Mathematical identities

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In the following, the rectangular system is characterized by coordinates (x, y, z) and unit vectors $\begin{bmatrix} \boldsymbol{x}_0 & \boldsymbol{y}_0 & \boldsymbol{z}_0 \end{bmatrix}$. The cylindrical system is characterized by coordinates (ρ, φ, z) and unit vectors $\begin{bmatrix} \boldsymbol{\rho}_0 & \boldsymbol{\varphi}_0 & \boldsymbol{z}_0 \end{bmatrix}$. The spherical system is characterized by coordinates (r, θ, φ) and unit vectors $\begin{bmatrix} \boldsymbol{r}_0 & \boldsymbol{\theta}_0 & \boldsymbol{\varphi}_0 \end{bmatrix}$.

0.1 Point Transformations

Rectangular - Cylindrical

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned} \tag{1}$$

Cylindrical - Rectangular

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \tan\varphi &= \frac{y}{x} \\ z &= z \end{aligned} \tag{2}$$

Rectangular - Spherical

$$\begin{aligned} x &= r\cos\varphi\sin\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\theta \end{aligned} \tag{3}$$

Spherical - Rectangular

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \tan\theta &= \frac{\sqrt{x^2 + y^2}}{z} \\ \tan\varphi &= \frac{y}{x} \end{aligned} \tag{4}$$

Cylindrical - Spherical

$$\begin{aligned} \rho &= r\sin\theta \\ z &= r\cos\theta \\ \varphi &= \varphi \end{aligned} \tag{5}$$

Spherical - Cylindrical

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2} \\ \tan\theta &= \frac{\rho}{z} \\ \varphi &= \varphi \end{aligned} \tag{6}$$

0.2 Vector Transformations

Rectangular - Cylindrical

$$\begin{aligned} \boldsymbol{x}_0 &= \boldsymbol{\rho}_0\cos\varphi - \boldsymbol{\varphi}_0\sin\varphi \\ \boldsymbol{y}_0 &= \boldsymbol{\rho}_0\sin\varphi + \boldsymbol{\varphi}_0\cos\varphi \\ \boldsymbol{z}_0 &= \boldsymbol{z}_0 \end{aligned} \tag{7}$$

Cylindrical - Rectangular

$$\begin{aligned} &\boldsymbol{\rho}_0 = \boldsymbol{x}_0 \cos\varphi + \boldsymbol{y}_0 \sin\varphi \\ &\boldsymbol{\varphi}_0 = -\boldsymbol{x}_0 \sin\varphi + \boldsymbol{y}_0 \cos\varphi \\ &\boldsymbol{z}_0 = \boldsymbol{z}_0 \end{aligned} \tag{8}$$

Rectangular - Spherical

$$\begin{aligned} &\boldsymbol{x}_0 = \boldsymbol{r}_0 \sin\theta\cos\varphi + \boldsymbol{\theta}_0 \cos\theta\cos\varphi - \boldsymbol{\varphi}_0 \sin\varphi \\ &\boldsymbol{y}_0 = \boldsymbol{r}_0 \sin\theta\sin\varphi + \boldsymbol{\theta}_0 \cos\theta\sin\varphi + \boldsymbol{\varphi}_0 \cos\varphi \\ &\boldsymbol{z}_0 = \boldsymbol{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta \end{aligned} \tag{9}$$

Spherical - Rectangular

$$\begin{aligned} &\boldsymbol{r}_0 = \boldsymbol{x}_0 \sin\theta\cos\varphi + \boldsymbol{y}_0 \sin\theta\sin\varphi + \boldsymbol{z}_0 \cos\theta \\ &\boldsymbol{\theta}_0 = \boldsymbol{x}_0 \cos\theta\cos\varphi + \boldsymbol{y}_0 \cos\theta\sin\varphi - \boldsymbol{z}_0 \sin\theta \\ &\boldsymbol{\varphi}_0 = -\boldsymbol{x}_0 \sin\varphi + \boldsymbol{y}_0 \cos\varphi \end{aligned} \tag{10}$$

Cylindrical - Spherical

$$\begin{aligned} &\boldsymbol{\rho}_0 = \boldsymbol{r}_0 \sin\theta + \boldsymbol{\theta}_0 \cos\theta \\ &\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0 \\ &\boldsymbol{z}_0 = \boldsymbol{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta \end{aligned} \tag{11}$$

Spherical - Cylindrical

$$\begin{aligned} &\boldsymbol{r}_0 = \boldsymbol{\rho}_0 \sin\theta + \boldsymbol{z}_0 \cos\theta \\ &\boldsymbol{\theta}_0 = \boldsymbol{\rho}_0 \cos\theta - \boldsymbol{z}_0 \sin\theta \\ &\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0 \end{aligned} \tag{12}$$

0.3 Differential Operators

Rectangular coordinate system

$$\begin{aligned} &\nabla f = \boldsymbol{x}_0 \frac{\partial f}{\partial x} + \boldsymbol{y}_0 \frac{\partial f}{\partial y} + \boldsymbol{z}_0 \frac{\partial f}{\partial z} \\ &\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &\nabla \times \boldsymbol{F} = \boldsymbol{x}_0 \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \boldsymbol{y}_0 \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \boldsymbol{z}_0 \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &\nabla^2 \boldsymbol{F} = \nabla\left(\nabla \cdot \boldsymbol{F}\right) - \nabla \times \nabla \times \boldsymbol{F} = \boldsymbol{x}_0 \nabla^2 F_x + \boldsymbol{y}_0 \nabla^2 F_y + \boldsymbol{z}_0 \nabla^2 F_z \end{aligned} \tag{13}$$

