Multiagent Systems

Two Lectures on Coalitional Game Theory

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Game Forms

1. Normal (Strategic)
2. Extensive
3. Coalitional

Coalitional games – assumptions

- Players maximizing their utility are allowed to form coalitions
- Coalitions are sets of players coordinating their strategies in order to maximize the utility of the coalition
- Strategic aspects of coalitional games are unimportant, since they are implicitly part of the deal among players

Which Situations Are Captured by Coalitional Games?

- Transactions among buyers and sellers in a market
- Voting in committees
- Cost-sharing in large investment projects

Applications

- Google Analytics
- Explainable AI algorithms
- Analysis of voting procedures
- Genetic analysis

Players and Coalitions

- Player set $N := \{1, \ldots, n\}$, where $n \in \mathbb{N}$
- Coalition is a subset $A \subseteq N$
- $\emptyset$ empty coalition, N grand coalition
- Powerset $\mathcal{P}(N)$ is the set of all coalitions:

$$\mathcal{P}(N) := \{A \mid A \subseteq N\}$$

Coalitional Game

Definition

Coalitional game is a pair (N, v), where v is a function

$$v\colon \mathcal{P}(N) \to \mathbb{R} \qquad \text{such that } v(\emptyset) = 0.$$

- Number $v(A)$ is called the worth of A and it can be interpreted as a utility/cost associated with the formation of A
- When N is fixed, we will identify a coalitional game (N, v) with function v and call v simply a game

Example 1

Gin & Tonic

5 friends arrive at a party, 3 of whom with a bottle of gin apiece. Each of the other 2 friends has 5 bottles of tonic. A price of cocktails made from 1 gin bottle and 5 tonic bottles is 2000 CZK.

$$G = \{1, 2, 3\}, \ \ T = \{4, 5\}, \ \ N = G \cup T$$

$$v(A) = 2000 \cdot \min\{|A \cap G|, |A \cap T|\}, \qquad A \subseteq N$$

Example 2

Security Council

UN Security Council has 5 permanent and 10 non-permanent members. The decision is approved by all the permanent members together with at least 4 non-permanent members.

$$N = \{1, \ldots, 5, 6, \ldots, 15\}$$

$$v(A) = \begin{cases} 1 & \text{if } A \supset \{1, \ldots, 5\} \text{ and } |A| \geq 9, \\ 0 & \text{otherwise.} \end{cases}$$

Properties of Games

Definition

We say that a coalitional game v is

- monotone if $v(A) \leq v(B)$ for all $A \subseteq B$,
- superadditive if $v(A \cup B) \geq v(A) + v(B)$ for all $A \cap B = \emptyset$,
- supermodular if $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$,
- additive if $v(A \cup B) = v(A) + v(B)$ for all $A \cap B = \emptyset$.

- Additive games are trivial, since the worth of A is

$$v(A) = \sum_{i \in A} v(\{i\})$$

- Supermodular (convex) games have very convenient computational properties

Coalitional games – questions

1. **Which** coalitions will form?
2. **What** is a worth allocation among individual players?
3. **How** to justify such a worth allocation?

Possible answers

1. It is extremely difficult (conceptually and computationally) to answer this question
2. We often assume that the grand coalition N was formed. This means that all the players eventually reach some agreement to distribute the worth $v(N)$
3. The problem of justification can be solved by adopting several basic principles, which determine the resulting allocation

Allocation

Definition

Allocation is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define

$$x(A) := \sum_{i \in A} x_i, \qquad A \subseteq N.$$

An allocation x in a coalitional game v is

- efficient if $x(N) = v(N)$
- coalitionally rational if $x(A) \geq v(A)$ for all $A \subseteq N$
- individually rational if $x_i \geq v(\{i\})$ for all $i \in N$

Solution Concepts

Definition

Let Γ be the set of all coalitional games with a fixed player set N. Solution is a mapping

$$\sigma\colon \Gamma \to \mathcal{P}(\mathbb{R}^n),$$

where $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$.

- The set $\sigma(v)$ contains allocation vectors $x = (x_1, \ldots, x_n)$
- Solution reflects various aspects of economic rationality, fairness assumptions, or stability
- Solutions can be single-valued or multi-valued

Agenda

We will discuss the following solution concepts:

- Shapley value
- Core
- Nucleolus

Shapley value

Value of Coalitional Games

Definition

Value is a mapping

$$\varphi\colon \Gamma \to \mathbb{R}^n$$

with components $\varphi = (\varphi_1, \ldots, \varphi_n)$. The number $\varphi_i(v)$ is called the value of player i.

