Data structures and algorithms

Part 11

Searching, mainly via Hash tables

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Topics

Searching

Hashing

- Hash function
- Resolving collisions
  - Hashing with chaining
  - Open addressing
    - Linear Probing
    - Double hashing

Dictionary

Many applications require:

– dynamic set

– with operations: Search, Insert, Delete

= dictionary

Ex. Table of symbols in a compiler

identifier	type	address
sum	int	0xFFFFDC09
...	...	...

Searching

Comparing the keys **$\Omega(\log n)$**

– Found when key of data item = searched key

– Ex: Sequential search, BST,...

associative

Indexing by the key (direct access) **$\Theta(1)$**

–The key value is the memory address of the item

– keys scope ~ indices scope

Hashing **on average $\Theta(1)$**

– The item address is computed using the key

address search

Hashing

= tradeoff between the speed and the memory usage

– ∞ time - sequential search

– ∞ memory - direct access (indexing by the key)

– few memory and few time:

- Hash table
- table size influences the search time

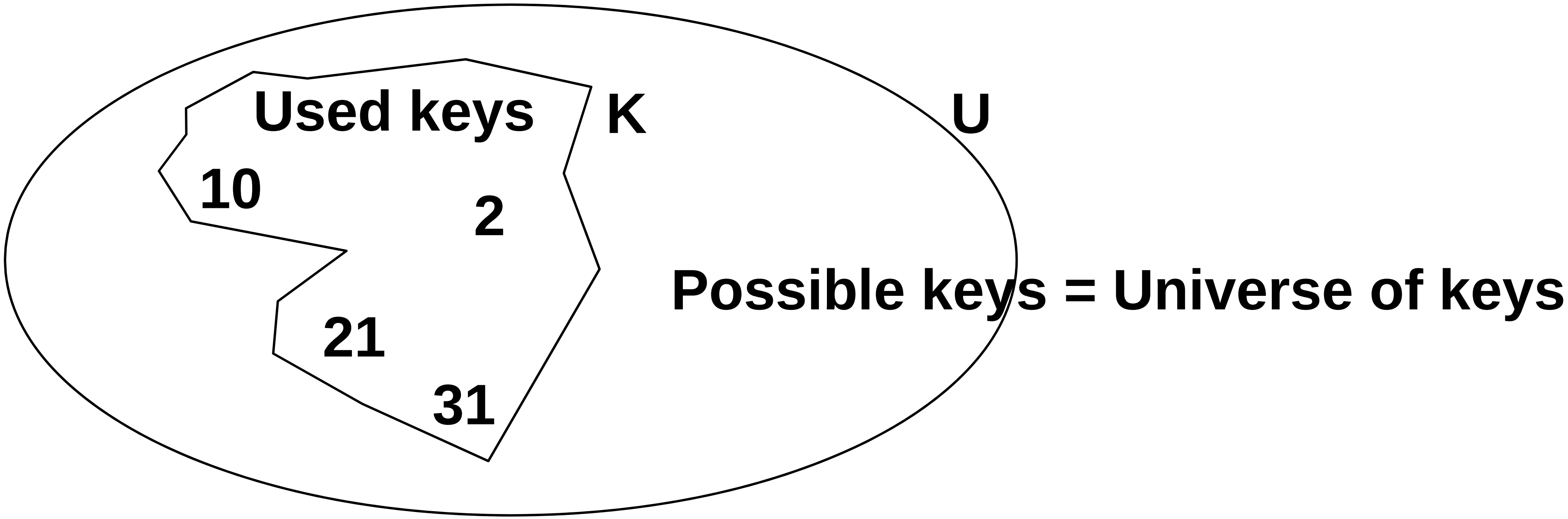
Hashing

Constant expected time of operations *search* and *insert* !!!

Tradeoff:

- Operation time ~ key length
- Hashing is not suitable for operations
  select a subset and *sort*

