x']$

else // follow the path to *x*, reporting points in subtrees right of the path

  t = t_{split}.left

  while(t is not a leaf)

    if($x \le$ t.x)

      ReportSubtree(*t.right*) // any kind of tree traversal

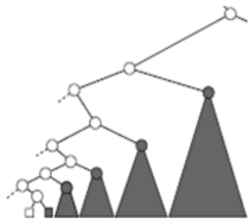
      t = t.left

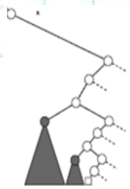
    else *t = t.right*

  check if the point in leaf *t* must be reported

  // Symmetrically follow the path to x’ reporting points left of the path

  t = t_{split}.right …



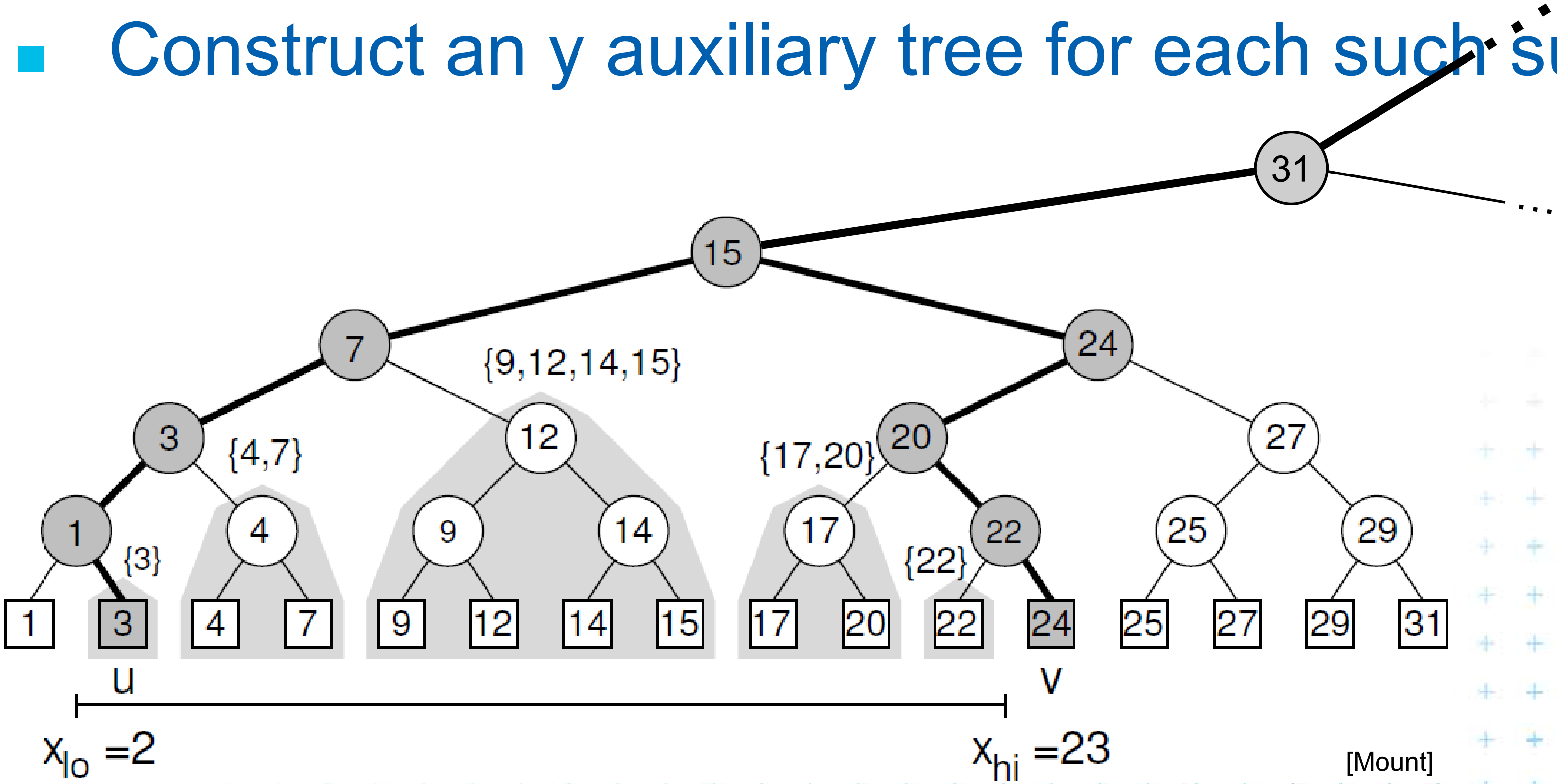


Multidimensional range searching

- Equal principle – find the largest subtrees contained within the range
- Separate one *n*-dimensional search into *n* 1-dimensional searches
- Different tree organization
  - Orthogonal (Multilevel) range search tree
    e.g. nd range tree
  - Kd tree

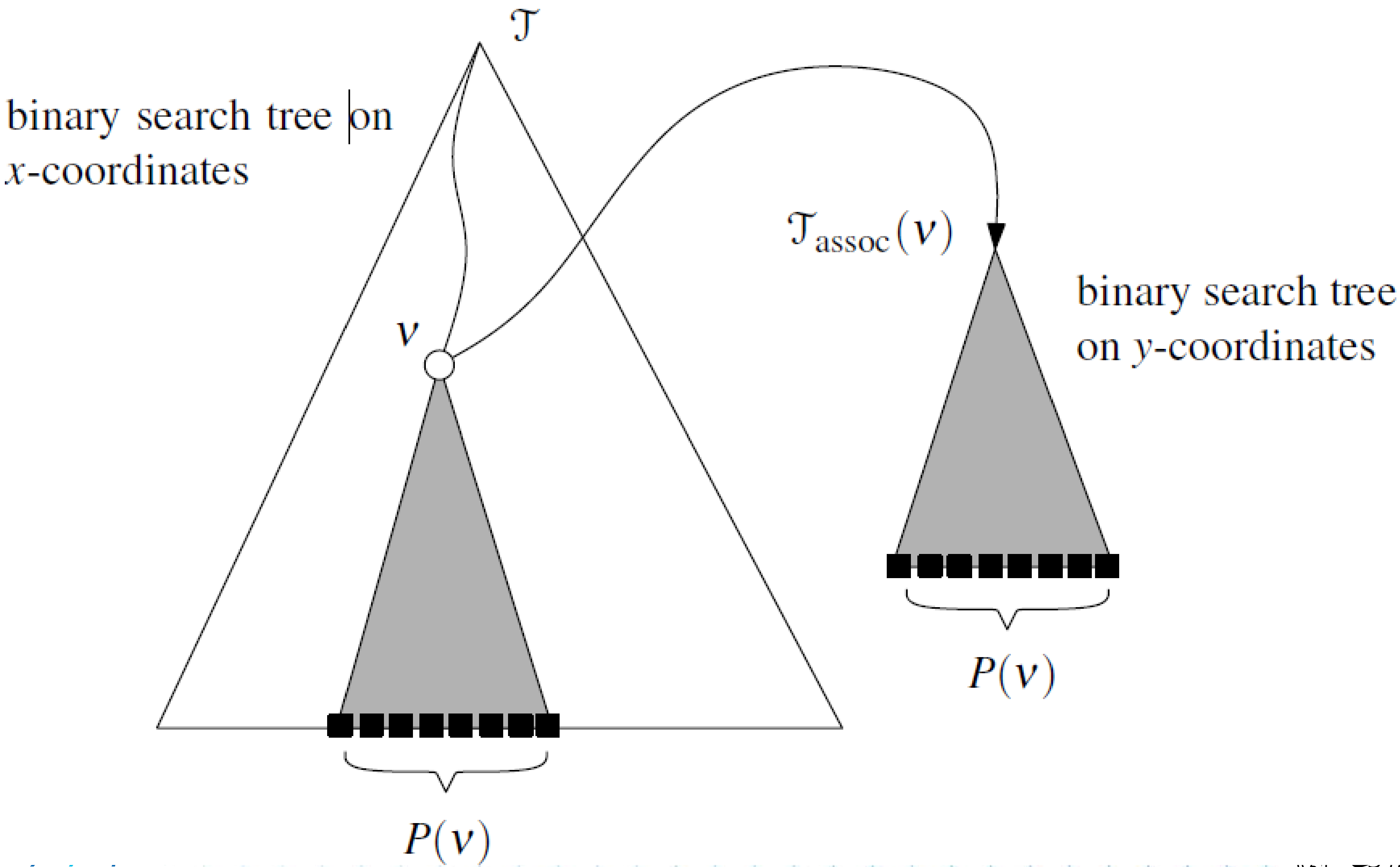
From 1D to 2D range tree

- Search points from $[Q.x_{lo}, Q.x_{hi}]$ $[Q.y_{lo}, Q.y_{hi}]$
- 1d range tree: log n canonical subsets based on x
- Construct an y auxiliary tree for each such subset