Some principles of a fair allocation:

- Distribute the total utility available
- The same reward for the same working contribution
- "He who does not work, neither shall he eat"

Axioms of Value

Definition

We say that a value φ

- is efficient, if $\sum_{i \in N} \varphi_i(v) = v(N)$ for each game v
- is symmetric, if $\varphi_i(v) = \varphi_j(v)$, for each game v and players $i, j \in N$ fullfilling the condition $v(A \cup \{i\}) = v(A \cup \{j\})$, for each coalition $A \subseteq N \setminus \{i, j\}$
- satisfies the null player property, if $\varphi_i(v) = 0$, for each game v and each $i \in N$ such that $v(A \cup \{i\}) = v(A)$, for all $A \subseteq N$
- is additive, if $\varphi(u + v) = \varphi(u) + \varphi(v)$, for $u, v \in \Gamma$

Shapley Value

Theorem (Shapley, 1953)

There is a unique value $\varphi^S : \Gamma \to \mathbb{R}^n$, which is efficient, additive, symmetric, and satisfies the null player property.

The value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{A \subseteq N \setminus \{i\}} \frac{|A|!(n - |A| - 1)!}{n!} \cdot \big(v(A \cup \{i\}) - v(A)\big).$$

Interpretation of Formula for Shapley Value

$$\varphi_i^S(v) = \sum_{A \subseteq N \setminus \{i\}} \frac{1}{n\binom{n-1}{|A|}} \cdot \underbrace{\big(v(A \cup \{i\}) - v(A)\big)}_{\text{marginal contribution of } i \text{ to } A}$$

- The Shapley value of player i is the expected value of the player's marginal contribution to every possible coalition
- The probability $\frac{1}{n\binom{n-1}{|A|}}$ is determined as follows:
    1. Player i randomly selects one of the sizes $0, 1, \ldots, n-1$ of a coalition to enter
    2. A coalition A of this size is then randomly chosen

Shapley Value as a Power Index

A game v is simple if

- $v(A) \in \{0, 1\}$
- v is monotone and $v(N) = 1$

Coalition $A \subseteq N$ is winning if $v(A) = 1$, and loosing if $v(A) = 0$.

Shapley–Shubik Index

$$\varphi_i^S(v) = \sum_{\substack{A \subseteq N \\ A \text{ loosing} \\ A \cup \{i\} \text{ winning}}} \frac{|A|!(n - |A| - 1)!}{n!}, \qquad i \in N.$$

Examples

Simple Majority Game

$$v(A) = \begin{cases} 1 & |A| > \frac{n}{2}, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

Efficiency and symmetry yield $\varphi_i^S(v) = \frac{1}{n}$ for each $i \in N$.

UN Security Council With $N = \{1, \ldots, 15\}$

We assume that $1, \ldots, 5$ are permanent members.

$v(A) = 1$ if $A \supset \{1, \ldots, 5\}$ and $|A| \geq 9$,

$v(A) = 0$ otherwise.

If $6 \leq i \leq 15$, we get $\varphi_i^S(v) = \binom{9}{3} \cdot \frac{8! \cdot 6!}{15!} \approx 0.0019$.

If $1 \leq j \leq 5$, we can proceed as follows:

$\varphi_j^S(v) = \frac{1}{5}(1 - 10\varphi_i^S(v)) \approx 0.1963$.

Power Indices for Voting

- The number of swings for player i in a simple game v is

$$s_i(v) := |\{A \subseteq N \mid v(A \cup \{i\}) - v(A) = 1\}|$$

- The Shapley-Shubik index uses the probability of a swing A proportional to its size, but there are alternative choices

Definition

Normalized Banzhaf index of player i is

$$\beta_i(v) = \frac{s_i(v)}{\sum_{i \in N} s_i(v)}$$

and Banzhaf index of player i is

$$\varphi_i^B(v) = \frac{s_i(v)}{2^{n-1}}$$

Example – UN Security Council

Old and new voting system with 5 permanent members

O 11 members, approval by at least 7 votes

N 15 members, approval by at least 9 votes

Shapley–Shubik indices:

O $\varphi_1^S(v) = 0.1974$, $\varphi_6^S(v) = 0.0022$ ratio $90 : 1$

N $\varphi_1^S(v) = 0.1963$, $\varphi_6^S(v) = 0.0019$ ratio $100 : 1$

Normalized Banzhaf indices:

O $\beta_1(v) = \frac{19}{105}$, $\beta_6(v) = \frac{1}{63}$ ratio $11 : 1$

N $\beta_1(v) = \frac{106}{635}$, $\beta_6(v) = \frac{21}{1270}$ ratio $10 : 1$

Random Order Approach

Let Π be the set of all permutations π of the player set N. Each number $\ell \in N$ is a ranking of player $\pi(\ell) \in N$.

