Randomized Sampling-based Motion Planning Methods

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Lecture 07

B4M36UIR – Artificial Intelligence in Robotics

Overview of the Lecture

- Part 1 – Randomized Sampling-based Motion Planning Methods
  - Sampling-Based Methods
  - Probabilistic Road Map (PRM)
  - Characteristics
  - Rapidly Exploring Random Tree (RRT)
- Part 2 – Optimal Sampling-based Motion Planning Methods
  - Optimal Motion Planners
  - Rapidly-exploring Random Graph (RRG)
  - Informed Sampling-based Methods
- Part 3 – Multi-Goal Motion Planning (MGMP)
  - Multi-Goal Motion Planning
  - Physical Orienteering Problem (POP)

Part I

Part 1 – Sampling-based Motion Planning

Outline

- Sampling-Based Methods
- Probabilistic Road Map (PRM)
- Characteristics
- Rapidly Exploring Random Tree (RRT)

(Randomized) Sampling-based Motion Planning

- It uses an explicit representation of the obstacles in $\mathcal{C}$-*space*.
  - A "black-box" function is used to evaluate if a configuration q is a collision-free, e.g.,
  - Based on geometrical models and testing collisions of the models.
  - 2D or 3D shapes of the robot and environment can be represented as sets of triangles, i.e., tesselated models.
  - Collision test is then a test of for the intersection of the triangles.

E.g., using RAPID library http://gamma.cs.unc.edu/OBB/

- **Creates a discrete representation of $\mathcal{C}_{free}$.**
- Configurations in $\mathcal{C}_{free}$ are sampled randomly and connected to a roadmap (**probabilistic roadmap**).
- Rather than the full completeness they provide **probabilistic completeness** or resolution completeness.

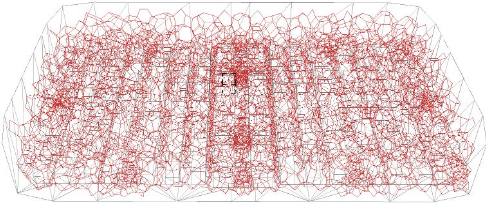
Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists).

Probabilistic Roadmaps

A discrete representation of the continuous $\mathcal{C}$-space generated by randomly sampled configurations in $\mathcal{C}_{free}$ that are connected into a graph.

- **Nodes** of the graph represent admissible configurations of the robot.
- **Edges** represent a feasible path (trajectory) between the particular configurations.

Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists).



Having the graph, the final path (trajectory) can be found by a graph search technique.

Incremental Sampling and Searching

- Single query sampling-based algorithms incrementally create a search graph (roadmap).
  1. **Initialization** – $G(V, E)$ an undirected search graph, V may contain q_{start}, q_{goal} and/or other points in $\mathcal{C}_{free}$.
  2. **Vertex selection method** – choose a vertex $q_{cur} \in V$ for the expansion.
  3. **Local planning method** – for some $q_{new} \in \mathcal{C}_{free}$, attempt to construct a path $\tau : [0, 1] \to \mathcal{C}_{free}$ such that $\tau(0) = q_{cur}$ and $\tau(1) = q_{new}$, τ must be checked to ensure it is collision free.
     - If τ is not a collision-free, go to Step 2.
  4. **Insert an edge in the graph** – Insert τ into E as an edge from q_{cur} to q_{new} and insert q_{new} to V if $q_{new} \notin V$. How to test q_{new} is in V?
  5. **Check for a solution** – Determine if G encodes a solution, e.g., using a single search tree or graph search technique.
  6. **Repeat Step 2** – iterate unless a solution has been found or a termination condition is satisfied.

LaValle, S. M.: Planning Algorithms (2006), Chapter 5.4

Probabilistic Roadmap Strategies

Multi-Query strategy is roadmap based.

- Generate a single roadmap that is then used for repeated planning queries.
- An representative technique is **Probabilistic RoadMap (PRM)**.

Kavraki, L., Svestka, P., Latombe, J.-C., Overmars, M. H.B (1996): Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces. T-RO.

Single-Query strategy is an incremental approach.

- For each planning problem, it constructs a new roadmap to characterize the subspace of $\mathcal{C}$-space that is relevant to the problem.
  - Rapidly-exploring Random Tree – RRT; *LaValle, 1998*
  - Expansive-Space Tree – EST; *Hsu et al., 1997*
  - Sampling-based Roadmap of Trees – SRT.

A combination of multiple–query and single–query approaches.
Plaku et al., 2005

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Multi-Query Strategy

Build a roadmap (graph) representing the environment.

1. Learning phase
   1.1 Sample n points in $\mathcal{C}_{free}$.
   1.2 Connect the random configurations using a local planner.
2. Query phase
   2.1 Connect start and goal configurations with the PRM.

   E.g., using a local planner.

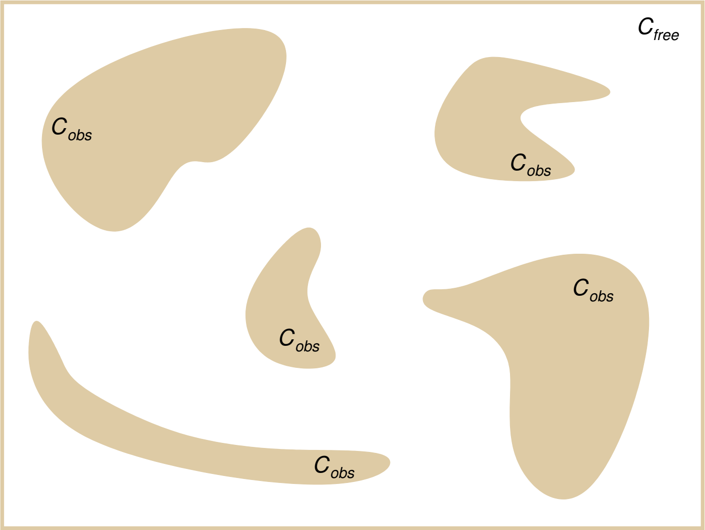
   2.2 Use the graph search to find the path.

Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces
Lydia E. Kavraki and Petr Svestka and Jean-Claude Latombe and Mark H. Overmars, IEEE Transactions on Robotics and Automation, 12(4):566–580, 1996.

First planner that demonstrates ability to solve general planning problems in more than 4-5 dimensions.

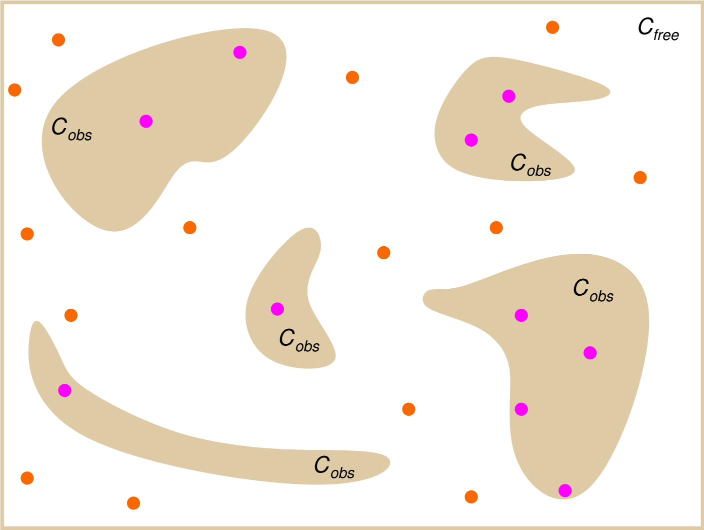
PRM Construction

Given problem domain:



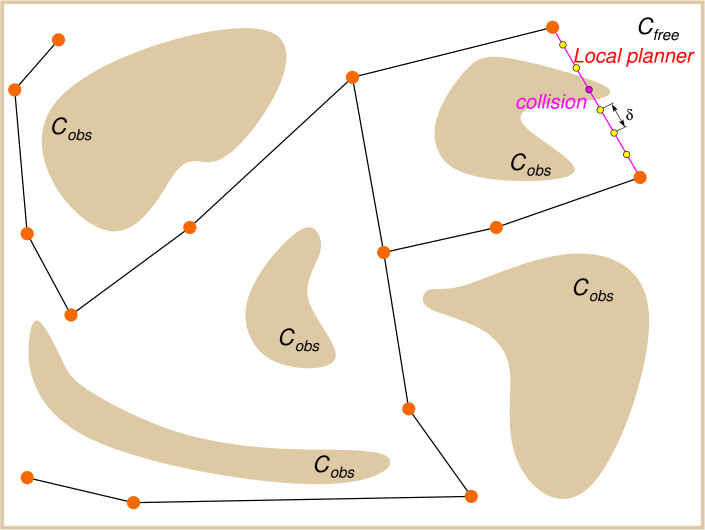
PRM Construction

Random configuration



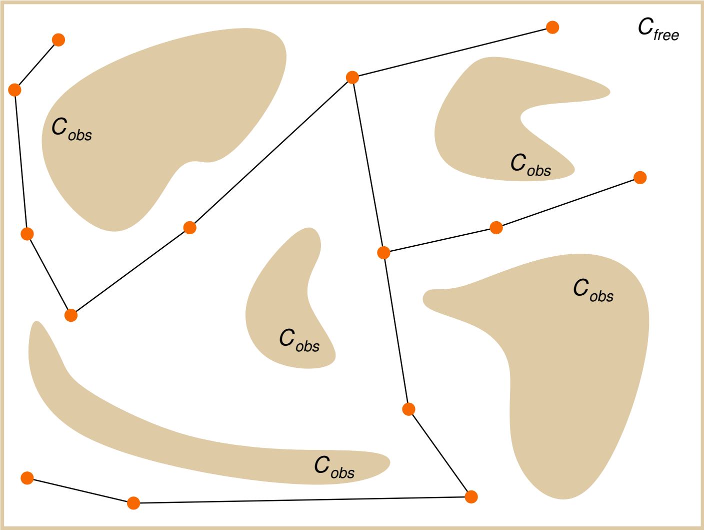
PRM Construction

Connecting random samples:



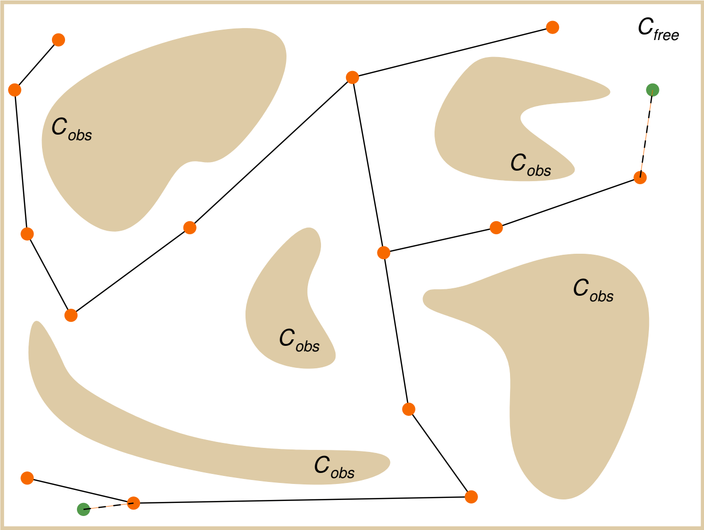
PRM Construction

Connected roadmap:



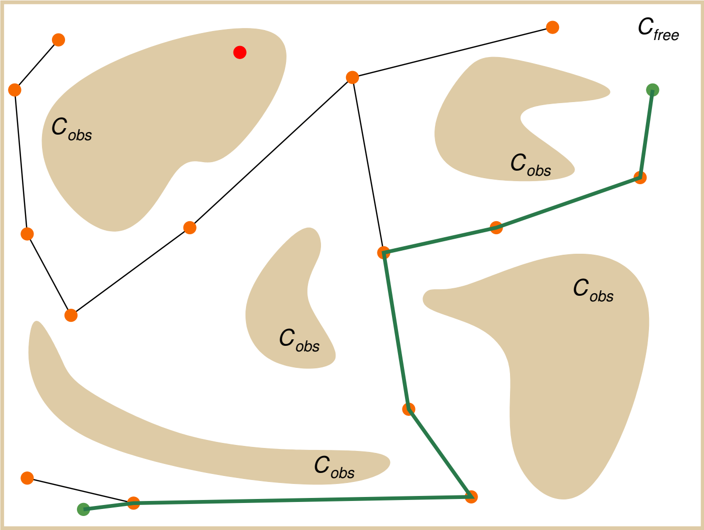
PRM Construction

Query configurations:



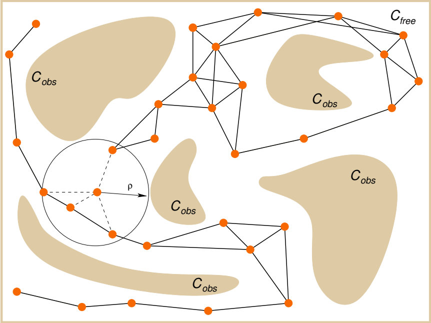
PRM Construction

Final found path:



Practical PRM

- Incremental construction.
- Connect nodes in a radius ρ.
- Local planner tests collisions up to selected resolution δ.
- Path can be found by Dijkstra's algorithm.



What are the properties of the PRM algorithm?

We need a couple of more formalisms.

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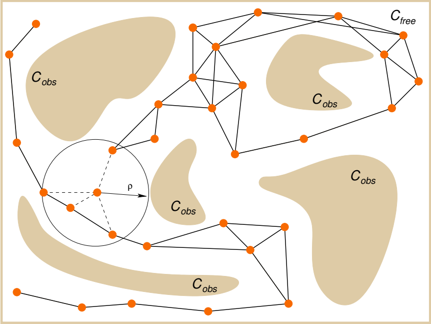

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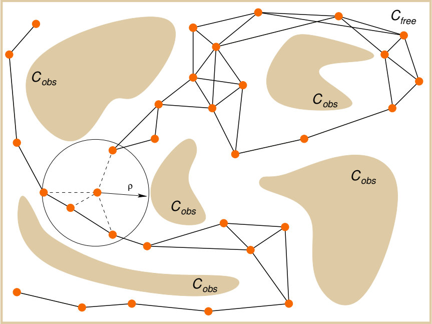

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We need a couple of more formalisms.

Path Planning Problem Formulation

- Path planning problem is defined by a triplet
$$\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal}), \text{ where}$$
  - $\mathcal{C}_{free} = \text{cl}(\mathcal{C} \setminus \mathcal{C}_{obs})$, $\mathcal{C} = (0,1)^d$, for $d \in \mathbb{N}$, $d \geq 2$; (scaling)
  - $q_{init} \in \mathcal{C}_{free}$ is the initial configuration (condition);
  - $\mathcal{Q}_{goal}$ is the goal region defined as an open subspace of $\mathcal{C}_{free}$.
- Function $\pi : [0,1] \to \mathbb{R}^d$ of *bounded variation* is called:
  - **path** if it is continuous;
  - **collision-free path** if it is a path and $\pi(\tau) \in \mathcal{C}_{free}$ for $\tau \in [0,1]$;
  - **feasible** if it is a collision-free path, and $\pi(0) = q_{init}$ and $\pi(1) \in \text{cl}(\mathcal{Q}_{goal})$.
- A function π with the total variation $\text{TV}(\pi) < \infty$ is said to have bounded variation, where $\text{TV}(\pi)$ is the total variation
$$\text{TV}(\pi) = \sup_{\{n \in \mathbb{N}, 0=\tau_0<\tau_1<\ldots<\tau_n=s\}} \sum_{i=1}^{n} |\pi(\tau_i) - \pi(\tau_{i-1})|.$$
- The total variation $\text{TV}(\pi)$ is de facto a path length.

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- Sampling-Based Methods
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Path Planning Problem

- **Feasible path planning**

  For a path planning problem $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$:
  - Find a feasible path $\pi : [0, 1] \rightarrow \mathcal{C}_{free}$ such that $\pi(0) = q_{init}$ and $\pi(1) \in \mathrm{cl}(\mathcal{Q}_{goal})$, if such path exists;
  - Report failure if no such path exists.

- **Optimal path planning**

  The optimality problem asks for a feasible path with the minimum cost.

  For $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$ and a cost function $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$:
  - Find a feasible path π^* such that $c(\pi^*) = \min\{c(\pi) : \pi \textit{ is feasible}\}$;
  - Report failure if no such path exists.

  The cost function is assumed to be monotonic and bounded, i.e., there exists k_c such that $c(\pi) \leq k_c \, \mathrm{TV}(\pi)$

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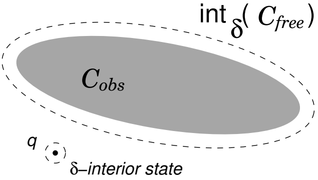
Probabilistic Completeness 1/2

First, we need **robustly feasible path planning problem** $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$.

- $q \in \mathcal{C}_{free}$ is ***δ-interior state* of $\mathcal{C}_{free}$** if the closed ball of radius δ centered at q lies entirely inside $\mathcal{C}_{free}$.