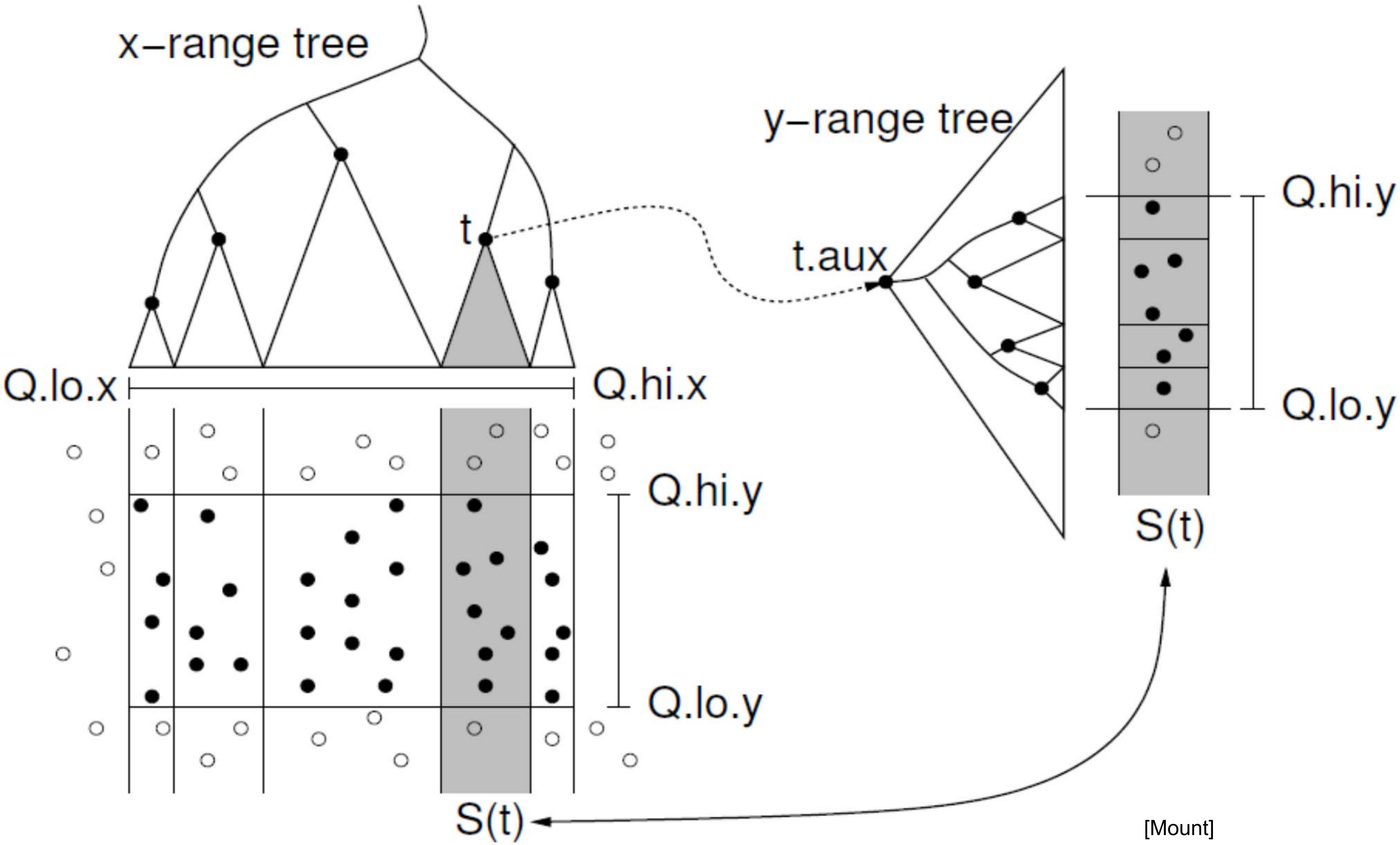


[Mount]

y-auxiliary tree for each canonical subset



2D range tree



[Mount]

2D range search

2dRangeQuery(*t*, [x:x'] × [y:y'])

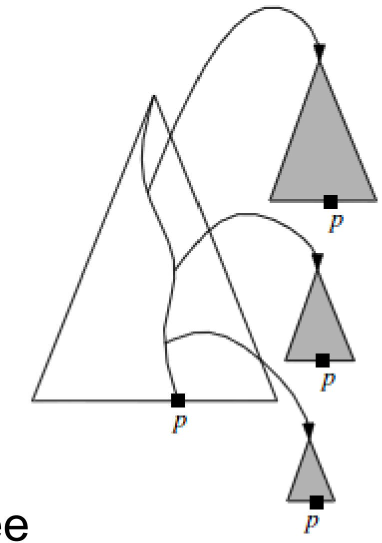
Input: 2d range tree *t* and Query range

Output: All points in *t* laying in the range

t_{split} = FindSplitNode(*t, x, x'*)
if(t_{split} is leaf)
    check if the point in t_{split} must be reported … t.x ∈ [x:x'], t.y ∈ [y:y']
else // follow the path to *x*, calling 1dRangeQuery on y
    t = t_{split}.left // path to the left
    while(t is not a leaf)
        if(*x* ≤ t.x)
            1dRangeQuerry(t_{assoc}(*t.right*), [*y:y'*]) // check associated subtree
            t = t.left
        else *t = t.right*
check if the point in leaf *t* must be reported … t.x ≤ x', t.y ∈ [y:y']
Similarly for the path to x' … // path to the right

2D range tree

- Search $O(\log^2 n + k)$ … $\log n$ in x, $\log n$ in y
- Space $O(n \log n)$
  - $O(n)$ the tree for x-coords
  - $O(n \log n)$ trees for y-coords
    - Point p is stored in all canonical subsets along the path from root to leaf with p,
    - once for x-tree level (only in one x-range)
    - each canonical subsets is stored in one auxiliary tree
    - $\log n$ levels of x-tree => $O(n \log n)$ space for y-trees



[Berg]

- Construction - $O(n \log n)$
  - Sort points (by x and by y). Bottom up construction

Canonical subsets

- Canonical subsets for this subtree are

Canonical subsets	#
{ {1}, {3}, …, {31},	16
{1, 3}, {4, 7}, …, {29, 31}	8
{1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31}	4
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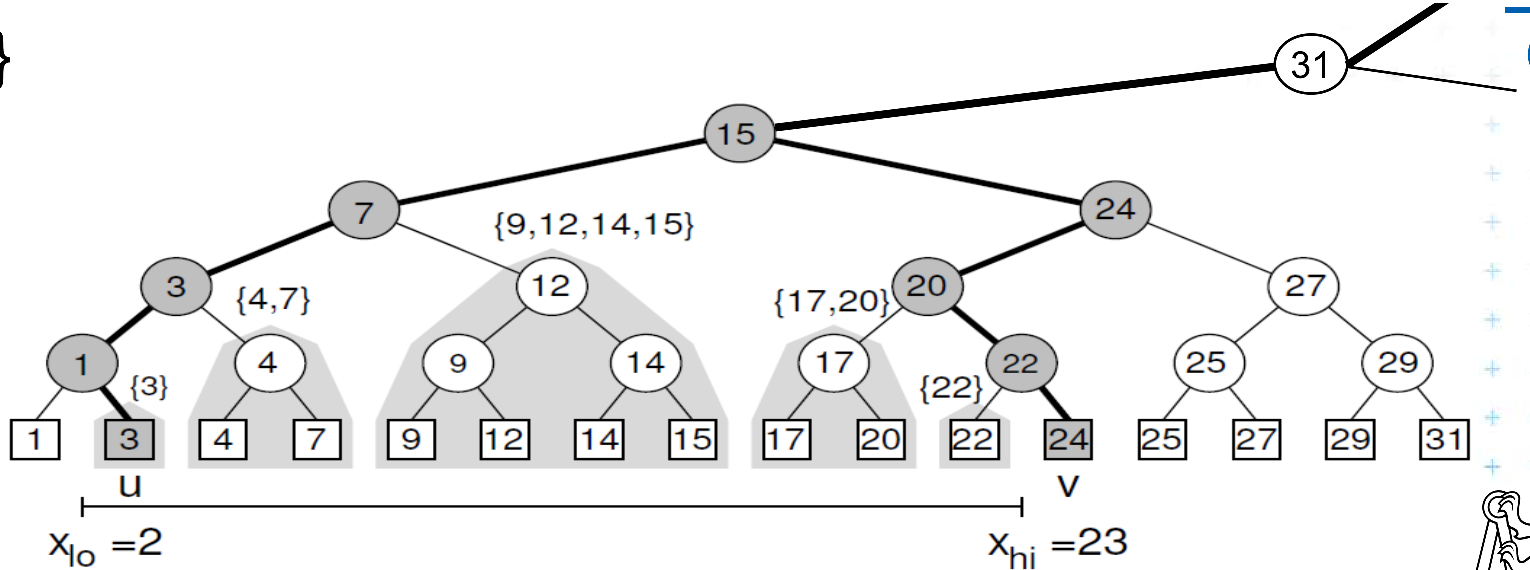
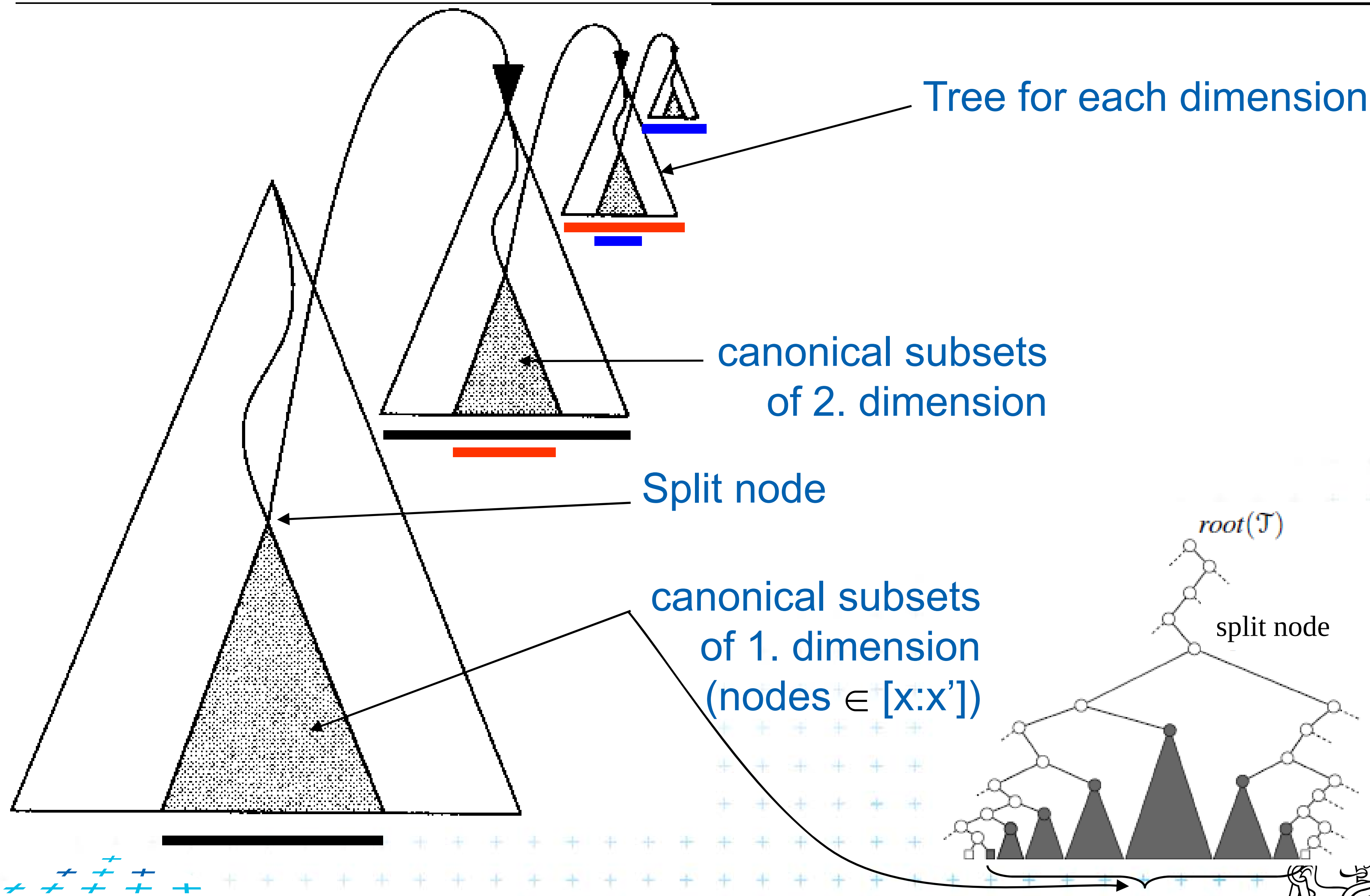
{1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31}	1
}	O(n)



