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1. Consider a coalitional game $v\colon \mathcal{P}(N) \to \mathbb{R}$ over the player set $N = \{1, 2, 3\}$ such that

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ 1 & A = \{1\}, \{2\}, \\ 2 & A = \{3\}, \\ 4 & |A| = 2, \\ 5 & A = N. \end{cases}$$

   Is v superadditive? What is its core?

   Solution: Game v is superadditive, if the inequality $v(A \cup B) \geq v(A) + v(B)$ holds for all $A, B \subseteq N$, $A \cap B = \emptyset$. Since $v(N) < v(\{1, 2\} + v(\{3\})$, game v is not superadditive. It is easy to see that $\mathcal{C}(v)$ is empty. Indeed, every vector $x \in \mathcal{C}(v)$ must satisfy the conditions $x_1 + x_2 + x_3 = 5$, $x_1 + x_2 \geq 4$, and $x_3 \geq 2$. But adding the last two inequalities yields $5 = x_1 + x_2 + x_3 \geq 6$, a contradiction.

2. Describe the core of a coalitional game v over the player set $N = \{1, 2, 3\}$, where

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ |A| - 1 & A \neq \emptyset. \end{cases}$$

   Solution: Using the identity $|A \cup B| = |A| + |B| - |A \cap B|$ we can easily verify that v is supermodular, that is, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. This implies that its core $\mathcal{C}(v)$ coincides with the convex hull of its marginal vectors x^π, where π is a permutation on N. For example, the permutation $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$ determines a marginal vector x^π whose coordinates are

$$\begin{aligned} x_2^\pi &= v(\{2\}) - v(\emptyset) = 0, \\ x_3^\pi &= v(\{2, 3\}) - v(\{2\}) = 1, \\ x_1^\pi &= v(\{1, 2, 3\}) - v(\{2, 3\}) = 1. \end{aligned}$$

   The remaining marginal vectors are computed analogously. This shows that the core is a triangle with vertices $(0, 1, 1), (1, 0, 1)$, and $(1, 1, 0)$, which is located in the plane given by the equation $x_1 + x_2 + x_3 = 2$.

3. Prove that the Shapley value $\varphi^S(v)$ of a supermodular game v belongs to its core: $\varphi^S(v) \in \mathcal{C}(v)$.

Solution: Let $N = \{1, \dots, n\}$ and v be a supermodular game over N. By supermodularity, the core of v is the convex hull of its marginal vectors,

$$\mathcal{C}(v) = \operatorname{conv}\{x^\pi \mid \pi \in \Pi\},$$

where Π is the set of all permutations over N. Hence, it suffices to show that $\varphi^S(v)$ can be written as $\varphi^S(v) = \sum_{\pi\in\Pi} c_\pi \cdot x^\pi$, where $c_\pi \geq 0$ and $\sum_{\pi\in\Pi} c_\pi = 1$. But one of the formulas for Shapley value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot \left(v(A^\pi_{\pi^{-1}(i)}) - v(A^\pi_{\pi^{-1}(i)-1}) \right).$$

Since marginal vector x^π has coordinates

$$x_i^\pi = v(A^\pi_{\pi^{-1}(i)}) - v(A^\pi_{\pi^{-1}(i)-1}),$$

is is enough to put $c_\pi := \frac{1}{n!}$ for each $\pi \in \Pi$. Note that we have even proved that $\varphi^S(v) \in \mathcal{C}(v)$ is a center of gravity of $\mathcal{C}(v)$.

4. A company has 3 shareholders whose shares are distributed in the following way. The first has 50 % shares and the remaining two have 25 % shares each. The three shareholders vote by using a weighted majority of votes. Describe precisely the resulting coalitional game. Compute the Shapley-Shubik index using the random order approach and then calculate the normalized Banzhaf index.