- *δ-interior* of $\mathcal{C}_{free}$ is $\text{int}_\delta(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} | \mathcal{B}_{/,\delta} \subseteq \mathcal{C}_{free}\}$.

A collection of all δ-interior states.

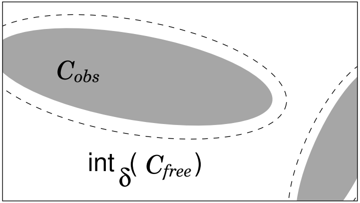
- A collision free path π has **strong δ-clearance**, if π lies entirely inside $\text{int}_\delta(\mathcal{C}_{free})$.
- $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$ is *robustly feasible* if a solution exists and it is a feasible path with *strong δ-clearance*, for $\delta > 0$.

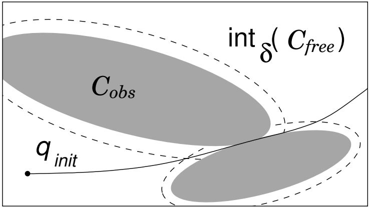
Probabilistic Completeness 2/2

An algorithm $\mathcal{ALG}$ is **probabilistically complete** if, for any *robustly feasible path planning problem* $\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$,

$$\lim_{n\to\infty} Pr(\mathcal{ALG} \text{ returns a solution to } \mathcal{P}) = 1.$$

- It is a *"relaxed"* notion of the completeness.
- Applicable only to problems with a **robust solution**.





We need some space, where random configurations can be sampled.

Asymptotic Optimality 1/4
Homotopy

Asymptotic optimality relies on a notion of **weak δ-clearance**.

Notice, we use strong δ-clearance for probabilistic completeness.

- We need to describe possibly improving paths (during the planning).
- Function $\psi : [0,1] \to \mathcal{C}_{free}$ is called **homotopy**, if $\psi(0) = \pi_1$ and $\psi(1) = \pi_2$ and $\psi(\tau)$ is collision-free path for all $\tau \in [0,1]$.
- A collision-free path π_1 is **homotopic** to π_2 if there exists homotopy function ψ.

A path homotopic to π can be continuously transformed to π through $\mathcal{C}_{free}$.

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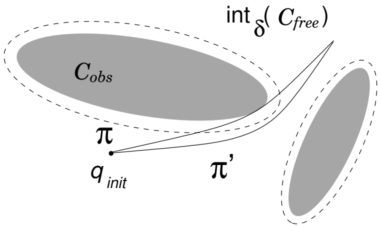
Asymptotic Optimality 2/4
Weak δ-clearance

- A collision-free path $\pi : [0, s] \rightarrow \mathcal{C}_{free}$ has **weak δ-clearance** if there exists a path π' that has **strong δ-clearance** and homotopy ψ with $\psi(0) = \pi$, $\psi(1) = \pi'$, and for all $\alpha \in (0, 1]$ there exists $\delta_\alpha > 0$ such that $\psi(\alpha)$ has strong δ-clearance.

Weak δ-clearance does not require points along a path to be at least a distance δ away from obstacles.



- A path π with a weak δ-clearance.
- π' lies in $\mathrm{int}_\delta(\mathcal{C}_{free})$ and it is the same homotopy class as π.

Asymptotic Optimality 3/4
Robust Optimal Solution

- It is applicable with a **robust optimal solution** that can be obtained as a limit of robust (non-optimal) solutions.
- A collision-free path π^* is **robustly optimal solution** if it has *weak δ-clearance* and for any sequence of collision free paths $\{\pi_n\}_{n\in\mathbb{N}}$, $\pi_n \in \mathcal{C}_{free}$ such that $\lim_{n\to\infty} \pi_n = \pi^*$,

$$\lim_{n\to\infty} c(\pi_n) = c(\pi^*).$$

There exists a path with strong δ-clearance, and π^* is homotopic to such path and π^* is of ***the lower cost****.*

- Weak δ-clearance implies robustly feasible solution problem.

Thus, it implies the probabilistic completeness.

Asymptotic Optimality 4/4
Asymptotically optimal algorithm

An algorithm $\mathcal{ALG}$ is **asymptotically optimal** if, for any path planning problem $\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$ and cost function c that admit a robust optimal solution with the finite cost c^*

$$Pr\left(\left\{\lim_{i\to\infty} Y_i^{\mathcal{ALG}} = c^*\right\}\right) = 1.$$

- $Y_i^{\mathcal{ALG}}$ is the extended random variable corresponding to the minimum-cost solution included in the graph returned by $\mathcal{ALG}$ at the end of the iteration i.

Properties of the PRM Algorithm

- Completeness for the standard PRM has not been provided when it was introduced.
- A simplified version of the PRM (called sPRM) has been mostly studied.
- sPRM is probabilistically complete.

What are the differences between PRM and sPRM?

PRM vs simplified PRM (sPRM)

Algorithm 1: PRM

Input: q_{init}, number of samples n, radius ρ
Output: PRM – $G = (V, E)$

$V \leftarrow \emptyset$; $E \leftarrow \emptyset$;
for $i = 0, \ldots, n$ **do**
  $q_{rand} \leftarrow$ SampleFree;
  $U \leftarrow$ Near$(G = (V, E), q_{rand}, \rho)$;
  $V \leftarrow V \cup \{q_{rand}\}$;
  foreach $u \in U$, *with increasing* $||u - q_r||$ **do**
    if q_{rand} *and* u *are not in the same connected component of* $G = (V, E)$ **then**
      if CollisionFree(q_{rand}, u) **then**
        $E \leftarrow E \cup \{(q_{rand}, u), (u, q_{rand})\}$;
return $G = (V, E)$;

Algorithm 2: sPRM

Input: q_{init}, number of samples n, radius ρ
Output: PRM – $G = (V, E)$

$V \leftarrow \{q_{init}\} \cup \{\text{SampleFree}_i\}_{i=1,\ldots,n-1}$; $E \leftarrow \emptyset$;
foreach $v \in V$ **do**
  $U \leftarrow$ Near$(G = (V, E), v, \rho) \setminus \{v\}$;
  foreach $u \in U$ **do**
    if CollisionFree(v, u) **then**
      $E \leftarrow E \cup \{(v, u), (u, v)\}$;
return $G = (V, E)$;

There are several ways for the set U of vertices to connect them:

- k-nearest neighbors to v;
- variable connection radius ρ as a function of n.

PRM – Properties

- sPRM (simplified PRM):
  - **Probabilistically complete and asymptotically optimal**.
  - Processing complexity can be bounded by $O(n^2)$.
  - Query complexity can be bounded by $O(n^2)$.
  - Space complexity can be bounded by $O(n^2)$.
- Heuristics practically used are usually not probabilistic complete.
  - k-nearest sPRM is not probabilistically complete.
  - Variable radius sPRM is not probabilistically complete.

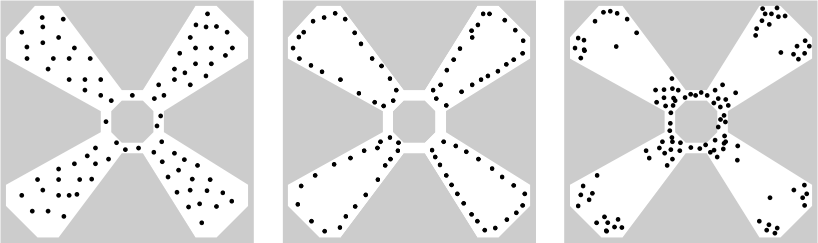
Based on analysis of Karaman and Frazzoli

PRM algorithm

+ It has very simple implementation.
+ It provides completeness (for sPRM).
– Differential constraints (car-like vehicles) are not straightforward.

Comments about Random Sampling 1/2

- Different sampling strategies (distributions) may be applied.



- Notice, one of the main issue of the randomized sampling-based approaches is the narrow passage.
- Several modifications of sampling based strategies have been proposed in the last decades.

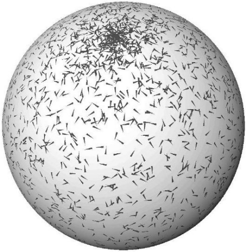
Comments about Random Sampling 2/2

- A solution can be found using only a few samples.

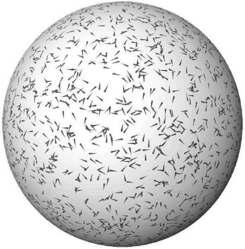
Do you know the Oraculum? (from Alice in Wonderland)

- Sampling strategies are important:
  - Near obstacles;
  - Narrow passages;
  - Grid-based;
  - Uniform sampling must be carefully considered.

James J. Kuffner (2004): Effective Sampling and Distance Metrics for 3D Rigid Body Path Planning. ICRA.