[Mount]

nD range tree (multilevel search tree)

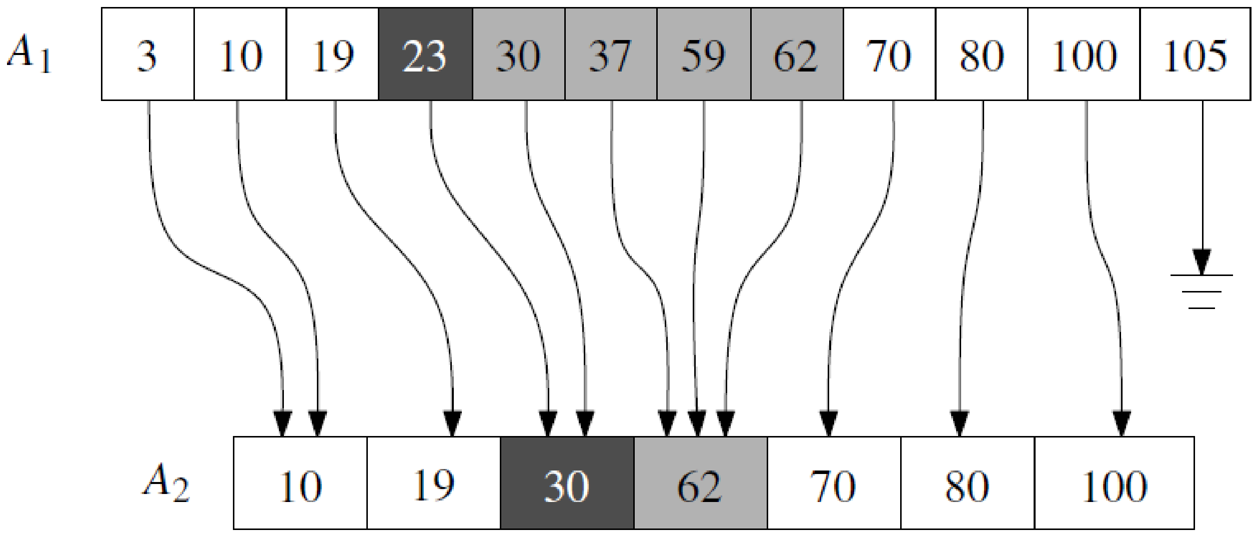


Fractional cascading - principle

- Two sets S_1, S_2 stored in sorted arrays A_1, A_2
- Report objects in both arrays whose keys in [y:y']
- Naïve approach – search twice independently
  - $O(\log n_1 + k_1)$ – search in A_1 + report k_1 elements
  - $O(\log n_2 + k_2)$ – search in A_2 + report k_2 elements
- Fractional cascading – adds pointers from A_1 to A_2
  - $O(\log n_1 + k_1)$ – search in A_1 + report k_1 elements
  - $O(1 + k_2)$ – jump to A_2 + report k_2 elements
  - Saves the $O(\log n_2)$ – search

Fractional cascading – principle for arrays

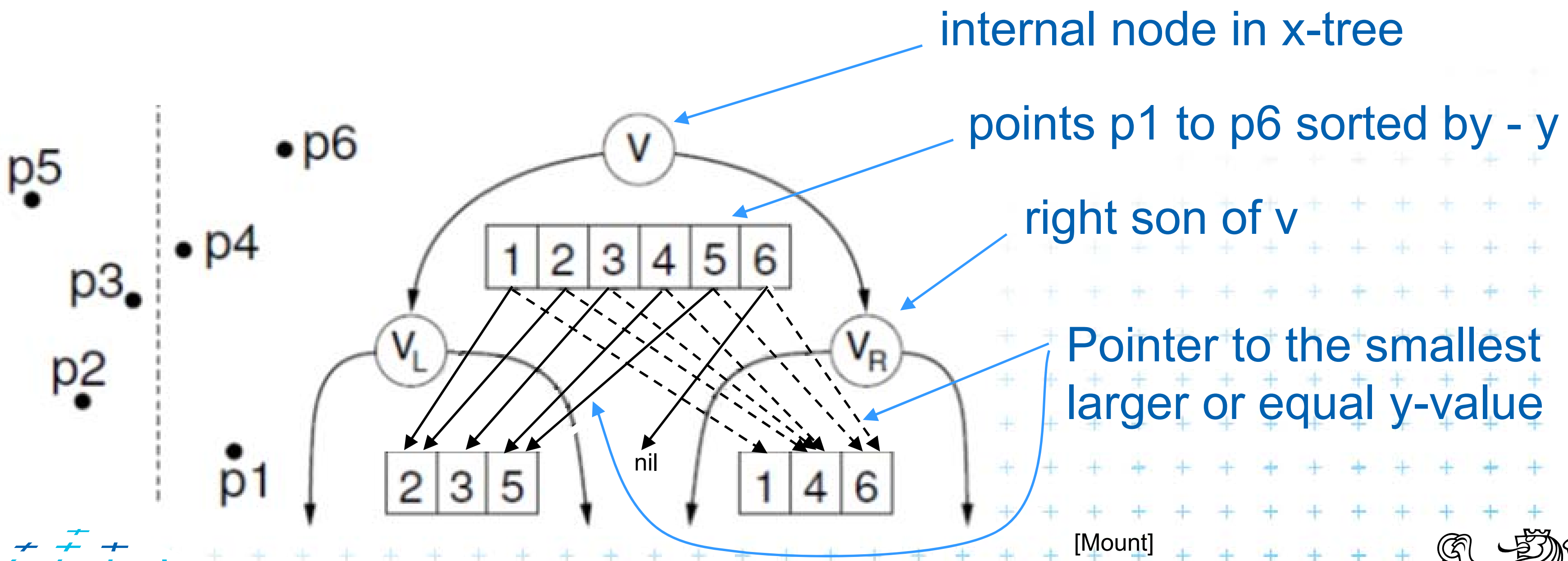
- Add pointers from A_1 to A_2
  - From element in A_1 with a key y_i point to the element in A_2 with the smallest key *larger or equal* to y_i
- Example query with the range [20 : 65]



[Berg]

Fractional cascading in the 2D range tree

- How to save one log n during last dim. search?
  - Store canonical subsets in arrays sorted by y
  - Pointers to subsets for both child nodes v_L and v_R
  - $\mathcal{O}(1)$ search in lower levels => in two dimensional search $\mathcal{O}(\log^2 n)$ time -> $\mathcal{O}(\log n)$



[Mount]

Orthogonal range tree - summary

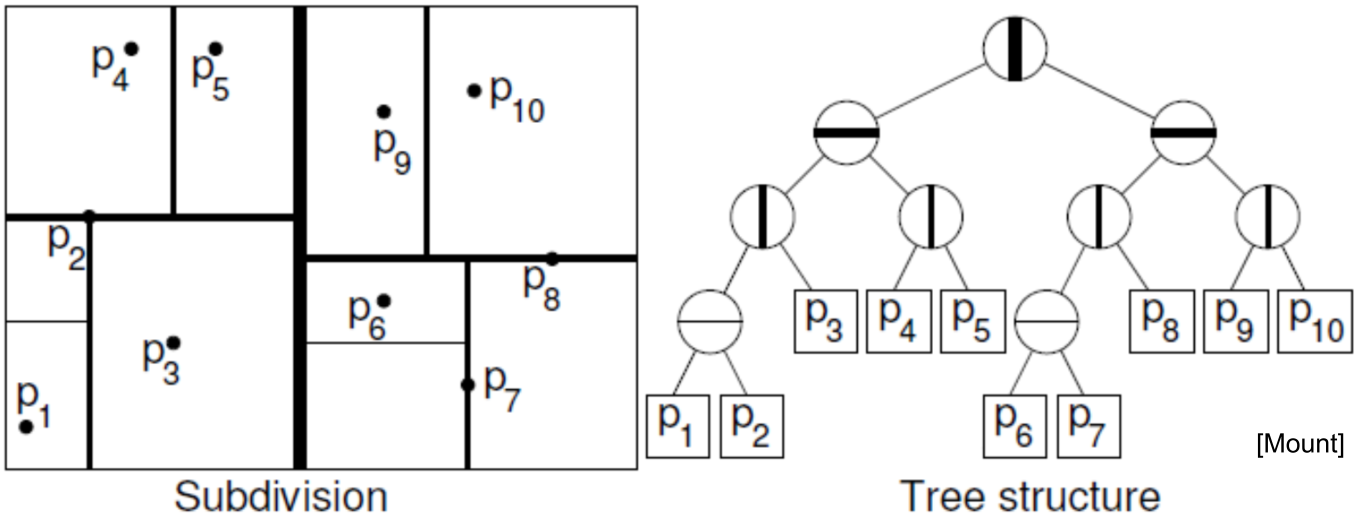
- Orthogonal range queries in plane
  - Counting queries $O(\log^2 n)$ time,
    or with fractional cascading $O(\log n)$ time
  - Reporting queries plus $O(k)$ time, for k reported points
  - Space $O(n \log n)$
  - Construction $O(n \log n)$
- Orthogonal range queries in d-dimensions, $d \geq 2$
  - Counting queries $O(log^d n)$ time,
    or with fractional cascading $O(\log^{d-1} n)$ time
  - Reporting queries plus $O(k)$ time, for k reported points
  - Space $O(n \log^{d-1} n)$
  - Construction $O(n \log^{d-1} n)$ time