Hashing

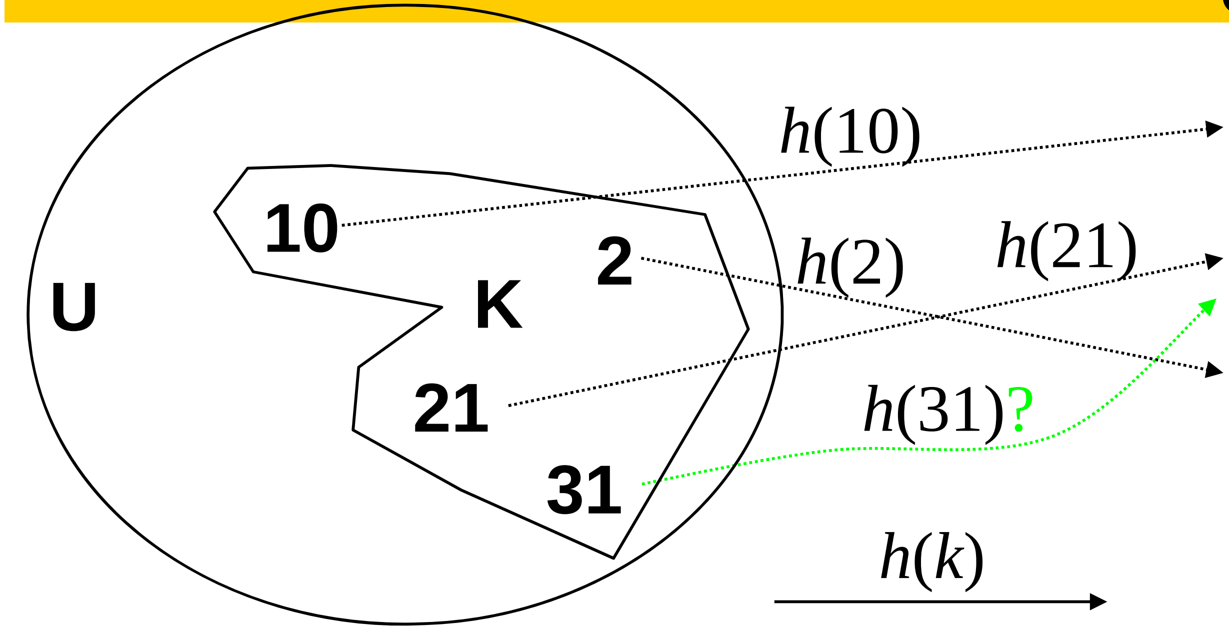


Hashing applicable when |K| << |U|

K Set of really used keys

U Universe of keys -- all possible (thinkable) keys, even if unused

hashing



0	**10**
1	**21**
2	**2**
...	
M-1	

Two phases

1. Compute hash function *h*(*k*)
   (*h*(*k*) produces item address based on the key value
2. Resolving collisions

h(31) ..... collision: index 1 is already occupied

1. Compute hash function $h(k)$

Hash function $h(k)$

Maps

set of keys $K_j \in U$

into the interval of addresses $A = \langle a_{min}, a_{max} \rangle$,

usually into $\langle 0, M-1 \rangle$

Synonyms: $k_1 \neq k_2$, $h(k_1) = h(k_2)$

= collision!!

Hash function *h*(*k*)

Depends very strongly on key properties and the memory representation of the keys

Ideally:

- simple calculation -- fast
- approximates well a random distribution
- exploits uniformly address space in memory
- generates minimum number of collisions
- Therefore: It uses all components of a key

Hash function *h*(*k*) - examples

Examples of *h*(*k*) for different key types

- Real (float) values
- integers
- bit strings
- strings

Hash function *h*(*k*) - examples

Real values from <0, 1>

- multiplicative: **h(k,M) = round(k * M)**

  (does not separate the clusters of similar values)

  M = table size

Hash function *h*(*k*) - examples

For *w-bit integers*

- multiplicative: (M is a prime)
  - **h(k,M) = round(k /2^{w} * M)**
- modular:
  - **h(k,M) = k % M**
- combined:
  - **h(k,M) = round(c * k) % M, c $\in$<0,1>**
  - **h(k,M) = (int)(0.616161 * (float) k) % M**
  - **h(k,M) = (16161 * (unsigned) k) % M**

Hash functions *h*(*k*) - examples

Fast but depends a lot on keys representation:

h(k) = k & (M-1) **for M = 2^x (not a prime) ,**

& = bit product

Hash function *h*(*k*) - examples

For *strings*:

```
int hash( char *k, int M )
{
    int h = 0, a = 127;
    for( ; *k != 0; k++ )
        h = ( a * h + *k ) % M;
    return h;
}
```

Horner scheme:

$$k_2 * a^2 + k_1 * a^1 + k_0 * a^0 =$$

$$((k_2 * a) + k_1) * a + k_0$$

Hash function *h*(*k*) - examples

For strings: (pseudo-) randomized

```
int hash( char *k, int M )
{  int h = 0, a = 31415; b = 27183;
   for( ; *k != 0; k++, a = a*b % (M-1) )
       h = ( a * h + *k ) % M;
   return h;
}
```

Universal hash function

- collision probability = 1/M
- different random constants applied to different positions in the string

Hash function *h*(*k*) - flaws

Frequent flaw: *h*(*k*) returns often the same value

- wrong type conversion
- works but generates many similar addresses
- therefore it produces many collisions

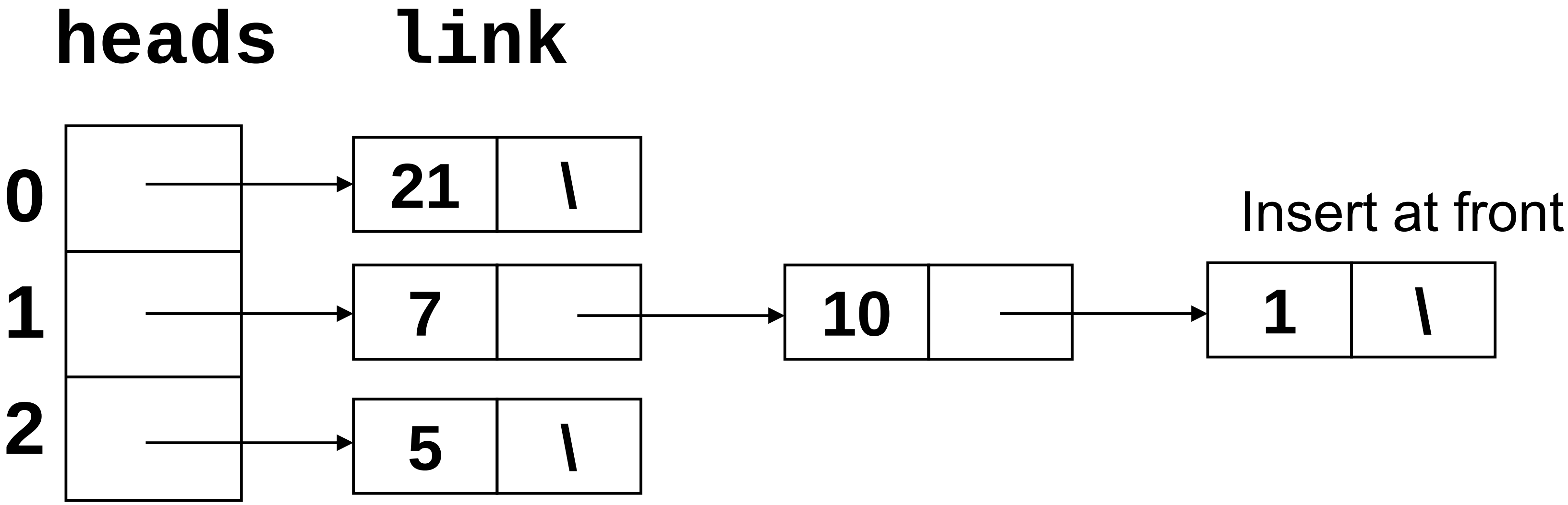
=> *the application is extremely slow*

2. Collision resolving

a) Chaining 1/5

$h(k) = k \bmod 3$

sequence: 1, 5, 21, 10, 7



lists of synonyms

a) Chaining 2/5

```
private:
  link* heads; int N,M;  [Sedgewick]

public:
  init( int maxN )          // initialization
  {
    N=0;                    // No.nodes
    M = maxN / 5;           // table size
    heads = new link[M]; // table with pointers
    for( int i = 0; i < M; i++ )
       heads[i] = null;
  }
  ...
```

a) Chaining 3/5

```
Item search( Key k )
{
  return searchList( heads[hash(k, M)], k );
}

void insert( Item item )                    // insert at front
{
  int i = hash( item.key(), M );
  heads[i] = new node( item, heads[i] );
  N++;
}
```

a) Chaining 4/5

synonyms chain has ideally length

$\alpha = n/m$, $\alpha > 1$ (load factor)