Naïve sampling



Uniform sampling of SO(3) using Euler angles

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Rapidly Exploring Random Tree (RRT)

Single–Query algorithm

- It incrementally builds a graph (tree) towards the goal area.

It does not guarantee precise path to the goal configuration.

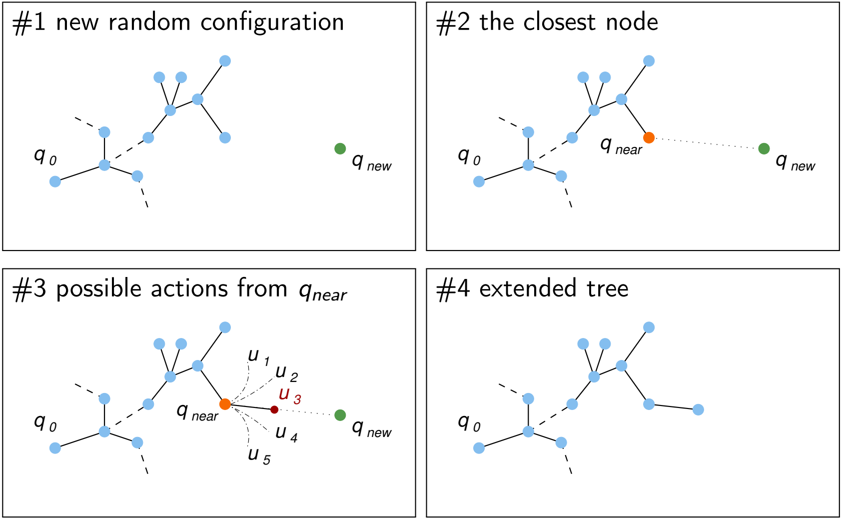
1. Start with the initial configuration q_0, which is a root of the constructed graph (tree).
2. Generate a new random configuration q_{new} in $\mathcal{C}_{free}$.
3. Find the closest node q_{near} to q_{new} in the tree.

   E.g., using KD-tree implementation like ANN or FLANN libraries.
4. Extend q_{near} towards q_{new}.

   Extend the tree by a small step, but often a direct control $u \in \mathcal{U}$ that will move robot the position closest to q_{new} is selected (applied for δt).
5. Go to Step 2, until the tree is within a sufficient distance from the goal configuration.

   Or terminates after dedicated running time.

RRT Construction



RRT Algorithm

- Motivation is a single query and *control-based* path finding
- It incrementally builds a graph (tree) towards the goal area

Algorithm 3: Rapidly Exploring Random Tree (RRT)

Input: q_{init}, number of samples n
Output: Roadmap $G = (V, E)$

$V \leftarrow \{q_{init}\}$; $E \leftarrow \emptyset$;
for $i = 1, \ldots, n$ **do**
  $q_{rand} \leftarrow$ SampleFree;
  $q_{nearest} \leftarrow$ Nearest$(G = (V, E), q_{rand})$;
  $q_{new} \leftarrow$ Steer$(q_{nearest}, q_{rand})$;
  if CollisionFree$(q_{nearest}, q_{new})$ **then**
    $V \leftarrow V \cup \{x_{new}\}$; $E \leftarrow E \cup \{(x_{nearest}, x_{new})\}$;
return $G = (V, E)$;

Extend the tree by a small step, but often a direct control $u \in \mathcal{U}$ that will move robot to the position closest to q_{new} is selected (applied for dt)

Rapidly-exploring random trees: A new tool for path planning
S. M. LaValle,
Technical Report 98-11, Computer Science Dept., Iowa State University, 1998.

Properties of RRT Algorithms

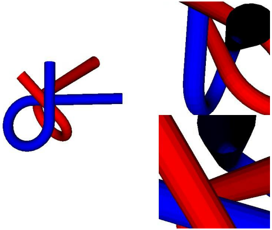
- The RRT algorithm rapidly explores the space.

  q_{new} will more likely be generated in large not yet covered parts.
- Allows considering kinodynamic/dynamic constraints (during the expansion).
- Can provide trajectory or a sequence of direct control commands for robot controllers.
- A collision detection test is usually used as a "black-box".

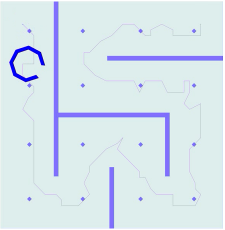
  E.g., RAPID, Bullet libraries.
- Similarly to PRM, RRT algorithms have poor performance in narrow passage problems.
- RRT algorithms provides feasible paths.

  It can be relatively far from optimal solution, e.g., according to the length of the path.
- Many variants of RRT have been proposed.

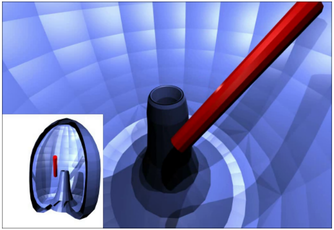
RRT – Examples 1/2



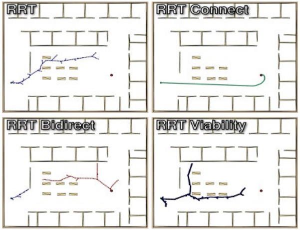
Alpha puzzle benchmark



Apply rotations to reach the goal



Bugtrap benchmark

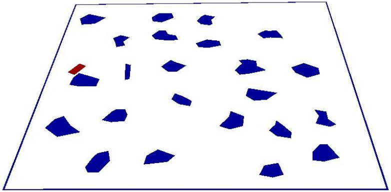


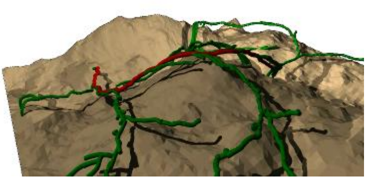
Variants of RRT algorithms

Courtesy of V. Vonásek and P. Vaněk

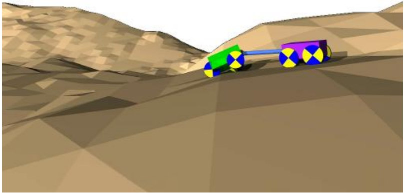
RRT – Examples 2/2

- Planning for a car-like robot





Planning on a 3D surface



Planning with dynamics (friction forces)

Courtesy of V. Vonásek and P. Vaněk.

Car-Like Robot

- Configuration

$$\vec{x} = \begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$$

position and orientation.

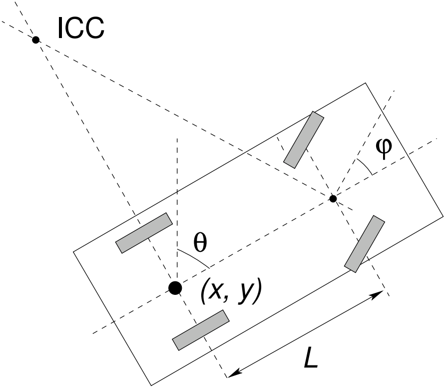
- Controls

$$\vec{u} = \begin{pmatrix} v \\ \varphi \end{pmatrix}$$

forward velocity, steering angle.

- System equation

$$\begin{aligned} \dot{x} &= v \cos \phi \\ \dot{y} &= v \sin \phi \\ \dot{\varphi} &= \frac{v}{L} \tan \varphi \end{aligned}$$



Kinematic constraints $\dim(\vec{u}) < \dim(\vec{x})$.

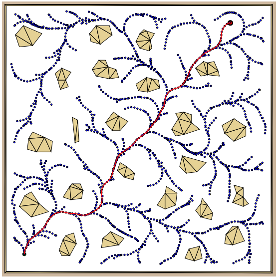
Differential constraints on possible $\dot{q}$:

$\dot{x} \sin(\phi) - \dot{y} \cos(\phi) = 0.$

Control-Based Sampling

- Select a configuration q from the tree T of the current configurations.
- Pick a control input $\vec{u} = (v, \varphi)$ and the integrate system (motion) equation over a short period Δt:

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \varphi \end{pmatrix} = \int_{t}^{t+\Delta t} \begin{pmatrix} v \cos \phi \\ v \sin \phi \\ \frac{v}{L} \tan \varphi \end{pmatrix} dt.$$



- If the motion is collision-free, add the endpoint to the tree.

E.g., considering k configurations for $k\delta t = dt$.

Part II

Part 2 – Optimal Sampling-based Motion Planning Methods

Outline

- Optimal Motion Planners
- Rapidly-exploring Random Graph (RRG)
- Informed Sampling-based Methods

Sampling-Based Motion Planning

- PRM and RRT are theoretically probabilistic complete.
- They provide a feasible solution without quality guarantee.