Kd-tree

- Easy to implement
- Good for different searching problems (counting queries, nearest neighbor,…)
- Designed by Jon Bentley as k-dimensional tree (2-dimensional kd-tree was a 2-d tree, …)
- Not the asymptotically best for orthogonal range search (=> range tree is better)
- Types of queries
  - Reporting – points in range
  - Counting – number of points in range

Kd-tree principle

- Subdivide space according to different dimension (x-coord, then y-coord, …)
- This subdivides space into rectangular cells => hierarchical decomposition of space
- In node t store: cutDim, cutVal, (size (for counting queries))
  = Cutting line



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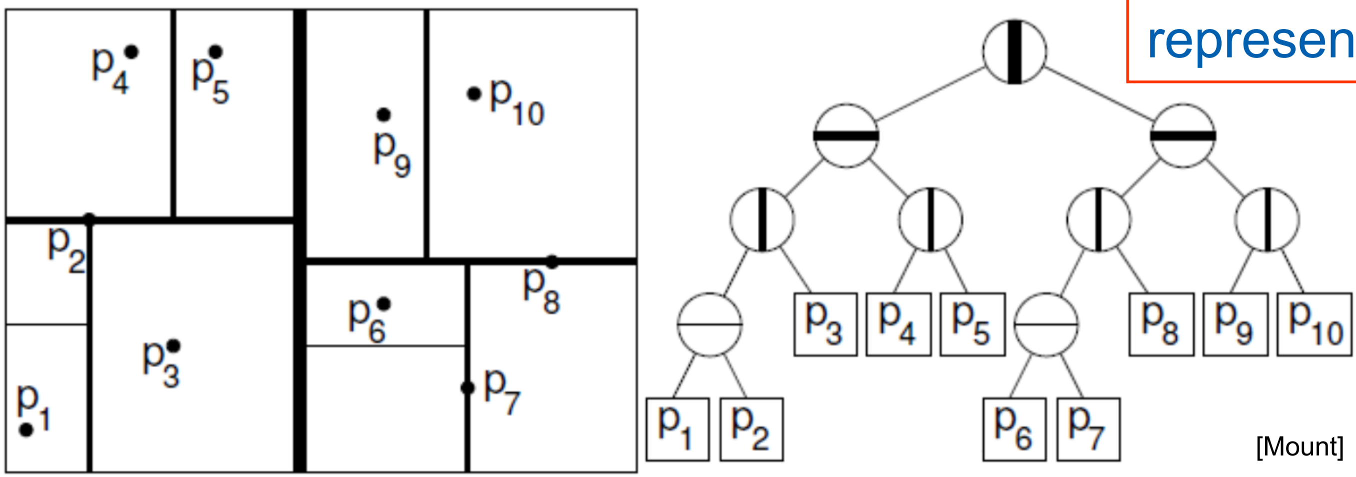
Each tree node represents a region



Subdivision Tree structure

Kd-tree principle

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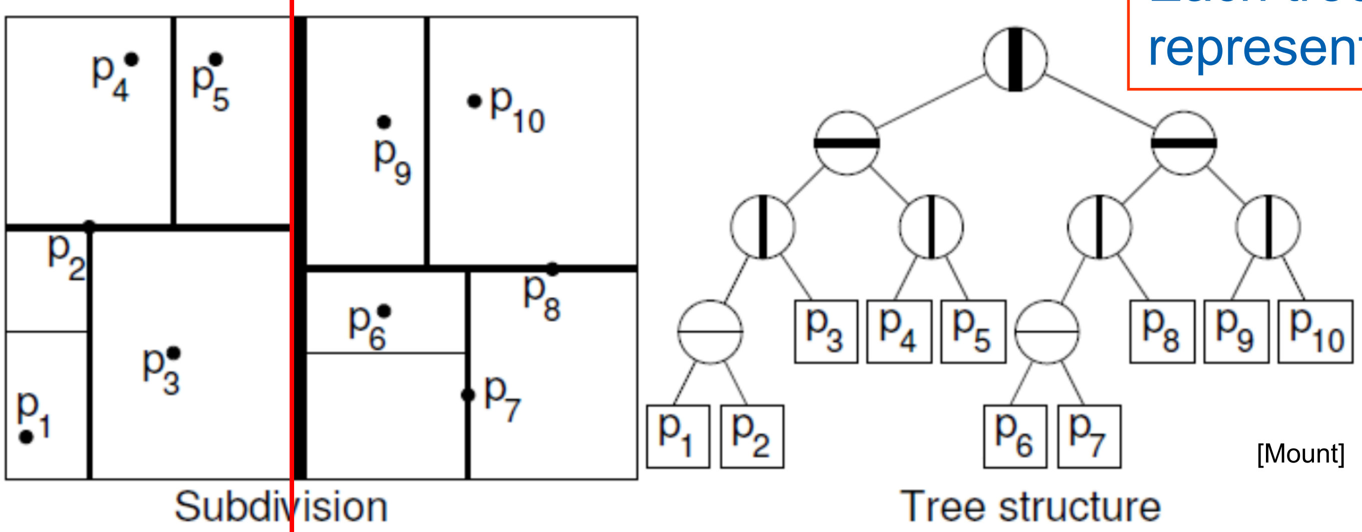

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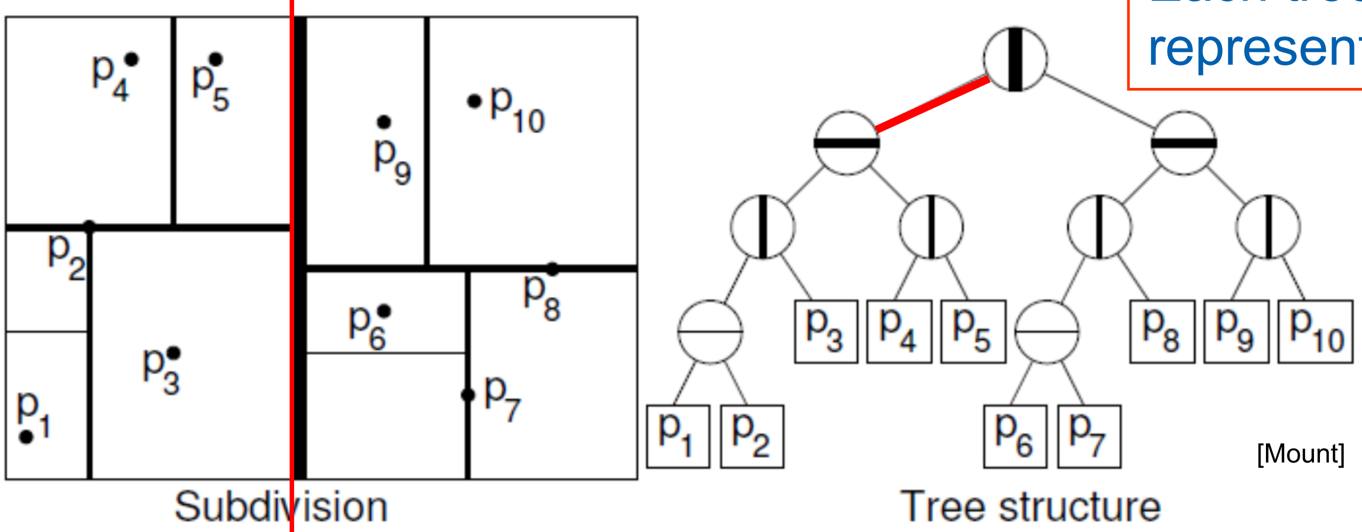

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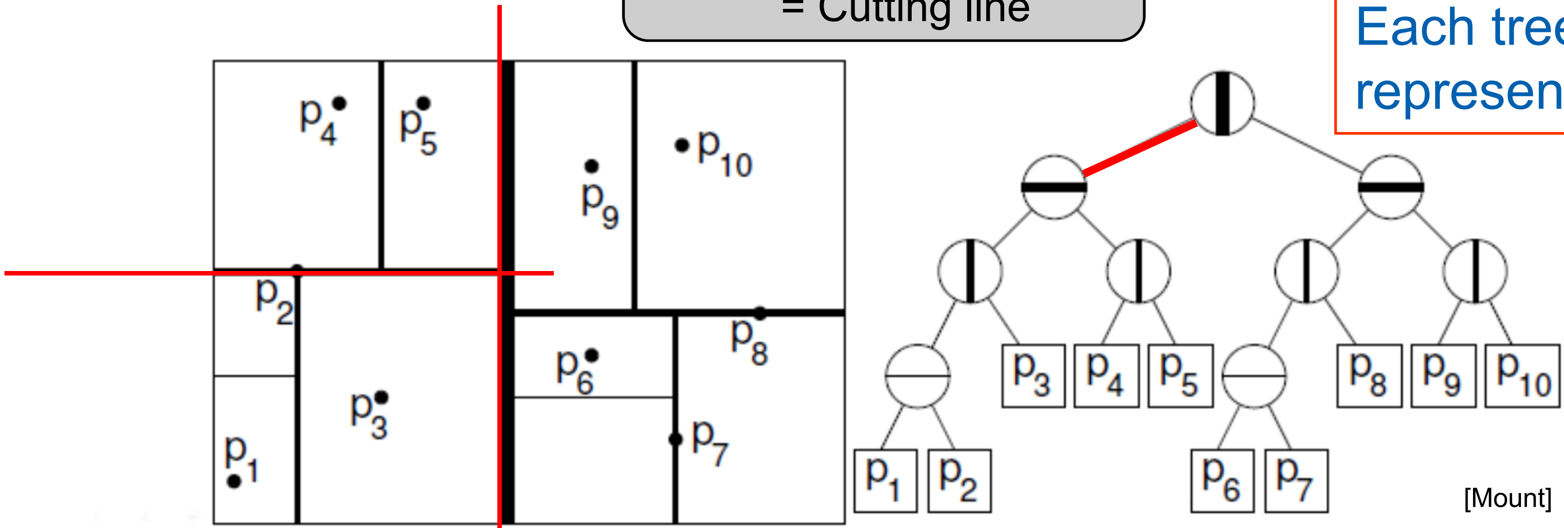