Solution: The player set is $N = \{1, 2, 3\}$. The coalitional game is v:

$$v(A) = \begin{cases} 1 & A = N, \{1,2\}, \{1,3\}, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

For the calculation of Shapley-Shubik index of i we enumerate all the permutations such that i makes the preceding coalition winning:

1 **2** 3   1 **3** 2   2 **1** 3   23 **1**   3 **1** 2   32 **1**

Thus,

$$\varphi_1^S(v) = \frac{2}{3}, \quad \varphi_2^S(v) = \varphi_3^S(v) = \frac{1}{6}.$$

In order to compute the normalized Banzhaf index $\beta(v)$, we enumerate the number of swings for each player:

1 **2**   **1** **3**   **1** 23

Hence, $s_1(v) = 3$, $s_2(v) = s_3(v) = 1$. These numbers are divided by the total number of swings:

$$\beta_1(v) = \frac{3}{5}, \quad \beta_1(v) = \beta_1(v) = \frac{1}{5}.$$

5. Consider a solution mapping $\psi\colon \Gamma \to \mathbb{R}^n$ over the set of all n-player coalitional games Γ defined by
$$\psi_i(v) = v(\{1,\dots,i\}) - v(\{1,\dots,i-1\}), \quad i \in N.$$
Show that ψ is efficient, additive, has the null player property, but fails symmetry.

Solution: First, we check efficiency:
$$\sum_{i\in N}\psi_i(v) = \sum_{i\in N}(v(\{1,\dots,i\}) - v(\{1,\dots,i-1\})) = v(N) - v(\emptyset) = v(N).$$
Aditivity: for all $v, w \in \Gamma$ we get
$$\begin{aligned}\psi_i(v+w) &= (v+w)(\{1,\dots,i\}) - (v+w)(\{1,\dots,i-1\})\\ &= (v(\{1,\dots,i\}) - v(\{1,\dots,i-1\})) + (w(\{1,\dots,i\}) - w(\{1,\dots,i-1\}))\\ &= \psi_i(v) + \psi_i(w).\end{aligned}$$
Null player property: let $i \in N$ be the null player. This means that $v(A \cup \{i\}) = v(A)$ for each coalition $A \subseteq N$. Then putting $A = \{1,\dots,i-1\}$ yields $\psi_i(v) = 0$.

We show that ψ fails symmetry. Letting $N = \{1,2,3\}$ we define a game
$$v(A) = \begin{cases}1 & A = \{2,3\}, N\\ 0 & \text{otherwise,}\end{cases} \qquad A \subseteq N.$$
Then $\psi(v) = (0,0,1)$. However, players 2 and 3 are symmetric in this game since $v(\{1,2\}) = v(\{1,3\})$. This implies that ψ fails symmetry.

6. *Spanning tree game.* The costs of connecting the cities denoted as $1, 2$, and 3 to the supplier of energy 0 are depicted in Figure 1. Construct the associated minimum cost spanning tree game and show that its core is nonempty.

Solution: We can easily find using Prim's algorithm that the corresponding cost game v with the player set $N = \{1,2,3\}$ is
$$v(A) = \begin{cases}0 & A = \emptyset,\\ 20 & A = \{1\},\\ 30 & A = \{3\},\\ 50 & A = N,\\ 40 & \text{otherwise.}\end{cases}$$
Since the numbers $v(A)$ are intepreted as costs, the core of v is precisely the set of allocations $x \in \mathbb{R}^3$ such that $x(N) = 50$ and
$$x_1 \le 20,\ x_3 \le 30,\ x(A) \le 40, \qquad \text{for all } A \in \{\{2\},\{1,2\},\{1,3\},\{2,3\}\}.$$

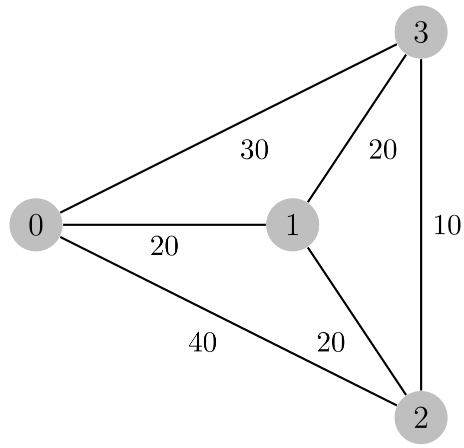

Figure 1: Graph from Example 6

It is easily shown that the core of v is nonempty. It suffices to take the minimum spanning tree associated with the grand coalition N together with the costs for the connection of individual cities in the resulting minimum spanning tree. For example,

$$x_1 \coloneqq 20, \quad x_2 \coloneqq 20, \quad x_3 \coloneqq 10.$$

Then $x = (x_1, x_2, x_3)$ belongs to the core of v.