  Despite that, they are successfully used in many practical applications.
- In 2011, a systematical study of the asymptotic behavior of randomized sampling-based planners has been published.

  It shows, that in some cases, they converge to a non-optimal value with a probability 1.
- Based on the study, new algorithms have been proposed: RRG and optimal RRT (RRT*).

Karaman, S., Frazzoli, E. (2011):Sampling-based algorithms for optimal motion planning. IJRR.

http://sertac.scripts.mit.edu/rrtstar

RRT and Quality of Solution 1/2

- Let Y_i^{RRT} be the cost of the best path in the RRT at the end of the iteration i.
- Y_i^{RRT} converges to a random variable

$$\lim_{i \to \infty} Y_i^{RRT} = Y_\infty^{RRT}.$$

- The random variable Y_∞^{RRT} is sampled from a distribution with zero mass at the optimum, and

$$Pr[Y_\infty^{RRT} > c^*] = 1.$$

Karaman and Frazzoli, 2011

- The best path in the RRT converges to a sub-optimal solution almost surely.

RRT and Quality of Solution 2/2

- RRT does not satisfy a necessary condition for the asymptotic optimality.
  - For $0 < R < \inf_{q \in \mathcal{Q}_{goal}} ||q - q_{init}||$, the event $\{\lim_{n\to\infty} Y_n^{RTT} = c^*\}$ occurs only if the k-th branch of the RRT contains vertices outside the R-ball centered at q_{init} for infinitely many k.

    See Appendix B in Karaman and Frazzoli, 2011
- It is required the root node will have infinitely many subtrees that extend at least a distance ϵ away from q_{init}.

  The sub-optimality is caused by disallowing new better paths to be discovered.

Outline

- Optimal Motion Planners
- Rapidly-exploring Random Graph (RRG)
- Informed Sampling-based Methods

Rapidly-exploring Random Graph (RRG)

Algorithm 4: Rapidly-exploring Random Graph (RRG)

Input: q_{init}, number of samples n
Output: $G = (V, E)$

```
V ← ∅; E ← ∅;
for i = 0, ..., n do
    q_rand ← SampleFree;
    q_nearest ← Nearest(G = (V, E), q_rand);
    q_new ← Steer(q_nearest, q_rand);
    if CollisionFree(q_nearest, q_new) then
        Q_near ← Near(G =
        (V, E), q_new, min{γ_RRG(log(card(V))/ card(V))^(1/d), η});
        V ← V ∪ {q_new};
        E ← E ∪ {(q_nearest, q_new), (q_new, q_nearest)};
        foreach q_near ∈ Q_near do
            if CollisionFree(q_near, q_new) then
                E ← E ∪ {(q_rand, u), (u, q_rand)};
return G = (V, E);
```

$$\mathcal{Q}_{near} \leftarrow \text{Near}\left(G = (V,E), q_{new}, \min\{\gamma_{RRG}(\log(\text{card}(V))/\text{card}(V))^{1/d}, \eta\}\right)$$

Proposed by Karaman and Frazzoli (2011). Theoretical results are related to properties of ***Random Geometric Graphs (RGG)*** *introduced by Gilbert (1961) and further studied by Penrose (1999).*

RRG Expansions

- At each iteration, RRG tries to connect new sample to all vertices in the r_n ball centered at it.
- The ball of radius

$$r(\mathrm{card}(V)) = \min\left\{\gamma_{RRG}\left(\frac{\log(\mathrm{card}(V))}{\mathrm{card}(V)}\right)^{1/d}, \eta\right\},$$

where

  - η is the constant of the local steering function;
  - $\gamma_{RRG} > \gamma^*_{RRG} = 2(1+1/d)^{1/d}(\mu(\mathcal{C}_{free})/\xi_d)^{1/d}$;
    - d – dimension of the space;
    - $\mu(\mathcal{C}_{free})$ – Lebesgue measure of the obstacle–free space;
    - ξ_d – volume of the unit ball in d-dimensional Euclidean space.
- The connection radius decreases with n.
- The rate of decay $\approx$ the average number of connections attempted is proportional to $\log(n)$.

RRG Properties

- Probabilistically complete;
- Asymptotically optimal;
- Complexity is $O(\log n)$.

(per one sample)

- Computational efficiency and optimality:
  - It attempts a connection to $\Theta(\log n)$ nodes at each iteration;

    in average
    - Reduce volume of the "connection" ball as $\log(n)/n$;
    - Increase the number of connections as $\log(n)$.

Other Variants of the Optimal Motion Planning

- **PRM*** follows the standard PRM algorithm where connections are attempted between roadmap vertices that are the within connection radius r as the function of n

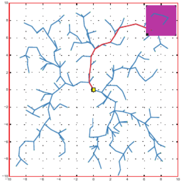
$$r(n) = \gamma_{PRM}(\log(n)/n)^{1/d}.$$

- **RRT*** is a modification of the RRG, where cycles are avoided.

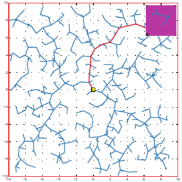
  It is a tree version of the RRG.

  - A tree roadmap allows to consider non-holonomic dynamics and kinodynamic constraints.
  - It is basically the RRG with "rerouting" the tree when a better path is discovered.

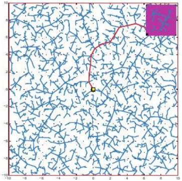
Example of Solution 1/3



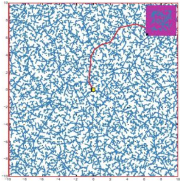
RRT, n=250



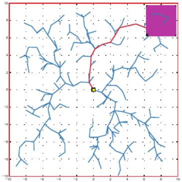
RRT, n=500



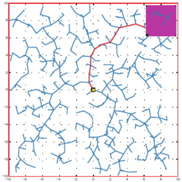
RRT, n=2500



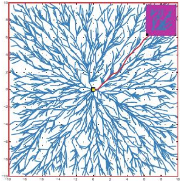
RRT, n=10000



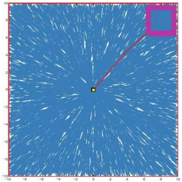
RRT*, n=250



RRT*, n=500



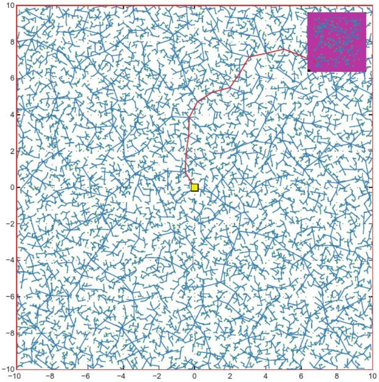
RRT*, n=2500



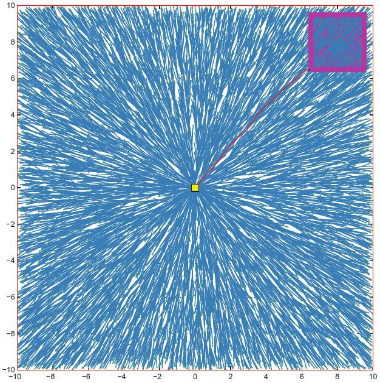
RRT*, n=10000

Karaman & Frazzoli, 2011

Example of Solution 2/3

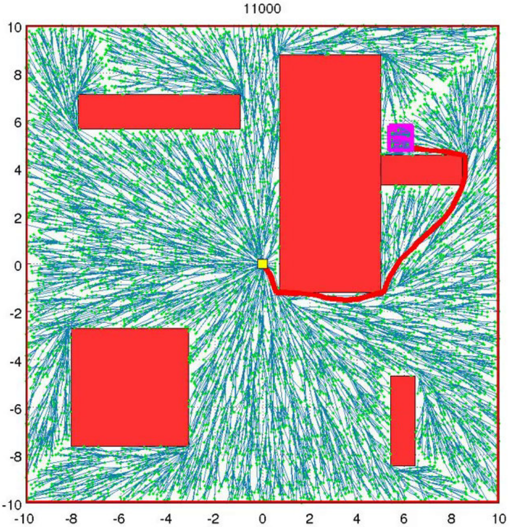


RRT, n=20000



RRT*, n=20000

Example of Solution 3/3



https://www.youtube.com/watch?v=YKiQTJpPFkA

Overview of Randomized Sampling-based Algorithms

Algorithm	Probabilistic Completeness	Asymptotic Optimality
PRM	✔	✘
sPRM	✔	✔
k-nearest sPRM	✘	✘
RRT	✔	✘
RRG	✔	✔
PRM*	✔	✔
RRT*	✔	✔

Notice, k-nearest variants of RRG, PRM, and RRT* are complete and optimal as well.*