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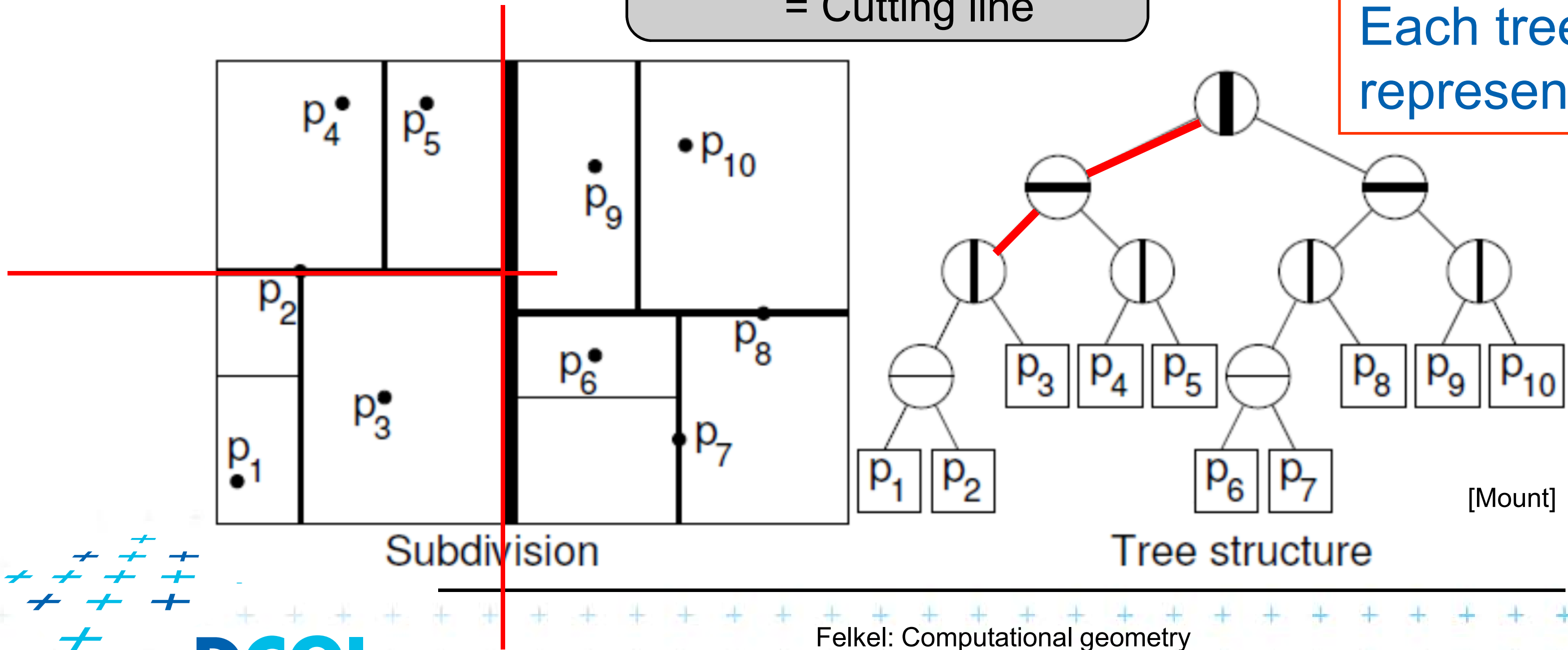

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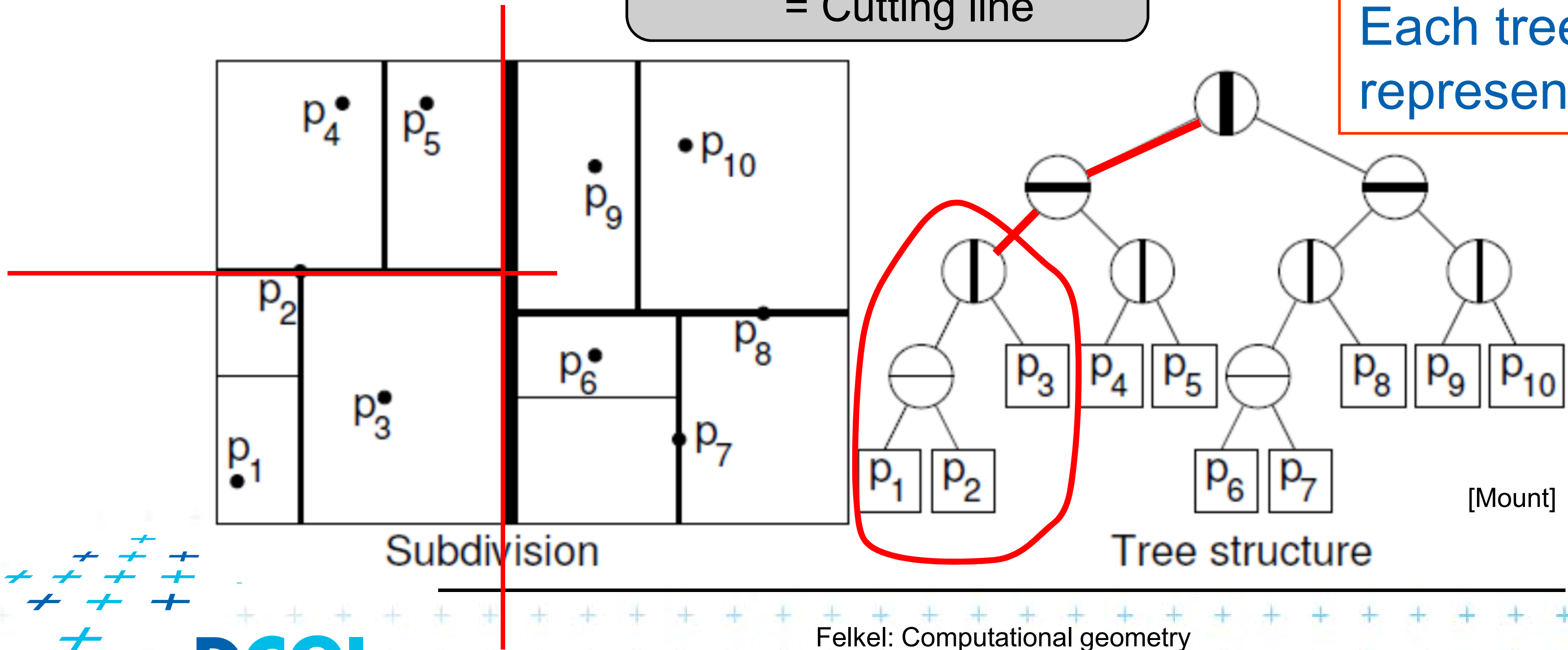


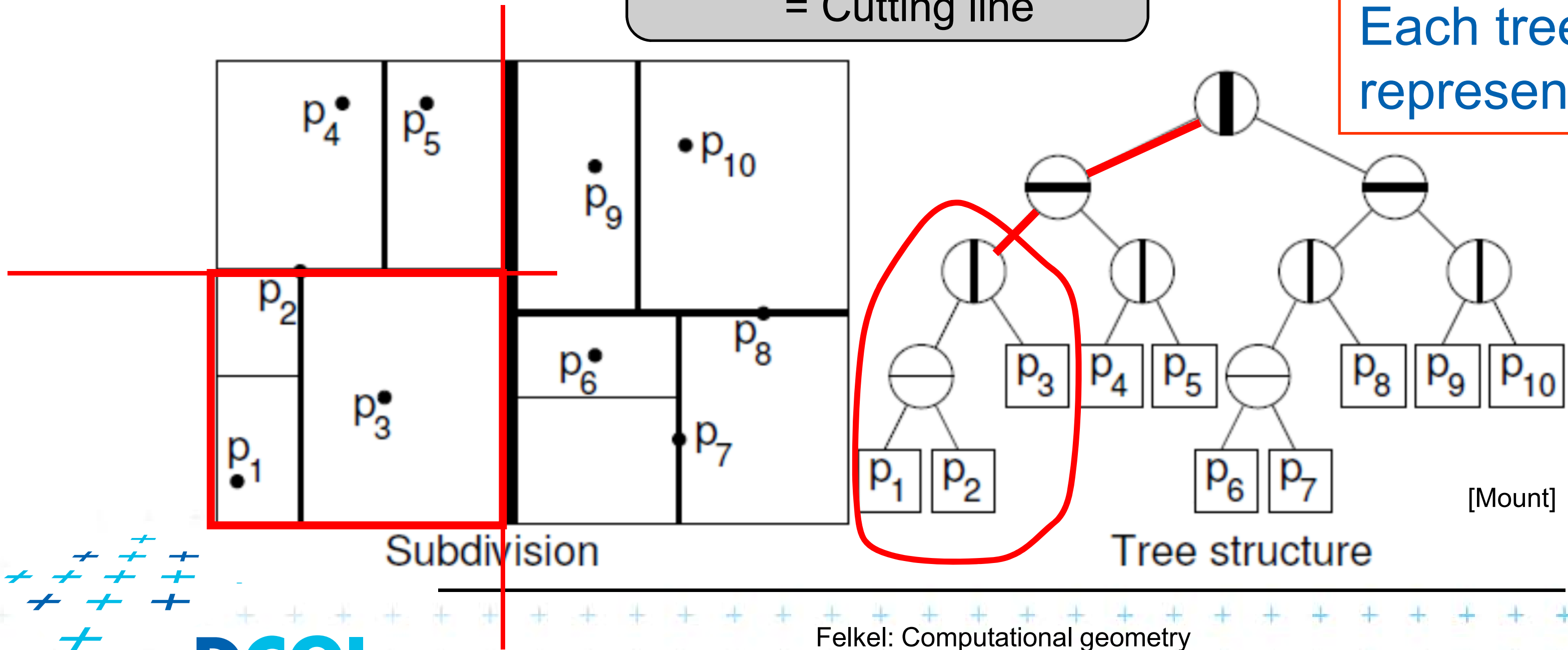
Subdivision Tree structure

Kd-tree principle

- Subdivide space according to different dimension (x-coord, then y-coord, …)
- This subdivides space into rectangular cells => hierarchical decomposition of space
- In node t store: cutDim, cutVal, (size (for counting queries))