Definition

- For each $\pi \in \Pi$ define

$$A_0^\pi := \emptyset, \qquad A_\ell^\pi := \{\pi(1), \dots, \pi(\ell)\}, \qquad \ell \in N.$$

- Marginal vector for a game v and a permutation π is an allocation vector $x^\pi \in \mathbb{R}^n$ with coordinates

$$x_i^\pi := v(A^\pi_{\pi^{-1}(i)}) - v(A^\pi_{\pi^{-1}(i)-1}), \quad i \in N.$$

Shapley Value From Random Order

$$\varphi_i^S(v) = \sum_{\pi \in \Pi} \frac{1}{n!} \cdot x_i^\pi$$

- The Shapley value $\varphi_i^S(v)$ of player i is an expected value of the marginal vectors of player i
- All the orders of players are equiprobable
- This formula becomes important for the approximate computation of Shapley value based on sampling

Estimation of the Shapley Value

Algorithm

Input: Coalitional game v and player i

1. Determine the size of the random sample $m \leq n!$
2. Sample (with replacement) permutations $(\pi_1, \ldots, \pi_m)$ from Π with uniform probability $\frac{1}{n!}$
3. Estimate the Shapley value by

$$\widehat{\varphi_i^S}(v) := \frac{1}{m} \sum_{k=1}^{m} x_i^{\pi_k}$$

The algorithm is polynomial, if the worth $v(A)$ of each coalition A can be calculated in polynomial time.

Core

Core

Definition

Core of a game v is the set of all efficient and coalitionally rational allocation vectors,

$$\mathcal{C}(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(A) \geq v(A),\ A \subseteq N\}.$$

- $\mathcal{C}(v)$ is convex polytope of dimension at most $n - 1$
- The core reflects the notion of stability:
  - $\mathcal{C}(v) \neq \emptyset$ if and only if the players are willing to form the grand coalition N
  - An allocation x is in $\mathcal{C}(v)$ if and only if no coalition has an incentive to break away from N

Core – Example (1)

Glove Game

Alice has a left glove. Bob and Cyril have one glove each.

The number of pairs of gloves collected by a coalition is its worth.

$N = \{1, 2, 3\}$

$$v(A) = \begin{cases} 1 & A \in \{\{1,2\}, \{1,3\}, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

Game v is monotone and superadditive, but not supermodular.

The core of v is

$$\mathcal{C}(v) = \{(1, 0, 0)\}.$$

Core – Example (2) and (3)

Majority voting

Three players vote by majority. This determines a game with the player set $N = \{1, 2, 3\}$, where

$$v(A) = \begin{cases} 1 & |A| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

There is no stable allocation in this game, $\mathcal{C}(v) = \emptyset$.

S. Zamir

$M := N \cup \{4\}$ and the game (M, u) is defined by

$$u(A) = \begin{cases} v(A) & A \subseteq N, \\ 0 & A \subset M, \\ \frac{3}{2} & A = M, \end{cases} \qquad \text{while } \mathcal{C}(v) = \{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0)\}.$$

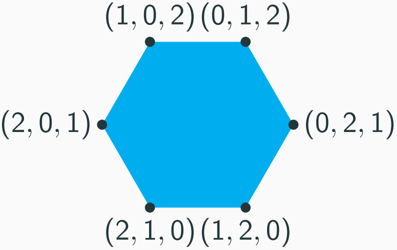
Core of Supermodular Games

Proposition

These assertions are equivalent for any game v.

- v is supermodular
- $x^\pi \in \mathcal{C}(v)$ for all $\pi \in \Pi$
- $\mathcal{C}(v) = \text{conv}\{x^\pi \mid \pi \in \Pi\}$

$$v(A) = \begin{cases} 0 & |A| = 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$$



Computation of the Core

- The shape of the core is known for special classes of games (simple, supermodular), but its structure can be fairly complicated ($n!$ extreme points)
- Checking whether the core is nonempty boils down to the feasibility problem with 2^n linear constraints
- In many situations it suffices to know that a given allocation belongs to the core, rather than to describe the core itself

Nucleolus

Measuring Deficit

What is a measure of dissatisfaction of a coalition $A \subseteq N$ in a coalitional game v, when an allocation $x \in \mathbb{R}^n$ is realized?