Outline

- Optimal Motion Planners
- Rapidly-exploring Random Graph (RRG)
- Informed Sampling-based Methods

Improved Sampling-based Motion Planners

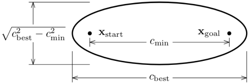
- Although asymptotically optimal sampling-based motion planners such as RRT* or RRG may provide high-quality or even optimal solutions of the complex problem, their performance in simple, e.g., 2D scenarios, is relatively poor

  In a comparison to the ordinary approaches (e.g., visibility graph)
- They are computationally demanding and performance can be improved similarly as for the RRT, e.g.,
  - Goal biasing, supporting sampling in narrow passages, multi-tree growing (Bidirectional RRT)
- The general idea of improvements is based on **informing** the sampling process
- Many modifications of the algorithms exists, selected representative modifications are
  - **Informed RRT***
  - Batch Informed Trees (**BIT***)
  - Regionally Accelerated BIT* (**RABIT***)

Informed RRT*

- Focused RRT* search to increase the convergence rate
- Use Euclidean distance as an admissible heuristic
- Ellipsoidal informed subset – the current best solution c_{best}

$$X_{\hat{f}} = \{\mathbf{x} \in X \,|\, ||\mathbf{x}_{start} - \mathbf{x}||_2 + ||\mathbf{x} - \mathbf{x}_{goal}||_2 \leq c_{best}\}$$



- Directly Based on the RRT*
- Having a feasible solution
- Sampling inside the ellipse

Algorithm 2: $\texttt{Sample}(\mathbf{x}_{start}, \mathbf{x}_{goal}, c_{max})$

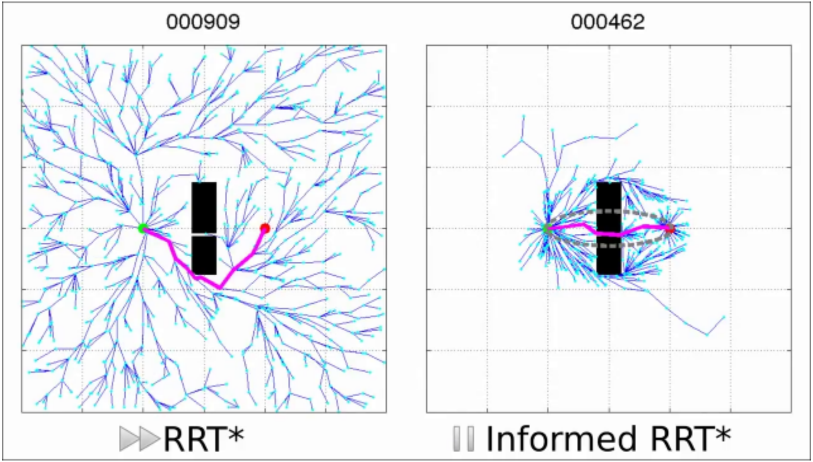
if $c_{max} < \infty$ **then**
  $c_{min} \leftarrow ||\mathbf{x}_{goal} - \mathbf{x}_{start}||_2$;
  $\mathbf{x}_{centre} \leftarrow (\mathbf{x}_{start} + \mathbf{x}_{goal})/2$;
  $\mathbf{C} \leftarrow \texttt{RotationToWorldFrame}(\mathbf{x}_{start}, \mathbf{x}_{goal})$;
  $r_1 \leftarrow c_{max}/2$;
  $\{r_i\}_{i=2,\ldots,n} \leftarrow \left(\sqrt{c_{max}^2 - c_{min}^2}\right)/2$;
  $\mathbf{L} \leftarrow \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$;
  $\mathbf{x}_{ball} \leftarrow \texttt{SampleUnitNBall}$;
  $\mathbf{x}_{rand} \leftarrow (\mathbf{CL}\mathbf{x}_{ball} + \mathbf{x}_{centre}) \cap X$;
else
  $\mathbf{x}_{rand} \sim \mathcal{U}(X)$;
return $\mathbf{x}_{rand}$;

Algorithm 1: Informed RRT*$(\mathbf{x}_{start}, \mathbf{x}_{goal})$

$V \leftarrow \{\mathbf{x}_{start}\}$;
$E \leftarrow \emptyset$;
$X_{soln} \leftarrow \emptyset$;
$\mathcal{T} = (V, E)$;
for iteration $= 1 \ldots N$ **do**
  $c_{best} \leftarrow \min_{\mathbf{x}_{soln} \in X_{soln}} \{\texttt{Cost}(\mathbf{x}_{soln})\}$;
  $\mathbf{x}_{rand} \leftarrow \texttt{Sample}(\mathbf{x}_{start}, \mathbf{x}_{goal}, c_{best})$;
  $\mathbf{x}_{nearest} \leftarrow \texttt{Nearest}(\mathcal{T}, \mathbf{x}_{rand})$;
  $\mathbf{x}_{new} \leftarrow \texttt{Steer}(\mathbf{x}_{nearest}, \mathbf{x}_{rand})$;
  if $\texttt{CollisionFree}(\mathbf{x}_{nearest}, \mathbf{x}_{new})$ **then**
    $V \leftarrow \cup \{\mathbf{x}_{new}\}$;
    $X_{near} \leftarrow \texttt{Near}(\mathcal{T}, \mathbf{x}_{new}, r_{RRT^*})$;
    $\mathbf{x}_{min} \leftarrow \mathbf{x}_{nearest}$;
    $c_{min} \leftarrow \texttt{Cost}(\mathbf{x}_{min}) + c \cdot \texttt{Line}(\mathbf{x}_{nearest}, \mathbf{x}_{new})$;
    for $\forall \mathbf{x}_{near} \in X_{near}$ **do**
      $c_{new} \leftarrow \texttt{Cost}(\mathbf{x}_{near}) + c \cdot \texttt{Line}(\mathbf{x}_{near}, \mathbf{x}_{new})$;
      if $c_{new} < c_{min}$ **then**
        if $\texttt{CollisionFree}(\mathbf{x}_{near}, \mathbf{x}_{new})$ **then**
          $\mathbf{x}_{min} \leftarrow \mathbf{x}_{near}$;
          $c_{min} \leftarrow c_{new}$;
    $E \leftarrow E \cup \{(\mathbf{x}_{min}, \mathbf{x}_{new})\}$;
    for $\forall \mathbf{x}_{near} \in X_{near}$ **do**
      $c_{near} \leftarrow \texttt{Cost}(\mathbf{x}_{near})$;
      $c_{new} \leftarrow \texttt{Cost}(\mathbf{x}_{new}) + c \cdot \texttt{Line}(\mathbf{x}_{new}, \mathbf{x}_{near})$;
      if $c_{new} < c_{near}$ **then**
        if $\texttt{CollisionFree}(\mathbf{x}_{new}, \mathbf{x}_{near})$ **then**
          $\mathbf{x}_{parent} \leftarrow \texttt{Parent}(\mathbf{x}_{near})$;
          $E \leftarrow E \setminus \{(\mathbf{x}_{parent}, \mathbf{x}_{near})\}$;
          $E \leftarrow E \cup \{(\mathbf{x}_{new}, \mathbf{x}_{near})\}$;
    if $\texttt{InGoalRegion}(\mathbf{x}_{new})$ **then**
      $X_{soln} \leftarrow X_{soln} \cup \{\mathbf{x}_{new}\}$;
return $\mathcal{T}$;

Gammell, J. B., Srinivasa, S. S., Barfoot, T. D. (2014): **Informed RRT***: Optimal Sampling-based Path Planning Focused via Direct Sampling of an Admissible Ellipsoidal Heuristic. IROS.

Informed RRT* – Demo



https://www.youtube.com/watch?v=d7dX5MvDYTc

Gammell, J. B., Srinivasa, S. S., Barfoot, T. D. (2014): **Informed RRT***: Optimal Sampling-based Path Planning Focused via Direct Sampling of an Admissible Ellipsoidal Heuristic. IROS.