= Cutting line

Each tree node represents a region



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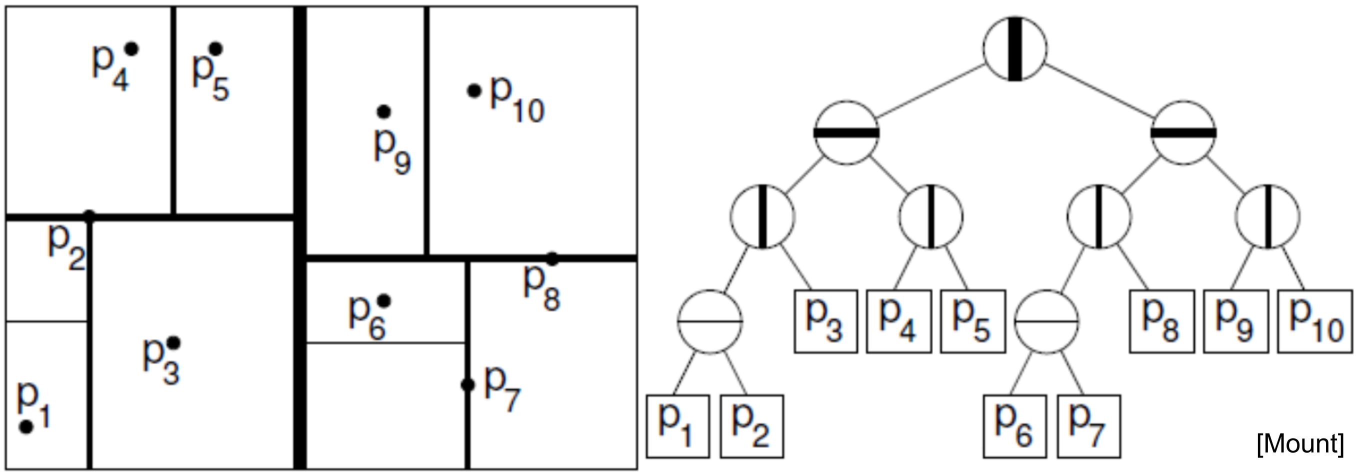

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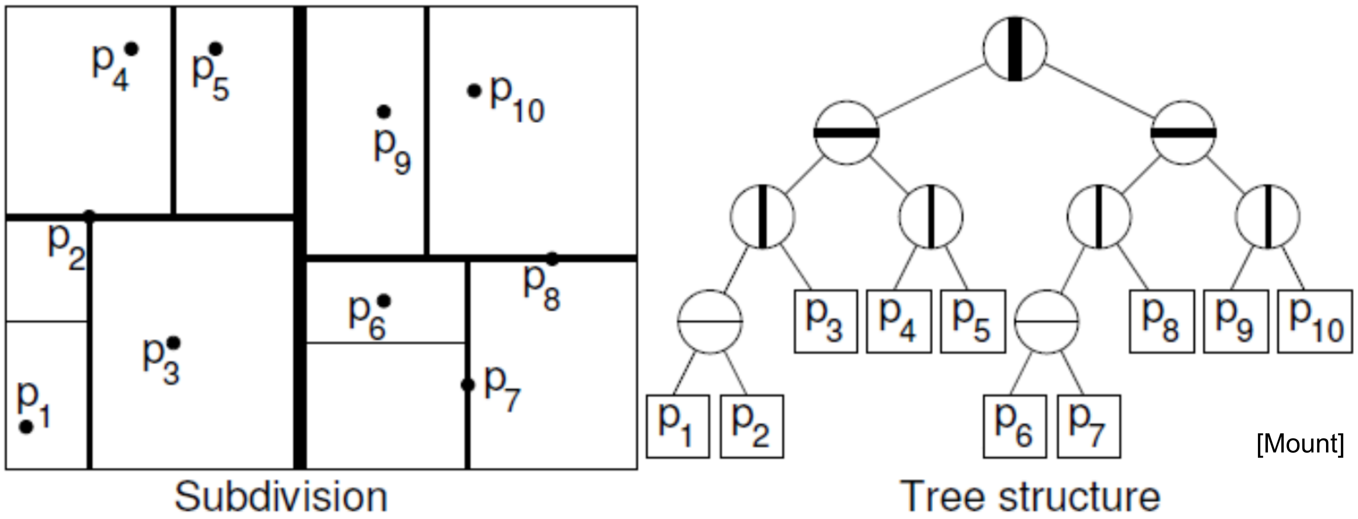


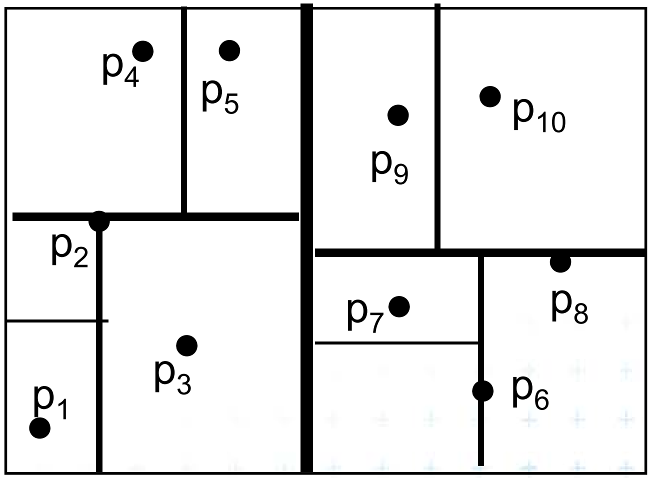
Where is a mistake in the figure?

Kd-tree principle

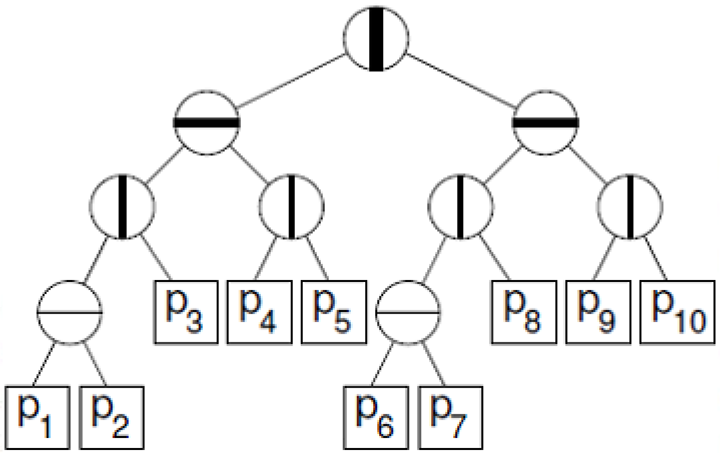
- Which dimension to cut? (cutDim)
  - Cycle through dimensions (round robin)
    - Save storage – cutDim is implicit ~ depth in the tree
    - May produce elongated cells (if uneven data distribution)
  - Greatest spread (the largest difference of coordinates)
    - Adaptive
    - Called "Optimal kd-tree"
- Where to cut? (cutVal)
  - Median, or midpoint between upper and lower median -> $O(n)$
  - Presort coords of points in each dimension $(x, y, \dots)$ for $O(1)$ median – resp. $O(d)$ for all d dimensions

Kd-tree principle

- What about points on the cell boundary?
  - Boundary belongs to the left child
  - Left: $p_{cutDim} \leq cutVal$
  - Right: $p_{cutDim} > cutVal$



Subdivision



Tree structure

[Mount]

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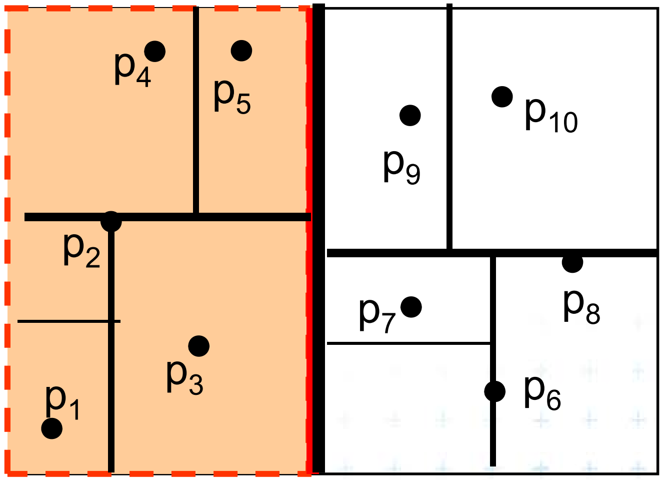
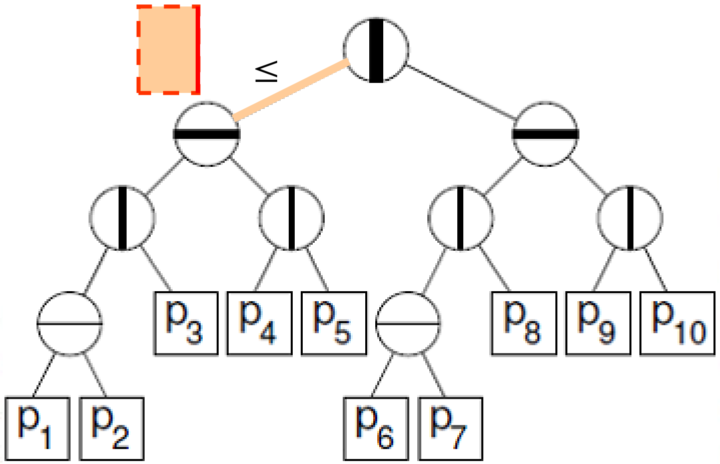