(n = no of elems, m = table size, m<n)

Highly improbable outcome

			on average
Insert	**$I(n) = t_{hash} + t_{link} = O(1)$**		
Search	**$Q(n) = t_{hash} + t_{search}$**		
	$= t_{hash} + t_c * n/(2m)$	**$= O(n)$**	**$O(1 + \alpha)$**
Delete	**$D(n) = t_{hash} + t_{search} + t_{link}$**	**$= O(n)$**	**$O(1 + \alpha)$**

for small α (and big m) it is close to O(1) !!!

for big α (and small m) m-times faster than sequential search

a) Chaining 5/5

Practical use:

choose $m = n/5 \ldots n/10$ => load factor $\alpha = 5 \ldots 10$

- sequential search in the chain is fast
- not many unused table slots

Pros & cons:

+ exact value of *n* needs not to be known in advance
– needs dynamic memory allocation
– needs additional memory for chain (list) pointers

b) Open-address hashing

The approximate number of elements is known

No additional pointers

=> Use 1D array

Hash function *h*(*k*) is tied with collision resolving

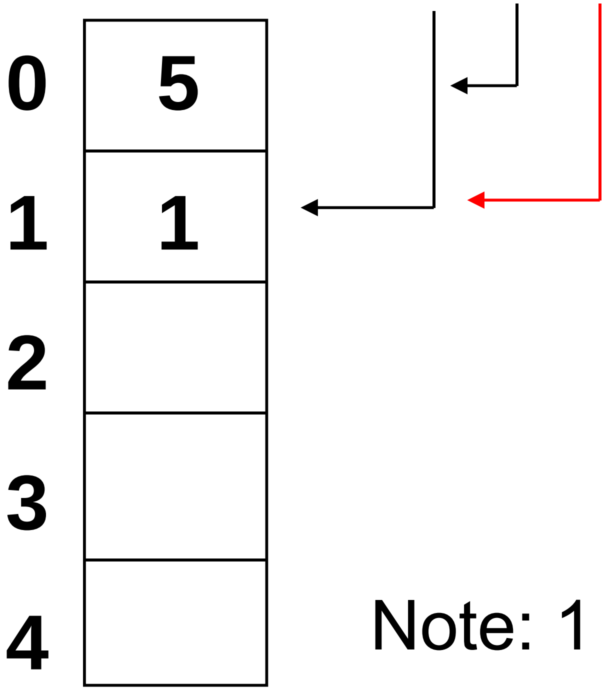
1. linear probing
2. double hashing

0	**5**
1	**1**
2	**21**
3	**10**
4	

b) Open-address hashing

$h(k) = k \bmod 5$ ($h(k) = k \bmod m$, m is array size)

sequence: 1, 5, 21, 10, 7



Problem:

collision - 1 already occupies the space for 21

1. linear probing
2. double hashing

Note: 1 and 21 are synonyms. The position is often occupied by a key which is not a synonym. Collision does not distinguish between synonyms and non-synonyms.

Probing

= check what is in the table at the position given by the hash function

- search hit = key found
- search miss = empty position, key not found
- else = position occupied by another key, continue searching

b) Open-address hashing

Methods of collision resolving

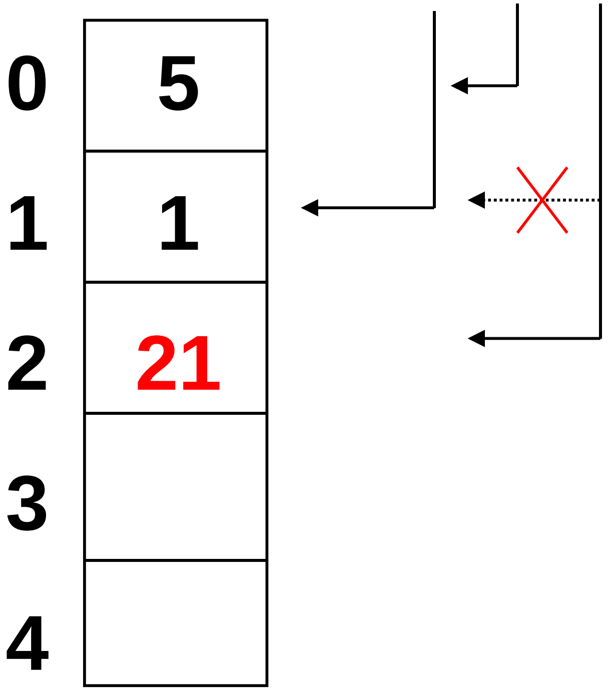
b1) Linear probing

b2) Double hashing

b1) Linear probing

$h(k)$ = [(k mod 5) + i] mod 5 = (k + i) mod 5

sequence: 1, 5, 21, 10, 7



collision!