Batch Informed Trees (BIT*)

- Combining RGG (Random Geometric Graph) with the heuristic in incremental graph search technique, e.g., Lifelong Planning A* (LPA*)
  - The properties of the RGG are used in the RRG and RRT*
- Batches of samples – a new batch starts with denser implicit RGG
- The search tree is updated using LPA* like incremental search to reuse existing information

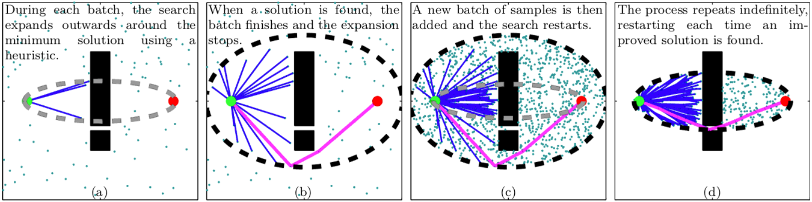
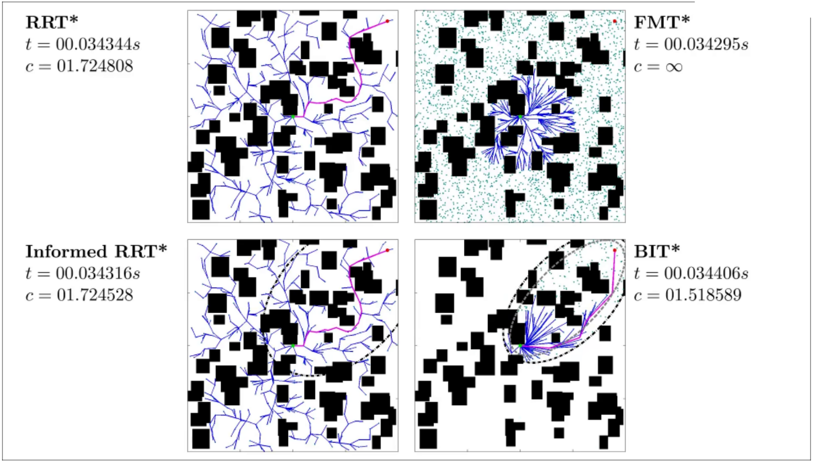


Fig. 3. An illustration of the informed search procedure used by BIT*. The start and goal states are shown as green and red, respectively. The current solution is highlighted in magenta. The subproblem that contains any better solutions is shown as a black dashed line, while the progress of the current batch is shown as a grey dashed line. Fig. (a) shows the growing search of the first batch of samples, and (b) shows the first search ending when a solution is found. After pruning and adding a second batch of samples, Fig. (c) shows the search restarting on a denser graph while (d) shows the second search ending when an improved solution is found. An animated illustration is available in the attached video.

Gammell, J. B., Srinivasa, S. S., Barfoot, T. D. (2015): Batch Informed Trees (BIT*): Sampling-based optimal planning via the heuristically guided search of implicit random geometric graphs. ICRA.

Batch Informed Trees (BIT*) – Demo



https://www.youtube.com/watch?v=TQIoCC48gp4

Gammell, J. B., Srinivasa, S. S., Barfoot, T. D. (2015): **Batch Informed Trees (BIT*)**: Sampling-based optimal planning via the heuristically guided search of implicit random geometric graphs. ICRA.

Regionally Accelerated BIT* (RABIT*)

- Use local optimizer with the BIT* to improve the convergence speed
- Local search Covariant Hamiltonian Optimization for Motion Planning (CHOMP) is utilized to connect edges in the search graphs using local information about the obstacles

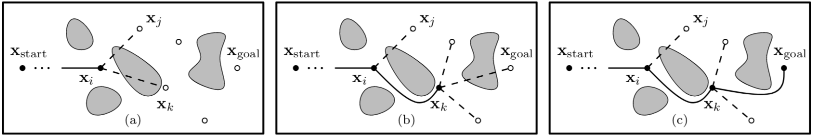
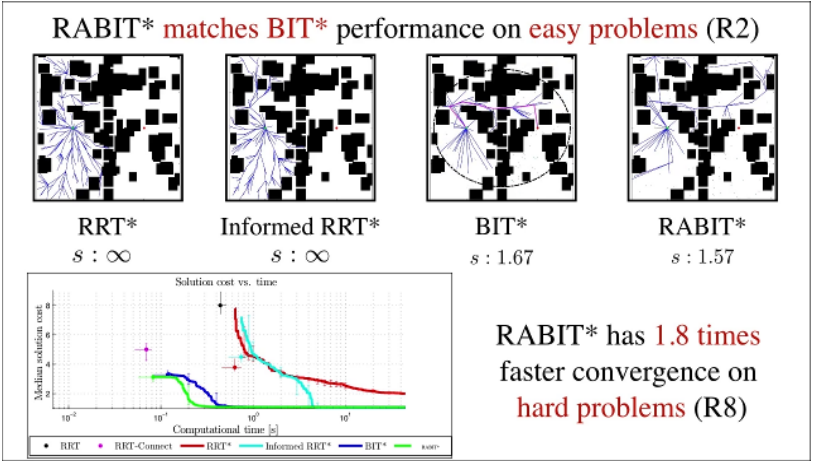


Fig. 2. An illustration of how the RABIT* algorithm uses a local optimizer to exploit obstacle information and improve a global search. The global search is performed, as in BIT*, by incrementally processing an edge queue (dashed lines) into a tree (a). Using heuristics, the potential edge from $\mathbf{x}_i$ to $\mathbf{x}_k$ is processed first as it could provide a better solution than an edge from $\mathbf{x}_i$ to $\mathbf{x}_j$. The initial straight-line edge is given to a local optimizer which uses information about obstacles to find a local optima between the specified states (b). If this edge is collision free, it is added to the tree and its potential outgoing edges are added to the queue. The next-best edge in the queue is then processed in the same fashion, using the local optimizer to once again propose a better edge than a straight-line (c).

Choudhury, S., Gammell, J. D., Barfoot, T. D., Srinivasa, S. S., Scherer, S. (2016): Regionally Accelerated Batch Informed Trees (RABIT*): A Framework to Integrate Local Information into Optimal Path Planning. ICRA.

Regionally Accelerated BIT* (RABIT*) – Demo



https://www.youtube.com/watch?v=mgq-DW36jSo

Choudhury, S., Gammell, J. D., Barfoot, T. D., Srinivasa, S. S., Scherer, S. (2016): Regionally Accelerated Batch Informed Trees (RABIT*): A Framework to Integrate Local Information into Optimal Path Planning. ICRA.

Overview of Improved Algorithm

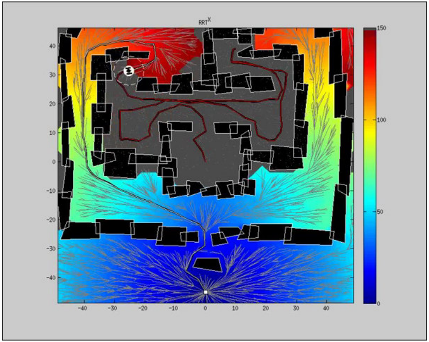
- Optimal path/motion planning is an active research field

Approaches	Constraints	Planning Mode	Kinematic Model	Sampling Strategy	Metric
1. RRT* [7]	Holonomic	Offline	Point	Uniform	Euclidean
2. Anytime RRT* [4]	Non-holonomic	Online	Dubin Car	Uniform	Euclidean + Velocity
3. B-RRT* [58]	Holonomic	Offline	Rigid Body	Local bias	Goal biased
4. RRT*FN [33]	Holonomic	Offline	Robotic Arm	Uniform	Cumulative Euclidean
5. RRT*-Smart [35]	Holonomic	Offline	Point	Intelligent	Euclidean
6. Optimal B-RRT* [36]	Holonomic	Offline	Point	Uniform	Euclidean
7. RRT# [50]	Holonomic	Offline	Point	Uniform	Euclidean
8. Adapted RRT* [64], [49]	Non-holonomic	Offline	Car-like and UAV	Uniform	A* Heuristic
9. SRRT* [44]	Non-holonomic	Offline	UAV	Uniform	Geometric + dynamic constraint
10. Informed RRT* [34]	Holonomic	Offline	Point	Direct Sampling	Euclidean
11. IB-RRT* [37]	Holonomic	Offline	Point	Intelligent	Greedy + Euclidean
12. DT-RRT [39]	Non-holonomic	Offline	Car-like	Hybrid	Angular + Euclidean
13. RRT*i [3]	Non-holonomic	Online	UAV	Local Sampling	A* Heuristic
14. RTR+CS* [43]	Non-holonomic	Offline	Car-like	Uniform + Local Planning	Angular + Euclidean
15. Mitsubishi RRT* [2]	Non-holonomic	Online	Autonomous Car	Two-stage sampling	Weighted Euclidean
16. CARRT* [65]	Non-holonomic	Online	Humanoid	Uniform	MW Energy Cost
17. PRRT* [48]	Non-holonomic	Offline	P3-DX	Uniform	Euclidean

Noreen, I., Khan, A., Habib, Z. (2016): **Optimal path planning using RRT* based approaches: a survey and future directions**. IJACSA.