Subdivision



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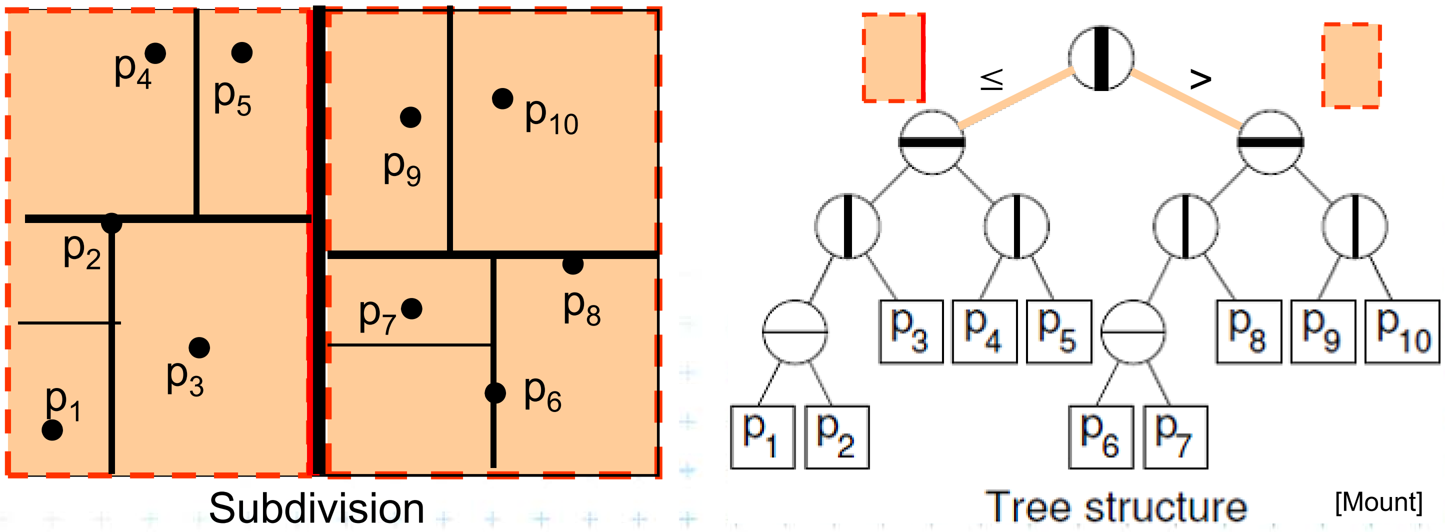

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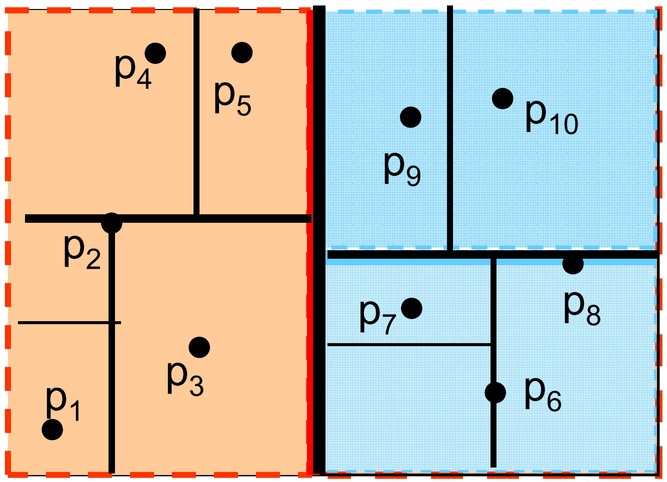
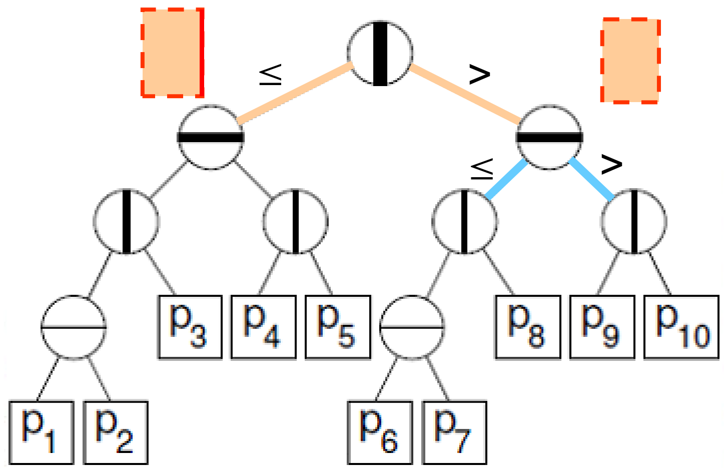


Subdivision



Tree structure

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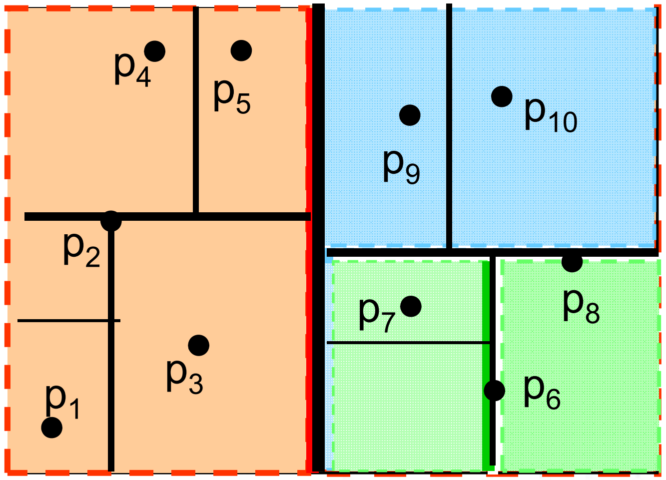
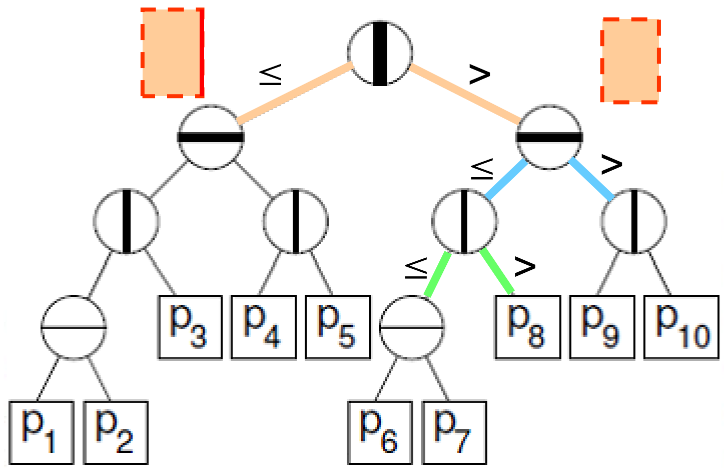


Subdivision



Tree structure [Mount]

Kd-tree construction in 2-dimensions

BuildKdTree(*P, depth*)

Input: A set of points *P* and current *depth.*

Output: The root of a kD tree storing P.

1. **If (***P* contains only one point) [or small set of (10 to 20) points]
2. **then return** a leaf storing this point
3. **else if (***depth* is even)
4. **then** split *P* with a vertical line *l* through median *x* into two subsets P_1 and P_2 (left and right from median)
5. **else** split *P* with a horiz. line *l* through median y into two subsets P_1 and P_2 (below and above the median)
6. t_{left} = BuildKdTree(P_1, *depth*+1)
7. t_{right} = BuildKdTree(P_2, *depth*+1)
8. create node *t* storing *l*, t_{left} and t_{right} children // l = cutDim, cutVal
9. **return** *t*

If median found in O(1) and array split in O(n)

T(n) = 2 T(n/2) + n => O(*n* log *n*) construction

Kd-tree construction in 2-dimensions

BuildKdTree(*P, depth*)

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Output: The root of a kD tree storing P.

If (*P* contains only one point) [or small set of (10 to 20) points]
    then return a leaf storing this point
    else if (*depth* is even)

Split according to (*depth%max_dim)* dimension

        then split *P* with a vertical line *l* through median *x* into two subsets P_1 and P_2 (left and right from median)
        else split *P* with a horiz. line *l* through median y into two subsets P_1 and P_2 (below and above the median)
    t_{left} = BuildKdTree(P_1, *depth+1*)
    t_{right} = BuildKdTree(P_2, *depth+1*)
    create node *t* storing *l*, t_{left} and t_{right} children // l = cutDim, cutVal
    return *t*

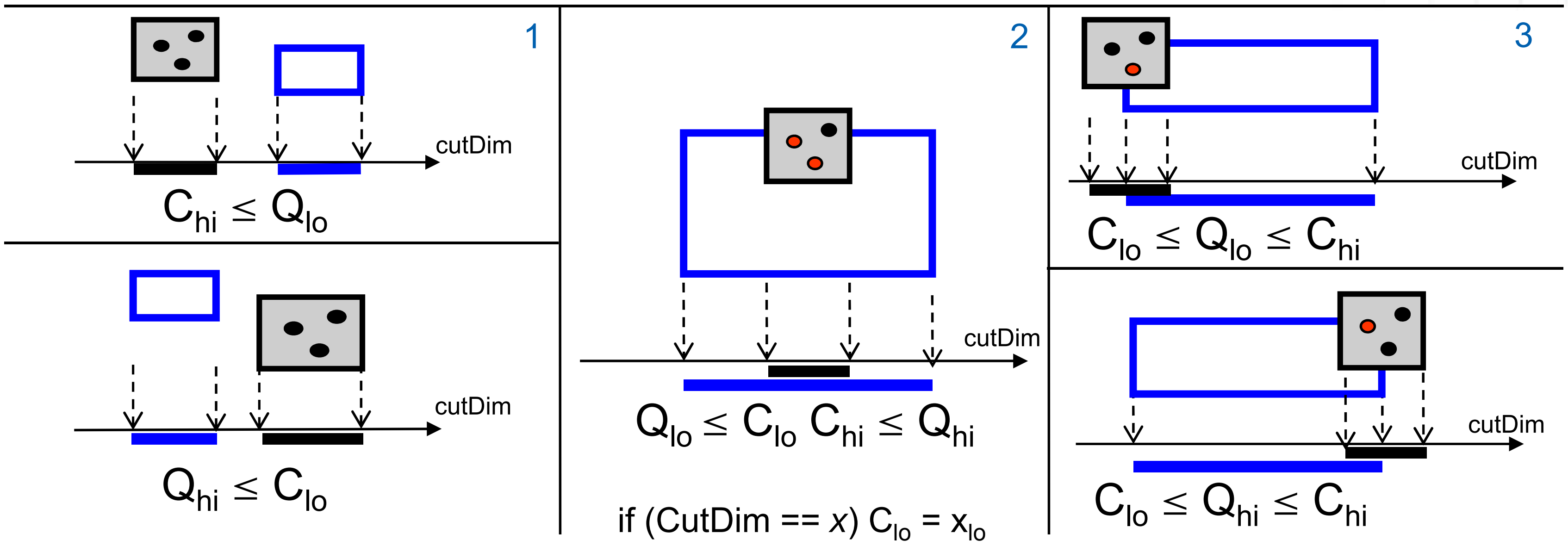
If median found in O(1) and array split in O(n)
T(n) = 2 T(n/2) + n => O(*n* log *n*) construction

Kd-tree test variants

Test interval-interval

a) Compare rectang. array Q with rectangular cells C

- Rectangle C:[x_{lo}, x_{hi}, y_{lo}, y_{hi}] – computed on the fly
- Test of kD node cell C against query Q (in one cutDim)
  1. if cell is disjoint with Q … $C \cap Q = \emptyset$ … stop
  2. If cell C completely inside Q … $C \subseteq Q$ … stop and report cell points
  3. else cell C overlaps Q … recurse on both children
- Recursion stops on the largest subtree (in/out)



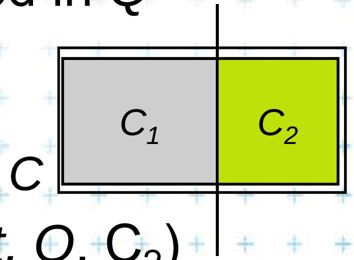
Kd-tree rangeCount (with rectangular cells)

int rangeCount(t, Q, C)

Input: The root t of kD tree, query range Q *and* t*'s* cell C.