=> 1. linear probing

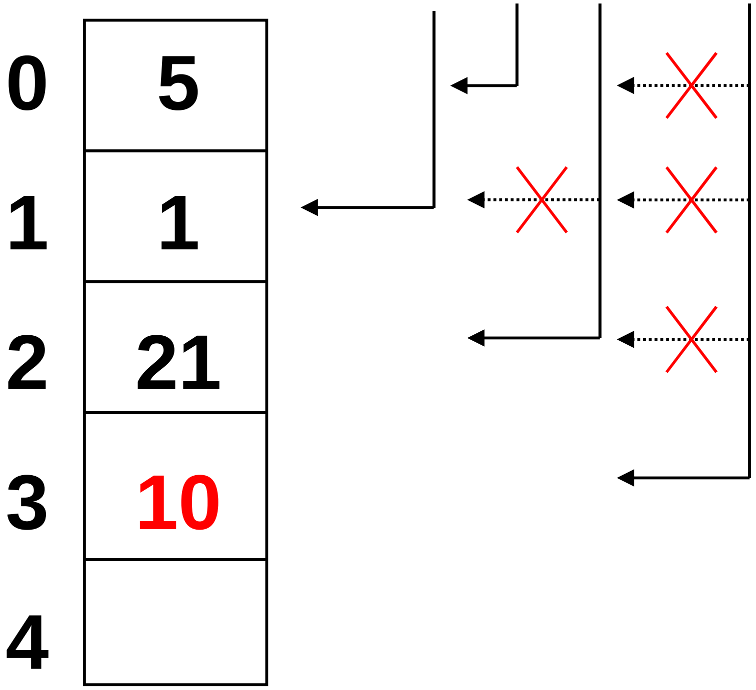
move forward

by one position (i++ => i = 1)

b1) Linear probing

$h(k)$ = (k + i) mod 5

sequence: 1, 5, 21, 10, 7



1. collision with 5 - move on
2. collision with1 - move on
3. collision with 21 - move on

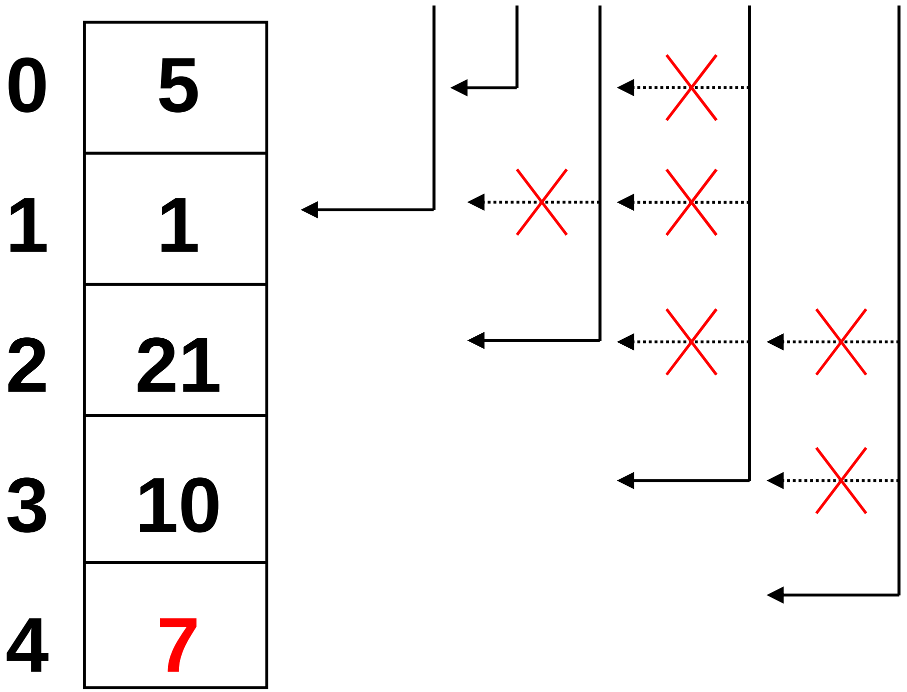
Inserted 3 positions further

in the table (i = 3)

b1) Linear probing

$h(k) = (k + i) \bmod 5$

sequence: 1, 5, 21, 10, 7



1. collision with 21 (i++)
2. collision with 10 (i++)

Inserted 3 positions further in the table (i = 2)

b1) Linear probing

$h(k) = (k + i) \bmod 5$

sequence: 1, 5, 21, 10, 7

0	**5**	i = 0
1	**1**	i = 0
2	**21**	i = 1
3	**10**	i = 3
4	**7**	i = 2

b1) Linear probing

```
private:
  Item *st; int N,M;  [Sedgewick]
  Item nullItem;
public:
  init( int maxN )        // initialization
  {
    N=0;                  // Number of stored items
    M = 2*maxN;           // load_factor < 1/2
    st = new Item[M];
    for( int i = 0; i < M; i++ )
       st[i] = nullItem;
  }...
```

b1) Linear probing

```
void insert( Item item )
{
  int i = hash( item.key(), M );

  while( !st[i].null() )
      i = (i+1) % M; // Linear probing

  st[i] = item;
  N++;
}
```

b1) Linear probing

```
Item search( Key k )
{
  int i = hash( k, M );

  while( !st[i].null() ) { // !cluster end
                           // sentinel
     if( k == st[i].key() )
          return st[i];
     else
          i = (i+1) % M; // Linear probing
    }
     return nullItem;
}
```