The deficit of A with respect to x is

$$d(A, x) \coloneqq v(A) - x(A).$$

Deficit vector

Order all the coalitions $A_1, \ldots, A_{2^n}$ according to the deficits:

$$d(A_1, x) \geq \cdots \geq d(A_{2^n}, x).$$

Then the deficit vector is

$$d(x) \coloneqq (d(A_1, x), \ldots, d(A_{2^n}, x)).$$

Lexicographic Order

- We want to compare deficit vectors $d(x)$ and $d(y)$ with respect to the initial coordinates (the highest deficits)
- $(10, 10, 10, 0)$ should be smaller than $(100, 0, -10, -10)$

Definition

Let $\alpha, \beta \in \mathbb{R}^m$. We write $\alpha \preceq \beta$ if

- there is k such that $\alpha_k < \beta_k$ and $\alpha_j = \beta_j$ for all $j < k$
- or $\alpha = \beta$

Example

Glove Game

$$N = \{1,2,3\} \qquad v(A) = \begin{cases} 1 & A \in \{\{1,2\},\{1,3\},N\}, \\ 0 & \text{otherwise.} \end{cases}$$

Allocations: $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (1,0,0)$, $z = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$

A	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$d(A,x)$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0
$d(A,y)$	0	-1	0	0	0	0	0	0
$d(A,z)$	0	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{2}{3}$	1

$$d(y) \preceq d(z) \preceq d(x)$$

Imputations

- The nucleolus is found among special allocations
- Imputation is an efficient and individually rational allocation
- The set of imputations in a game v is

$$\mathcal{I}(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(\{i\}),\ i \in N\}$$

Claim

$\mathcal{I}(v) \neq \emptyset$ for every superadditive game v

Nucleolus

Which imputations lexicographically minimize the deficit vector?

Definition

Let v be a game with $\mathcal{I}(v) \neq \emptyset$. The nucleolus of v is the set

$$\mathcal{N}(v) := \{x \in \mathcal{I}(v) \mid d(x) \preceq d(y) \quad \text{for all } y \in \mathcal{I}(v)\}$$

1. Is $\mathcal{N}(v)$ nonempty?
2. Is $\mathcal{N}(v)$ single-valued?
3. How to compute $\mathcal{N}(v)$?

Existence of Nucleolus

Theorem (Schmeidler, 1969)

The nucleolus $\mathcal{N}(v)$ contains a unique allocation for any game v with $\mathcal{I}(v) \neq \emptyset$.

- The nucleolus with respect to a set of allocations different from imputations may be empty or not unique, in general
- Computing the nucleolus in many classes of games is NP-hard

How to Compute $\mathcal{N}(v)$?

This algorithm finds an imputation whose deficit vector is lexicographically minimal.

Algorithm

Input: Coalitional game v such that $\mathcal{I}(v) \neq \emptyset$

1. Find all the imputations X_1 whose maximal deficit is minimal
2. Find all the imputations X_2 among X_1 whose second largest deficit is minimal
3. Continue this procedure...
4. ...until it halts and yields a single imputation, the nucleolus

Note that the complexity is exponential in the number of players n.

Minimizing the Maximal Deficit

Solve the LP with variables $x = (x_1, \ldots, x_n), t$

$$\begin{aligned} \text{Minimize} \quad & t \\ \text{subject to} \quad & d(A, x) \leq t, \quad \emptyset \neq A \subset N, \\ & x \in \mathcal{I}(v). \end{aligned}$$

- t_1 := the value of the LP
- $X_1 \times \{t_1\}$:= the set of optimal solutions
- If X_1 is a singleton, then $X_1 = \mathcal{N}(v)$
- Otherwise put

$$\mathcal{F}_1 := \{A \subset N \mid d(A, x) = t_1, \ \ x \in X_1\} \neq \emptyset$$

Minimizing the Second-largest Deficit

Solve the LP with variables $x = (x_1, \ldots, x_n), t$

$$\begin{aligned} \text{Minimize} \quad & t \\ \text{subject to} \quad & d(A, x) \leq t, \quad A \notin \mathcal{F}_1, \;\; \emptyset \neq A \subset N \\ & x \in X_1. \end{aligned}$$

- $t_2 :=$ the value of the LP
- $X_2 \times \{t_2\} :=$ the set of optimal solutions
- If X_2 is a singleton, then $X_2 = \mathcal{N}(v)$
- Otherwise put

$$\mathcal{F}_2 := \{A \subset N \mid d(A, x) = t_2, \;\; x \in X_2\} \neq \emptyset$$

Minimizing the kth Largest Deficit

The algorithm eventually continues until the set X_k is a singleton:

- Each t_k is the kth largest deficit
- Each $\mathcal{F}_k$ is the collection of coalitions with the deficit t_k
- At each step, $\mathcal{F}_k$ contains at least one new coalition
- The procedure halts after at most $2^n - 1$ steps

Properties of Solution Concepts

	Shapley value	core	nucleolus
Nonemptiness	✓	–	–
Efficiency	✓	✓	✓
Individual rationality	–	✓	✓
Symmetry	✓	–	✓
Null player property	✓	–	✓
Additivity	✓	–	–