Coalitional Games – Exercises for Multiagent Systems 2019

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1. Consider a coalitional game $v\colon \mathcal{P}(N) \to \mathbb{R}$ over the player set $N = \{1,2,3\}$ such that
$$v(A) = \begin{cases} 0 & A = \emptyset, \\ 1 & A = \{1\}, \{2\}, \\ 2 & A = \{3\}, \\ 4 & |A| = 2, \\ 5 & A = N. \end{cases}$$
Is v superadditive? What is its core?

2. Describe the core of a coalitional game v over the player set $N = \{1,2,3\}$, where
$$v(A) = \begin{cases} 0 & A = \emptyset, \\ |A| - 1 & A \neq \emptyset. \end{cases}$$

3. Prove that the Shapley value $\varphi^S(v)$ of a supermodular game v belongs to its core: $\varphi^S(v) \in \mathcal{C}(v)$.

4. A company has 3 shareholders whose shares are distributed in the following way. The first has 50 % shares and the remaining two have 25 % shares each. The three shareholders vote by using a weighted majority of votes. Describe precisely the resulting coalitional game. Compute the Shapley-Shubik index using the random order approach and then calculate the normalized Banzhaf index.

5. Consider a solution mapping $\psi\colon \Gamma \to \mathbb{R}^n$ over the set of all n-player coalitional games Γ defined by
$$\psi_i(v) = v(\{1,\dots,i\}) - v(\{1,\dots,i-1\}), \quad i \in N.$$
Show that ψ is efficient, additive, has the null player property, but fails symmetry.

6. *Spanning tree game.* The costs of connecting the cities denoted as $1, 2$, and 3 to the supplier of energy 0 are depicted in Figure 1. Construct the associated minimum cost spanning tree game and show that its core is nonempty.

7. A *simple game* is a coalitional game $v\colon \mathcal{P}(N) \to \{0,1\}$ that is monotone and $v(N) = 1$. We call a player $i \in N$ in a simple game v a *veto player*, if for each coalition $A \subseteq N$ holds $v(A \setminus \{i\}) = 0$. Show that the following hold for any simple game v:

   (a) Player i is veto if and only if $v(N \setminus \{i\}) = 0$.

   (b) $\mathcal{C}(v) \neq \emptyset$ if and only if v has a veto player.

   (c) If the set of veto players $W \subseteq N$ is nonempty, then the core is
$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(W) = 1,\ x_i \geq 0 \text{ pro } i \in W \text{ a } x_j = 0 \text{ pro } j \in N \setminus W\}. \tag{1}$$

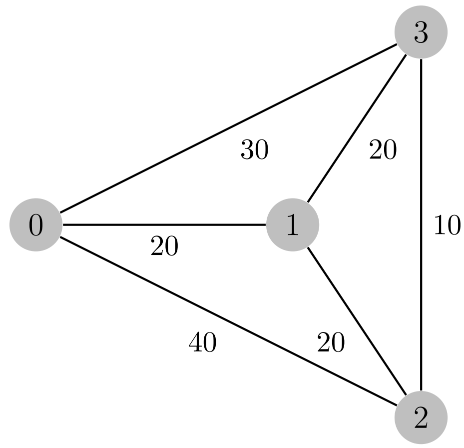

Figure 1: Graph from Example 6

8. Decide if the assertions below are true or false.

   (a) If the core is nonempty, it contains the Shapley value.

   (b) If marginal contributions of players i and j to every coalition are the same, then their Shapley values coincides.

   (c) The core of every monotone coalitional game is nonempty.

   (d) Nucleolus satisfies the properties of efficiency, symmetry, and null player property.

   (e) Nucleolus is an additive solution concept.

   (f) Shapley value $\varphi^S(v) = (\varphi_1^S(v), \dots, \varphi_n^S(v))$ of every n-player coalitional game v is uniquely determined by the $(n-1)$-tuple $(\varphi_1^S(v), \dots, \varphi_{n-1}^S(v))$.

   (g) Shapley value is individually rational, that is, $\varphi_i^S(v) \geq v(\{i\})$.

   (h) The nucleolus is individually rational.

References

[1] J. González-Díaz, I. García-Jurado, and M. G. Fiestras-Janeiro. *An Introductory Course on Mathematical Game Theory*, volume 115 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.

[2] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.

[3] G. Owen. *Game theory*. Academic Press Inc., San Diego, CA, third edition, 1995.