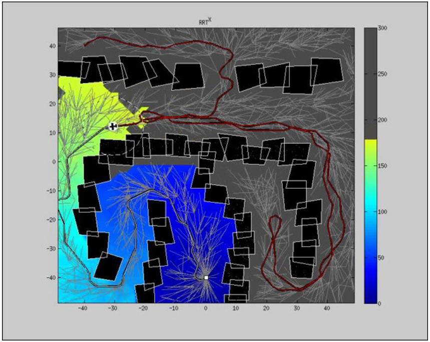
Motion Planning for Dynamic Environments – RRT$^{\text{X}}$

- Refinement and repair of the search graph during the navigation (quick rewiring of the shortest path)



RRT$^{\text{X}}$ – Robot in 2D

https://www.youtube.com/watch?v=S9pguCPUo3M



RRT$^{\text{X}}$ – Robot in 2D

https://www.youtube.com/watch?v=KxFivNgTV4o

Otte, M., & Frazzoli, E. (2016). RRT$^{\text{X}}$: Asymptotically optimal single-query sampling-based motion planning with quick replanning. The International Journal of Robotics Research, 35(7), 797-–822.

Part III

Part 3 – Multi-goal Motion Planning (MGMP)

Outline

- Multi-Goal Motion Planning
- Physical Orienteering Problem (POP)

Multi-Goal Motion Planning

- In the previous cases, we consider existing roadmap or relatively "simple" collision free (shortest) paths in the polygonal domain
- However, determination of the collision-free path in high dimensional configuration space ($\mathcal{C}$-space) can be a challenging problem itself
- Therefore, we can generalize the MTP to multi-goal **motion** planning (MGMP) considering motion planners using the notion of $\mathcal{C}$-space for avoiding collisions.

- An example of MGMP can be

Plan a cost efficient trajectory for hexapod walking robot to visit a set of target locations.

Problem Statement – MGMP Problem

- The working environment $\mathcal{W} \subset \mathbb{R}^3$ is represented as a set of obstacles $\mathcal{O} \subset \mathcal{W}$ and the robot configuration space $\mathcal{C}$ describes all possible configurations of the robot in $\mathcal{W}$
- For $q \in \mathcal{C}$, the robot body $\mathcal{A}(q)$ at q is collision free if $\mathcal{A}(q) \cap \mathcal{O} = \emptyset$ and all collision free configurations are denoted as $\mathcal{C}_{free}$
- Set of n goal locations is $\mathcal{G} = (g_1, \ldots, g_n)$, $g_i \in \mathcal{C}_{free}$
- Collision free path from q_{start} to q_{goal} is $\kappa : [0, 1] \to \mathcal{C}_{free}$ with $\kappa(0) = q_{start}$ and $d(\kappa(1), q_{end}) < \epsilon$, for an admissible distance ϵ
- Multi–goal path τ is **admissible** if $\tau : [0, 1] \to \mathcal{C}_{free}$, $\tau(0) = \tau(1)$ and there are n points such that $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, $d(\tau(t_i), v_i) < \epsilon$, and $\bigcup_{1<i\leq n} v_i = \mathcal{G}$

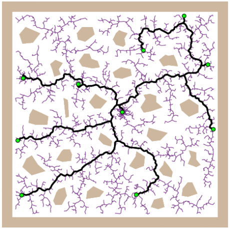
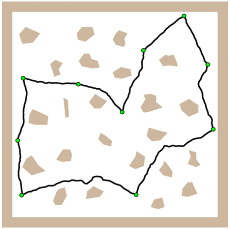
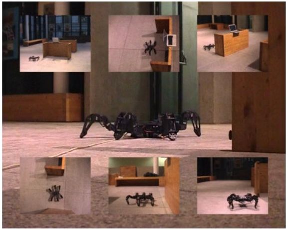
- **The problem is to find the path τ^* for a cost function c such that $c(\tau^*) = \min\{c(\tau) \mid \tau$ *is admissible multi–goal path*$\}$**

MGMP – Existing Approches

- Determining all paths connecting any two locations $g_i, g_j \in \mathcal{G}$ is usually very computationally demanding; and several approaches exist
  - Considering Euclidean distance as an approximation in the solution of the TSP as the Minimum Spanning Tree (MST) – Edges in the MST are iteratively refined using optimal motion planner until all edges represent a feasible solution

    Saha, M., Roughgarden, T., Latombe, J.-C., Sánchez-Ante, G. (2006): **Planning Tours of Robotic Arms among Partitioned Goals.** IJRR.
  - **Synergistic Combination of Layers of Planning** (**SyCLoP**) – A combination of route and trajectory planning

    Plaku, E., Kavraki, L.E., Vardi, M.Y. (2010): Motion Planning With Dynamics by a Synergistic Combination of Layers of Planning. T-RO.
  - Steering RRG roadmap expansion by unsupervised learning for the TSP
  - Steering PRM* expansion using VNS-based routing planning in the **Physical Orienteering Problem** (POP)

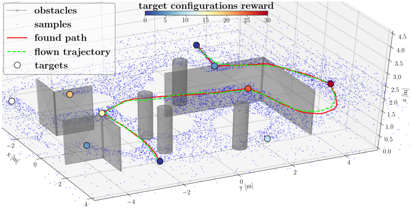




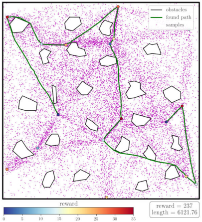
Outline

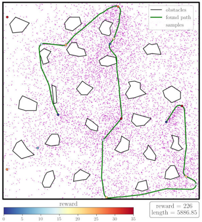
- Multi-Goal Motion Planning
- Physical Orienteering Problem (POP)

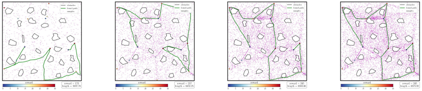
Multi-Goal Trajectory Planning with Limited Travel Budget Physical Orienteering Problem (POP)

- **Orienteering Problem** (**OP**) in an environment with obstacles and motion constraints of the data collecting vehicle.
- A combination of motion planning and routing problem with profits.
- **VNS-PRM*** – **VNS**-based routing and motion planning is addressed by **PRM***
  - An initial low-dense roadmap is continuously expanded during the VNS-based POP optimization to shorten paths of promising solutions.
  - Shorten trajectories allow visiting more locations within T_{max}.



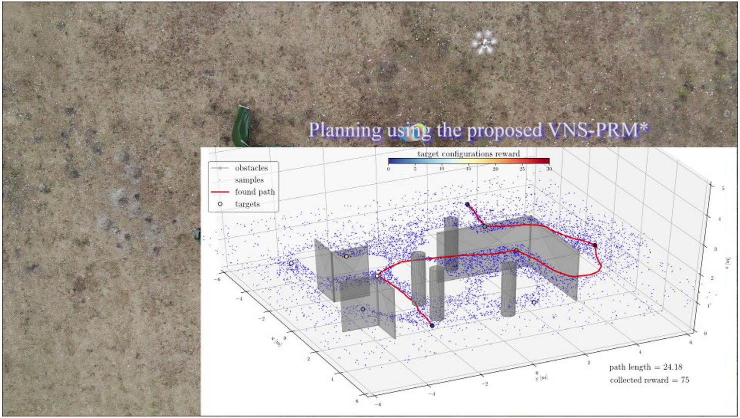






- Pěnička, Faigl and Saska: *Physical Orienteering Problem for Unmanned Aerial Vehicle Data Collection Planning in Environments with Obstacles*. **IEEE Robotics and Automation Letters** 4(3):3005–3012, 2019.

Multi-Goal Trajectory Planning with Limited Travel Budget Physical Orienteering Problem (POP) – Real Experimental Verification



Summary of the Lecture

Topics Discussed – Randomized Sampling-based Methods

- Single and multi-query approaches – Probabilistic Roadmap Method (**PRM**); Rapidly Exploring Random Tree (**RRT**)
- Optimal sampling-based planning – Rapidly-exploring Random Graph (**RRG**)
- Properties of the sampling-based motion planning algorithms
  - **Path**, **collision-free path**, **feasible path**
  - **Feasible path planning** and **optimal path planning**
  - **Probabilistic completeness**, **strong δ-clearance**, **robustly feasible** path planning problem
  - **Asymptotic optimality**, **homotopy**, **weak δ-clearance**, **robust optimal solution**
  - **PRM**, **RRT**, **RRG**, **PRM***, **RRT***
- Improved randomized sampling-based methods
  - Informed sampling – Informed RRT*; Improving by batches of samples and reusing previous searches using Lifelong Planning A* (LPA*)
  - Improving local search strategy to improve convergence speed
  - Planning in dynamic environments – RRT^{X}
- Multi-goal motion planning (**MGMP**) problems are further variants of the robotic TSP

- Next: Game Theory in Robotics