7. A *simple game* is a coalitional game $v \colon \mathcal{P}(N) \to \{0,1\}$ that is monotone and $v(N) = 1$. We call a player $i \in N$ in a simple game v a *veto player*, if for each coalition $A \subseteq N$ holds $v(A \setminus \{i\}) = 0$. Show that the following hold for any simple game v:
   (a) Player i is veto if and only if $v(N \setminus \{i\}) = 0$.
   (b) $\mathcal{C}(v) \neq \emptyset$ if and only if v has a veto player.
   (c) If the set of veto players $W \subseteq N$ is nonempty, then the core is

$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(W) = 1,\ x_i \geq 0 \text{ pro } i \in W \text{ a } x_j = 0 \text{ pro } j \in N \setminus W\}. \qquad (1)$$

Solution: (a) Necessity is obvious. Assume $v(N \setminus \{i\}) = 0$. Then monotonicity gives $v(A \setminus \{i\}) = 0$ for each coalition A.

(b) Let $k \in N$ be a veto player. We define an allocation vector $x \in \mathbb{R}^n$ as follows:

$$x_i = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

Since v is non-constant, $v(N) = 1 = \sum_{i \in N} x_i = x(N)$. Choose $A \subseteq N$. If $k \in A$, then $x(A) = 1 \geq v(A)$. If $k \notin A$, then $x(A) = 0 = v(A)$, since k is veto. We have shown that $x \in \mathcal{C}(v)$.

Conversely, assume that v has no veto players. We want to conclude that v has empty core. By way of contradiction, let $x \in \mathcal{C}(v)$. Then the condition $x(N) = 1$ implies that there exists $i \in N$ such that $x_i > 0$, hence $x(N \setminus \{i\}) = 1 - x_i < 1$. Since i is not veto, $v(N \setminus \{i\}) = 1 > x(N \setminus \{i\})$, which contradicts our assumption $x \in \mathcal{C}(v)$.

(c) Observe that if $A \subseteq N$ is winning ($v(A) = 1$), then $A \supseteq W$. Let x meet the condition on the right-hand side of (1). Obviously, $x(N) = x(W) = 1$. If $A \subseteq N$ is loosing ($v(A) = 0$), then $x(A) \geq 0$. Let $v(A) = 1$. Then $A \supseteq W$, which gives

$$x(A) \geq x(W) = 1 = v(A).$$

Thus, $x \in \mathcal{C}(v)$.

Conversely, let $x \in \mathcal{C}(v)$. Then $x_i \geq 0$ for all $i \in N$ and $x(N) = 1$. We need to show that $x_i = 0$ for all $i \in N \setminus W$. Pick $i \in N \setminus W$. Player i is not veto and hence

$$1 = x(N) \geq x(N \setminus \{i\}) \geq v(N \setminus \{i\}) = 1,$$

which implies $x(N) = x(N \setminus \{i\})$, so that $x_i = 0$.

8. Decide if the assertions below are true or false.

   (a) If the core is nonempty, it contains the Shapley value.

   (b) If marginal contributions of players i and j to every coalition are the same, then their Shapley values coincides.

   (c) The core of every monotone coalitional game is nonempty.

   (d) Nucleolus satisfies the properties of efficiency, symmetry, and null player property.

   (e) Nucleolus is an additive solution concept.

   (f) Shapley value $\varphi^S(v) = (\varphi_1^S(v), \ldots, \varphi_n^S(v))$ of every n-player coalitional game v is uniquely determined by the $(n-1)$-tuple $(\varphi_1^S(v), \ldots, \varphi_{n-1}^S(v))$.

   (g) Shapley value is individually rational, that is, $\varphi_i^S(v) \geq v(\{i\})$.

   (h) The nucleolus is individually rational.

Solution:

(a) False. For example, the Shapley value of the 3-player glove game is not an element of its core.

(b) True. This is exactly the symmetry of Shapley value.

(c) False. For example, take a two-player game $v(1) = v(2) = v(12) = 1$.

(d) True.

(e) False. Shapley value is the only single-valued solution satisfying these four properties: efficiency, symmetry, null player property, and additivity. Since the nucleolus has the first three properties and it is different from the Shapley value, it cannot be additive.

(f) True. By efficiency of Shapley value, $\varphi_n^S(v) = v(N) - \sum_{i=1}^{n-1} \varphi_i^S(v)$.

(g) False. Consider a 2-player game v that is not superadditive. Such a game satisfies the inequality $v(12) < v(1) + v(2)$, which implies $\varphi_1(v) < v(1)$.

(h) The nucleolus is an imputation, hence individual rationality.

References

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