b) Open-address hashing

Methods of collision resolving

b1) Linear probing

b2) Double hashing

b2) Double hashing

Hash function $h(k) = [h_1(k) + i.h_2(k)] \bmod m$

$h_1(k) = k \bmod m$ // initial position

$h_2(k) = 1 + (k \bmod m')$ // offset

Both depend on k => Each key has different probe sequence

m = prime number or m = power of 2

m' = slightly less m' = odd

If d = greatest common divisor => search 1/d slots only

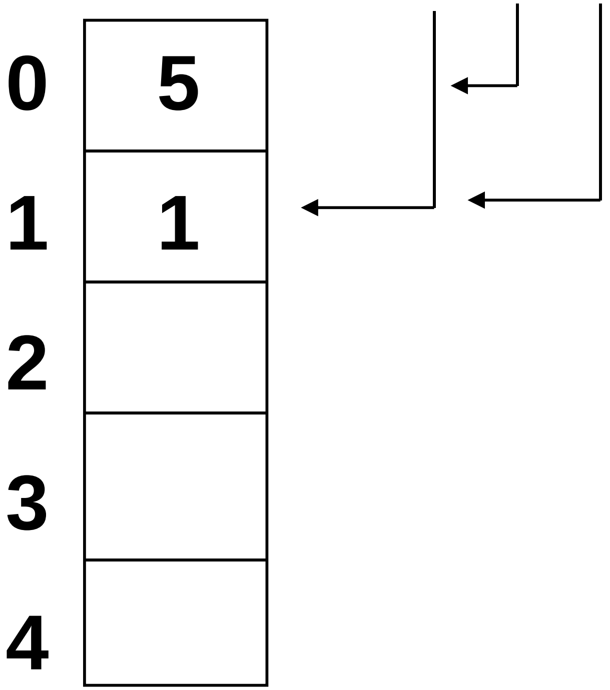
Ex: $k = 123456$, $m = 701$, $m' = 700$

$h_1(k) = 80$, $h_2(k) = 257$ Starts at 80, and every 257 % 701

b2) Double hashing

$h(k) = k \bmod 5$

sequence: 1, 5, 20, 25, 18



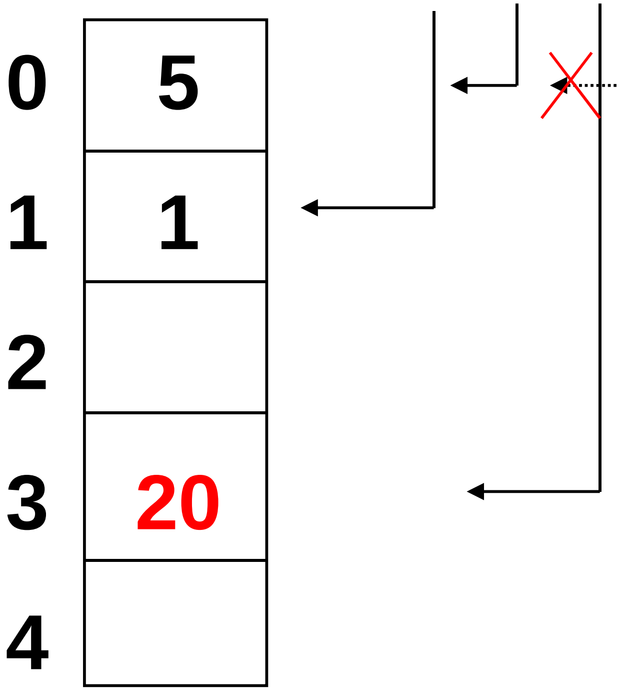
collision

=> 2. double hashing

b2) Double hashing

$h(k)$ = [(k mod 5) + i.h_2(k)] mod 5, $h_2(k)$ = 1 + k mod 3

sequence: 1, 5, 20, 25, 18



collision,

$h_2(20)$ = 1 + 20 mod 3 = 3,

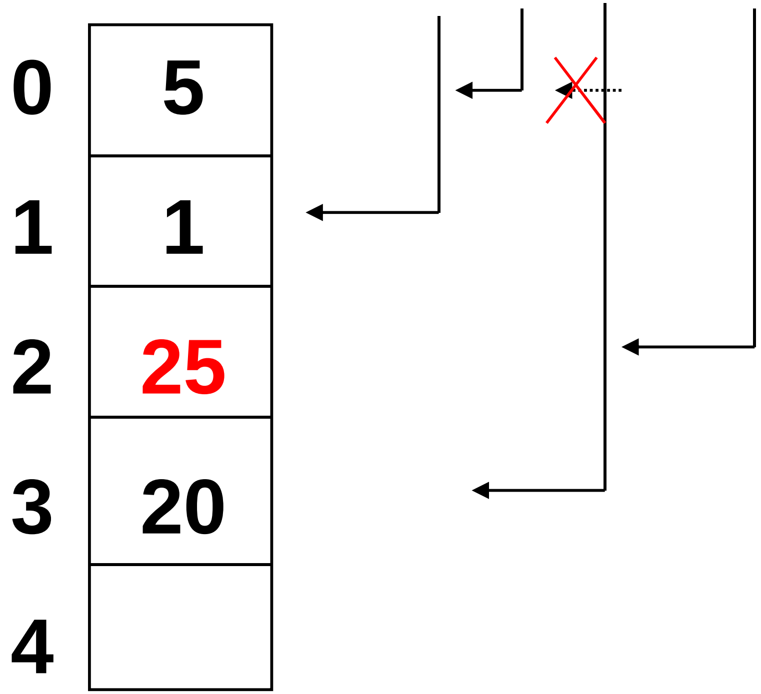
store 20 at position

0 + 3

b2) Double hashing

$h(k) = [(k \bmod 5) + i.h_2(k)] \bmod 5$, $h_2(k) = 1 + k \bmod 3$

sequence: 1, 5, 20, 25, 18



collision,

$h_2(25) = 1 + 25 \bmod 3 = 2$,

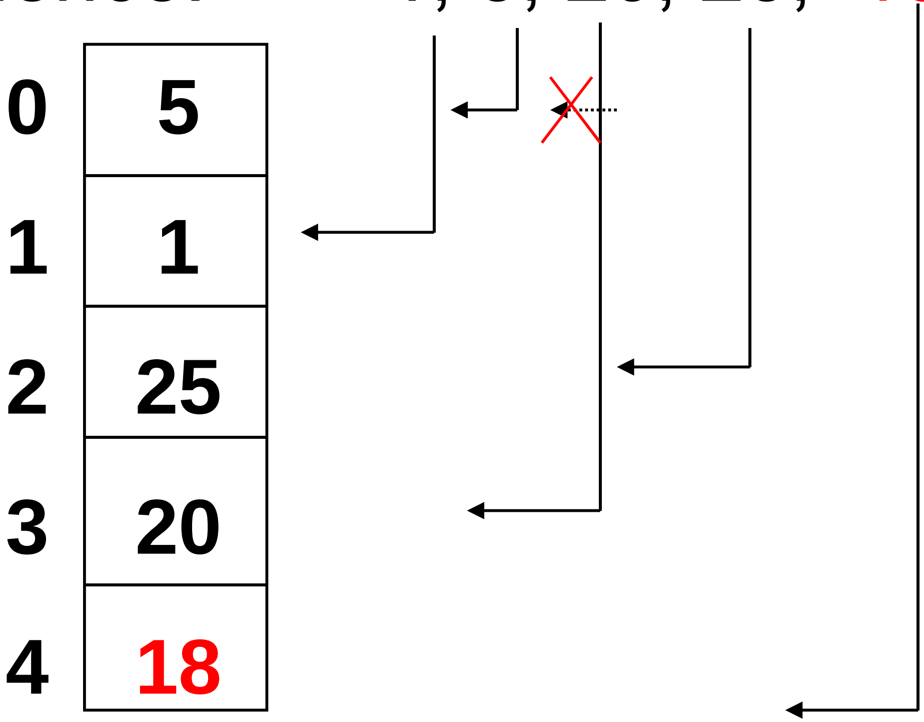
store 25 at position

0 + 2

b2) Double hashing

$h(k) = [(k \bmod 5) + i.h_2(k)] \bmod 5$, $h_2(k) = 1 + k \bmod 3$

sequence: 1, 5, 20, 25, 18



collision,

$h_2(18) = 1 + 18 \bmod 3 = 1$,

store 18 at position

3 + 1 = 4

b2) Double hashing