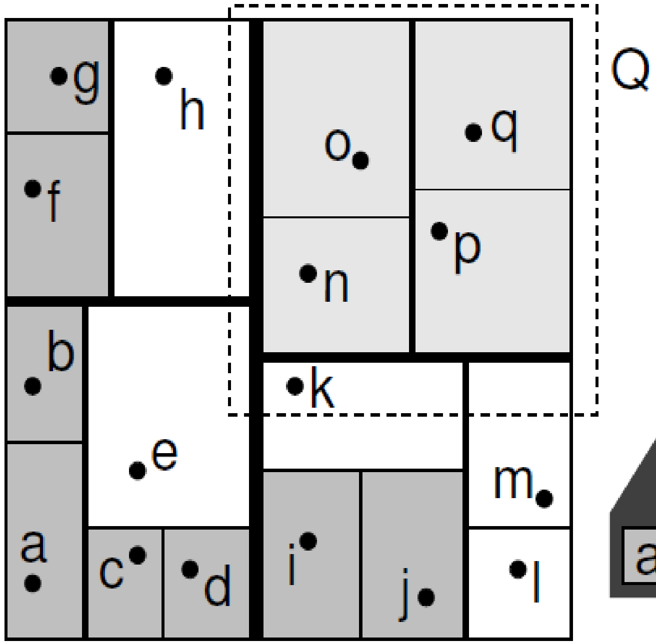
Output: Number of points at leaves below t that lie in the range.

```
if (t is a leaf)
    if (t.point lies in Q) return 1    // or loop this test for all points in leaf
    else return 0                      // visited, not counted
else  // (t is not a leaf)
    if (C ∩ Q = ∅)  return 0          … disjoint
    else if (C ⊆ Q)  return t.size     … C is fully contained in Q
    else
        split C along t's cutting value and dimension,
         creating two rectangles C1 and C2.
        return rangeCount(t.left, Q, C1) + rangeCount(t.right, Q, C2)
```



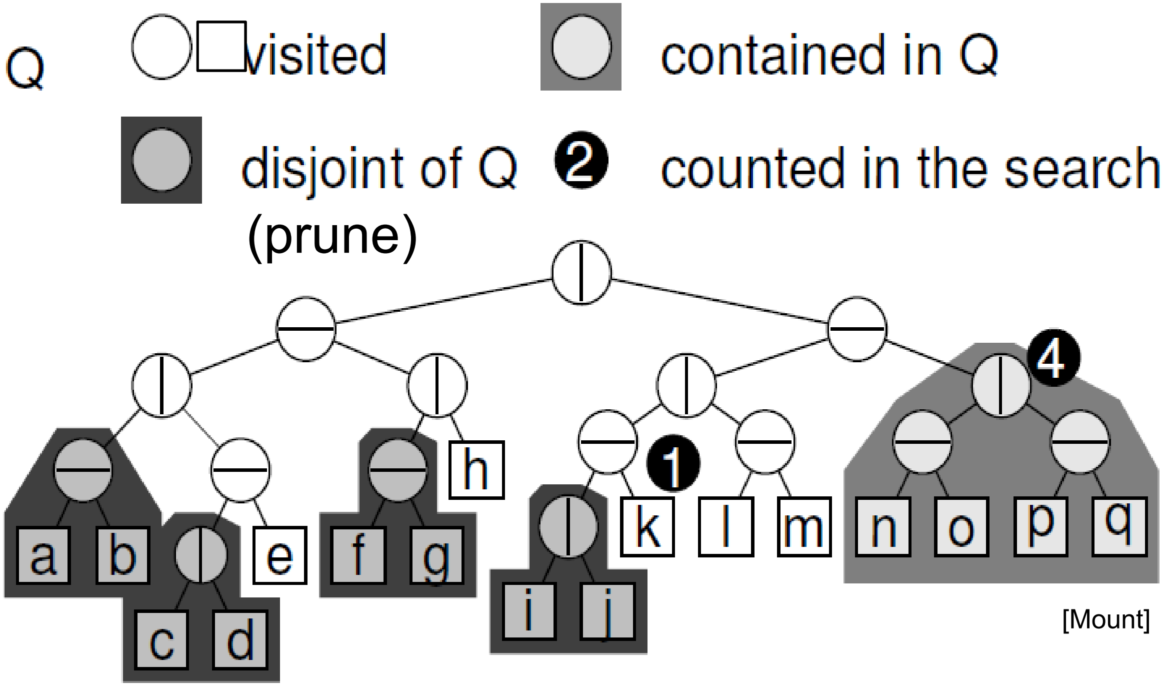
// (pictograms refer to the next slide)

Kd-tree rangeCount example



kd–tree subdivision

Tree node (rectangular region)



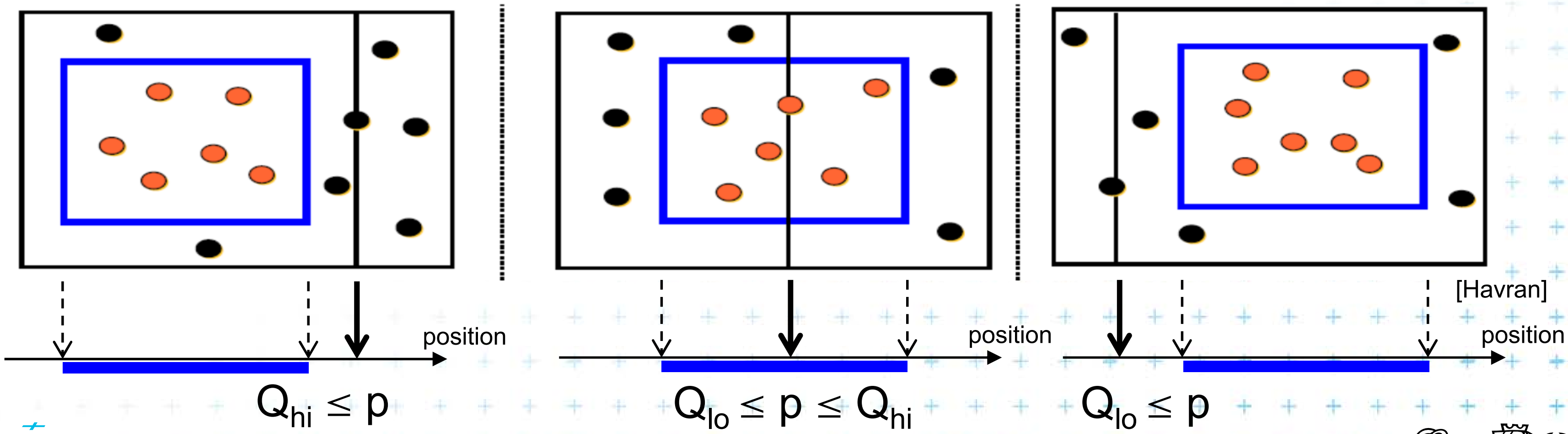
Nodes visited in range search

Kd-tree test variants

Test point-interval

b) Compare Q with cutting lines

- Line = Splitting value *p* in one of the dimensions
- Test of single position given by dimension against Q
  1. Line *p* is right from Q … recurse on left child only (prune right child)
  2. Line *p* intersects Q … recurse on both children
  3. Line *p* is left from Q … recurse on right child only (prune left ch.)
- Recursion stops in leaves - traverses the whole tree



Kd-tree rangeSearch (with cutting lines)

int rangeSearch(t, Q)

Input: The root t of (a subtree of a) kD tree and query range Q.

Output: Points at leaves below t that lie in the range.

1. **if (**t is a leaf)
2.   **if** (*t.point* lies in *Q) report t.point* // or loop test for all points in leaf
3.   *else return*
4. **else (**t is not a leaf)
5.   **if (**$Q_{hi} \le$ *t.cutVal)* rangeSearch(*t.left*, Q) // go left only
6.   **if (**$Q_{lo} >$ *t.cutVal)* rangeSearch(*t.right*, Q) // go right only
7.   **else**
8.   rangeSearch(*t.left*, Q) // go to both
9.   rangeSearch(*t.right*, Q)

Kd-tree - summary

- Orthogonal range queries in the plane (in balanced 2d-tree)
  - Counting queries O(√n) time
  - Reporting queries O(√n + k) time, where k = No. of reported points
  - Space O(n)
  - Preprocessing: Construction O(n log n) time (Proof: if presorted points to arrays in dimensions. Median in O(1) and split in O(n) per level, log n levels of the tree)
- For d≥2:
  - Construction O(d n log n), space O(dn), Search O(d n^(1-1/d) + k)

Proof sqrt(n)

Každé sudé patro se testuje osa x.

- V patře 0 je jeden uzel a jde se do obou synů (v patře 1 se jde taky do obou)
- v patře 2 jsou 4 uzly, z nich jsou ale 2 bud úplně mimo, nebo úplně in => stab jen 2
- v 4. patře stab 4 z 8, …
- v i-tém patře stab 2^i uzlů

Výška stromu je log n

Proto tedy sčítám sudé členy z 0..log n z 2^i. Je to exponenciála, proto dominuje poslední člen

2^(log n /2) = 2^log (sqrt(n)) = sqrt(n)

Orthogonal range tree (RT)

- DS highly tuned for orthogonal range queries
- Query times in plane

2d tree	versus	2d range tree
O($\sqrt{n}$ + k) time of Kd	>	O(log *n*) time query
O(*n*) space of Kd	<	O(*n* log *n*) space

n = number of points

k = number of reported points

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