$h(k)$ = [(k mod 5) + i.h_2(k)] mod 5, $h_2(k)$ = 1 + k mod 3

sequence: 1, 5, 20, 25, 18

0	**5**	i = 0
1	**1**	i = 0
2	**25**	i = 0
3	**20**	i = 1
4	**18**	i = 1

Linear probing x Double hashing

$h(k) = (k + i) \bmod 5$

0	5	i = 0
1	1	i = 0
2	21	i = 1
3	10	i = 3 !
4	7	i = 2

long clusters

$h(k) = [(k \bmod 5) + i.h_2(k)] \bmod 5,$

$h_2(k) = 1 + k \bmod 3$

0	5	i = 0
1	1	i = 0
2	25	i = 1
3	20	i = 1
4	18	i = 1

mixed probe sequences

b2) Double hashing

```
void insert( Item item )
{
  Key k = item.key();
  int i = hash( k, M ),
      j = hashTwo( k, M );  // Double Hashing!

  while( !st[i].null() )
     i = (i+j) % M;          //Double Hashing

  st[i] = item; N++;
}
```

b2) Double hashing

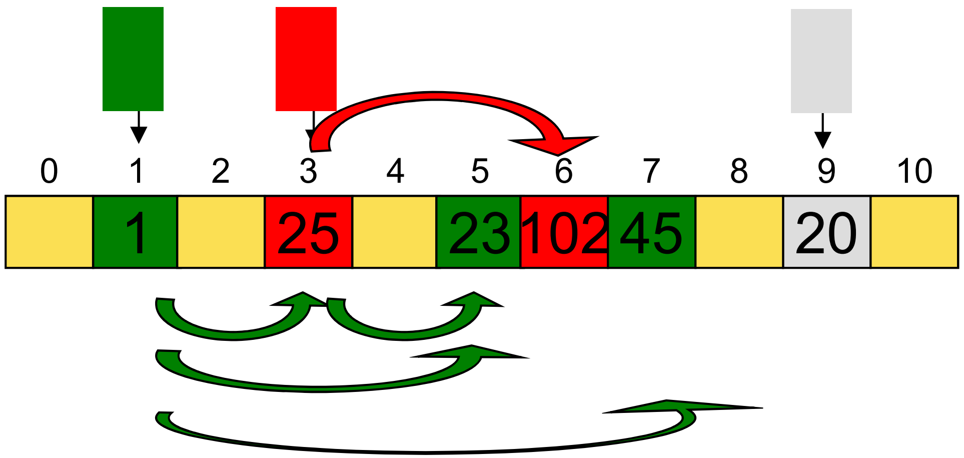
```
Item search( Key k )
{
  int i = hash( k, M ),
      j = hashTwo( k, M ); // Double Hashing

  while( !st[i].null() )
  {
     if( k == st[i].key() )
          return st[i];
     else
          i = (i+j) % M; // Double Hashing
   }
   return nullItem;
}
```

Double hashing - example

b2) Double hashing $h(k) = [h_1(k) + i.h_2(k)] \bmod m$

Input	$h_1(k)$= k %11	$h_2(k)$= 1+k %10	i	$h(k)$
1	1	2	0	1
25	3	6	0	3
23	1	4	0,1	1,5
45	1	6	0,1	1,7
102	3	3	0,1	3,6
20	9	1	0	9



$h_1(k) = k \% 11$

$h_2(k) = 1 + (k \% 10)$

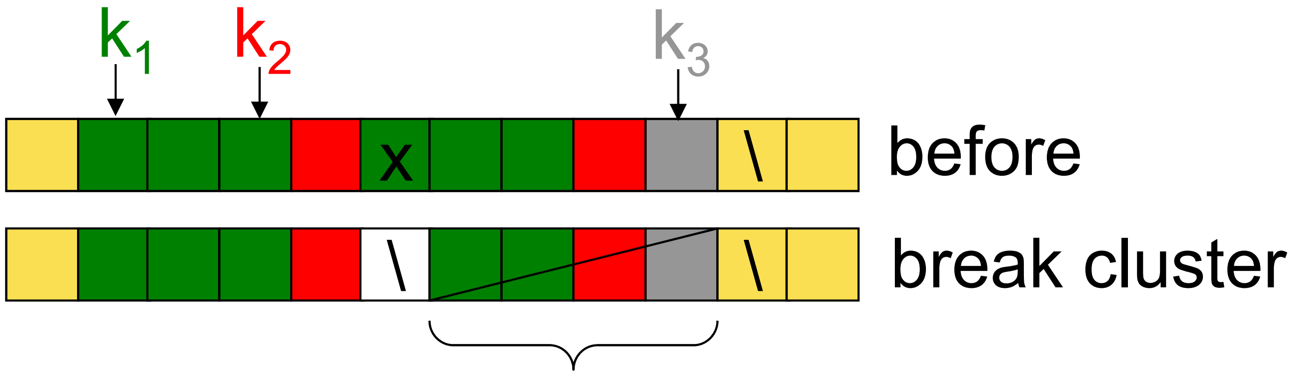
Item removal (delete)

Item ‘x’ removal

x replaced by null

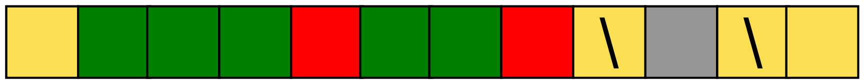
null breaks cluster(s) !!!

=> do not leave the hole after delete



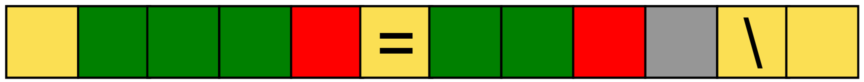
Correction different for linear probing and double hashing

b1) in linear probing



=> reinsert the items after x (to the first null = to cluster end)

b2) in double hashing

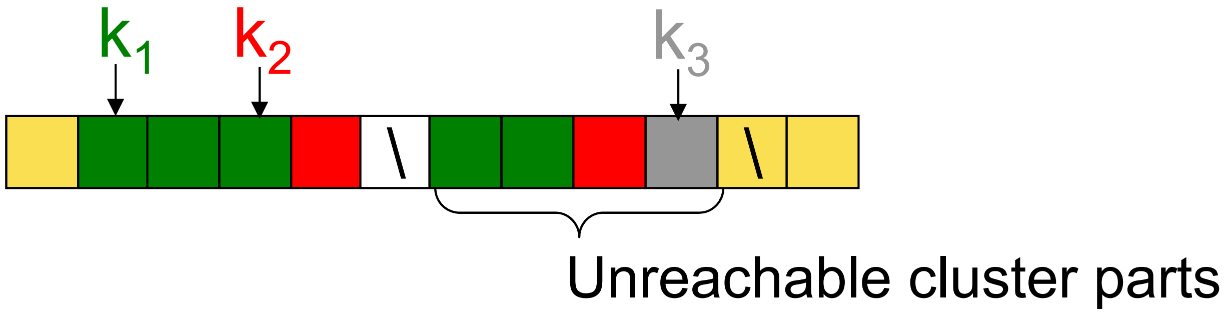


=> fill the hole up by a special sentinel

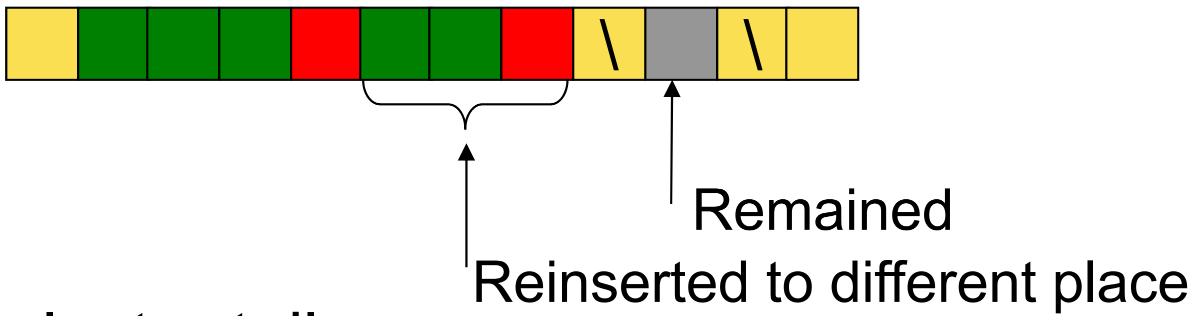
skipped by search, replaced by insert

Item removal (delete)

b1) in linear probing

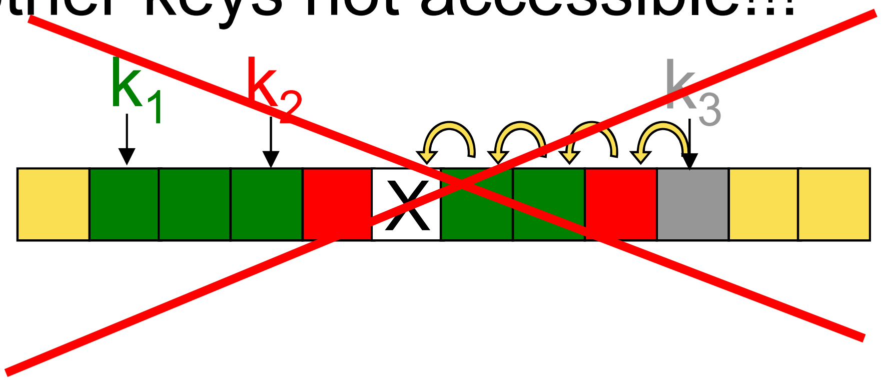


=> reinsert the items behined the cluster break (to the null)



=> avoid simple move of cluster tail

it can make other keys not accessible!!!

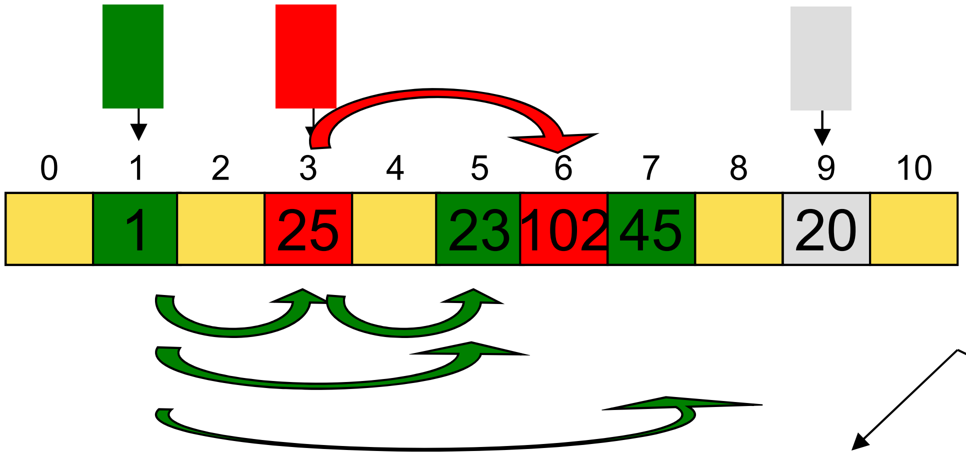


Linear-probing Item Removal

```
// do not leave the hole - can break a cluster
void remove( Item item )
{ Key k = item.key();
  int i = hash( k, M ), j;
  while( !st[i].null() )// find item to remove
      if( item.key() == st[i].key() ) break;
      else i = (i+1) % M;
  if( st[i].null() ) return; // not found
  st[i] = nullItem; N--;     //delete,reinsert
  for(j = i+1; !st[j].null(); j=(j+1)%M, N--)
  {  Item v = st[j]; st[j] = nullItem;
     insert(v);  //reinsert elements after deleted
  }
}
```

Item removal (delete)

b2) Double hashing $h(k) = [h_1(k) + i.h_2(k)] \bmod m$

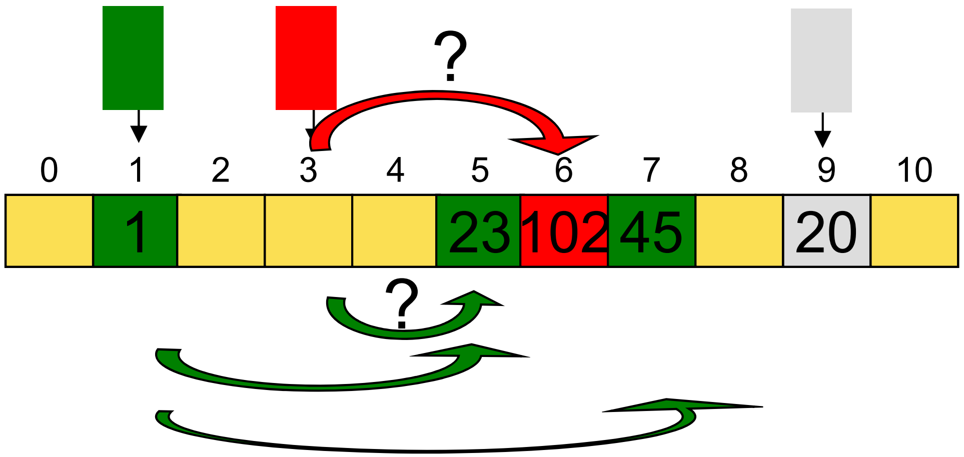


$h_1(k) = k \% 11$

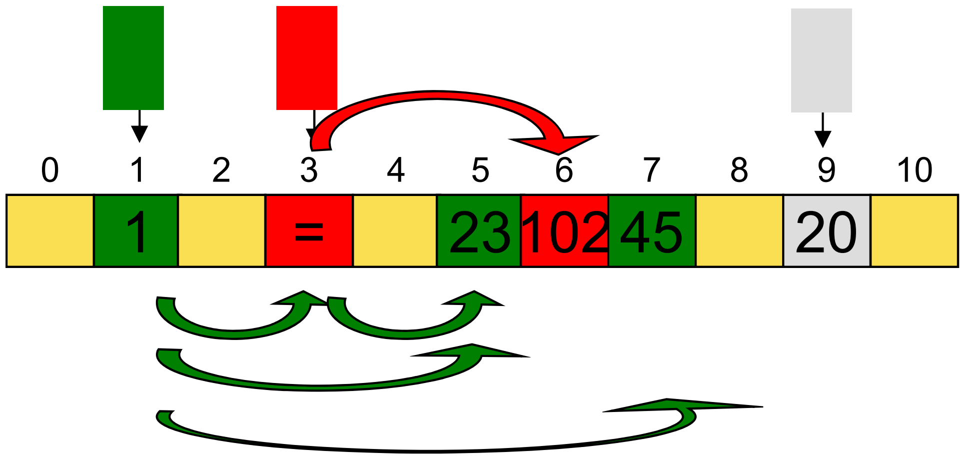
$h_2(k) = 1 + (k \% 10)$

Remove 25

null – breaks paths to 23 and 102



Sentinel is correct



Double-hashing Item Removal

```
// Double Hashing - overlapping search sequences
//      - fill up the hole by sentinel
//   - skipped by search, replaced by insert
void remove( Item item )
{ Key k = item.key();
  int i = hash( k, M ), j = hashTwo( k, M );
  while( !st[i].null() ) // find item to remove
     if( item.key() == st[i].key() ) break;
     else i = (i+j) % M;
  if( st[i].null() ) return; // not found
  st[i] = sentinelItem; N--; // "delete" = replace
}
```

b) Open-addressing hashing

α = ***load factor of the table***

α **= n/m,** $\alpha \in \langle 0,1 \rangle$

n = number of items in the table

***m = table size,* m>n**

b) Open-addressing hashing

Average number of probes [Sedgewick]

Linear probing:

Search hits	$0.5 (1 + 1 / (1 - \alpha))$	**found**
Search misses	$0.5 (1 + 1 / (1 - \alpha)^2)$	**not found**

Double hashing:

Search hits	$(1 / \alpha) \ln (1 / (1 - \alpha)) + (1 / \alpha)$
Search misses	$1 / (1 - \alpha)$

$\alpha = n/m, \alpha \in \langle 0,1 \rangle$

b) Expected number of tests

Linear probing:

load factor α	**1/2**	**2/3**	**3/4**	**9/10**
Search hit	**1.5**	**2.0**	**3.0**	**5.5**
Search miss	**2.5**	**5.0**	**8.5**	**55.5**

Double hashing:

load factor α	**1/2**	**2/3**	**3/4**	**9/10**
Search hit	**1.4**	**1.6**	**1.8**	**2.6**
Search miss	**1.5**	**2.0**	**3.0**	**5.5**

Table can be more loaded before the effectivity starts decaying.

Same effectivity can be achieved with smaller table.

References

[Cormen]

Cormen, Leiserson, Rivest: Introduction to Algorithms, Chapter 12, McGraw Hill, 1990