Cylindrical coordinate system

$$\nabla f = \boldsymbol{\rho}_0 \frac{\partial f}{\partial \rho} + \boldsymbol{\varphi}_0 \frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \boldsymbol{z}_0 \frac{\partial f}{\partial z}$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{\rho}\frac{\partial \left(\rho F_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \boldsymbol{F} = \boldsymbol{\rho}_0 \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right) + \boldsymbol{\varphi}_0 \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right) + \boldsymbol{z}_0 \left(\frac{1}{\rho}\frac{\partial \left(\rho F_\varphi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \varphi}\right)$$

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{14}$$

$$\nabla^2 \boldsymbol{F} = \nabla\left(\nabla \cdot \boldsymbol{F}\right) - \nabla \times \nabla \times \boldsymbol{F} =$$

$$\boldsymbol{\rho}_0 \left(\frac{\partial^2 F_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_\rho}{\partial \rho} - \frac{F_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 F_\rho}{\partial \varphi^2} - \frac{2}{\rho^2}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial^2 F_\rho}{\partial z^2}\right) +$$

$$\boldsymbol{\varphi}_0 \left(\frac{\partial^2 F_\varphi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \rho} - \frac{F_\varphi}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 F_\varphi}{\partial \varphi^2} + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial \varphi} + \frac{\partial^2 F_\varphi}{\partial z^2}\right) +$$

$$\boldsymbol{z}_0 \left(\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F_z}{\partial \varphi^2} + \frac{\partial^2 F_z}{\partial z^2}\right)$$

Spherical coordinate system

$$\nabla f = \boldsymbol{r}_0 \frac{\partial f}{\partial r} + \boldsymbol{\theta}_0 \frac{1}{r}\frac{\partial f}{\partial \theta} + \boldsymbol{\varphi}_0 \frac{1}{r \sin\theta}\frac{\partial f}{\partial \varphi}$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial \left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

$$\nabla \times \boldsymbol{F} = \frac{\boldsymbol{r}_0}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(F_\varphi \sin\theta\right) - \frac{\partial F_\theta}{\partial \varphi}\right) + \frac{\boldsymbol{\theta}_0}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \varphi} - \frac{\partial \left(r F_\varphi\right)}{\partial r}\right) +$$

$$\frac{\boldsymbol{\varphi}_0}{r}\left(\frac{\partial \left(r F_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)$$

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

$$\nabla^2 \boldsymbol{F} = \nabla\left(\nabla \cdot \boldsymbol{F}\right) - \nabla \times \nabla \times \boldsymbol{F} = \tag{15}$$

$$\boldsymbol{r}_0 \left(\begin{array}{l} \frac{\partial^2 F_r}{\partial r^2} + \frac{2}{\rho}\frac{\partial F_r}{\partial r} - \frac{2}{r^2}F_r + \frac{1}{r^2}\frac{\partial^2 F_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial F_r}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 F_r}{\partial \varphi^2} \\ -\frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta} - \frac{2\cot\theta}{r^2}F_\theta - \frac{2}{r^2\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} \end{array}\right) +$$

$$\boldsymbol{\theta}_0 \left(\begin{array}{l} \frac{\partial^2 F_\theta}{\partial r^2} + \frac{2}{\rho}\frac{\partial F_\theta}{\partial r} - \frac{1}{r^2 \sin^2\theta}F_\theta + \frac{1}{r^2}\frac{\partial^2 F_\theta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial F_\theta}{\partial \theta} \\ +\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 F_\theta}{\partial \varphi^2} + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial F_\varphi}{\partial \varphi} \end{array}\right) +$$

$$\boldsymbol{\varphi}_0 \left(\begin{array}{l} \frac{\partial^2 F_\varphi}{\partial r^2} + \frac{2}{\rho}\frac{\partial F_\varphi}{\partial r} - \frac{1}{r^2 \sin^2\theta}F_\varphi + \frac{1}{r^2}\frac{\partial^2 F_\varphi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial F_\varphi}{\partial \theta} \\ +\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 F_\varphi}{\partial \varphi^2} + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial \varphi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial F_\theta}{\partial \varphi} \end{array}\right)$$

0.4 Differential Identities

$$\begin{aligned}
&\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \\
&\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c} \\
&(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c}) \\
&\nabla \times (\nabla f) = 0 \\
&\nabla \cdot (\nabla \times \boldsymbol{F}) = 0 \\
&\nabla\,(f \cdot g) = g \nabla f + f \nabla g \\
&\nabla \cdot (\boldsymbol{F} f) = \boldsymbol{F} \cdot (\nabla f) + f\,(\nabla \cdot \boldsymbol{F}) \\
&\nabla \times (\boldsymbol{F} f) = (\nabla f) \times \boldsymbol{F} + f\,(\nabla \times \boldsymbol{F}) \\
&\nabla \left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2} \\
&\nabla \cdot \left(\frac{\boldsymbol{F}}{f}\right) = \frac{f\,(\nabla \cdot \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla f)}{f^2} \\
&\nabla \times \left(\frac{\boldsymbol{F}}{f}\right) = \frac{f\,(\nabla \times \boldsymbol{F}) - (\nabla f) \times \boldsymbol{F}}{f^2} \\
&\nabla\,(\boldsymbol{F} \cdot \boldsymbol{G}) = (\boldsymbol{F} \cdot \nabla)\, \boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\, \boldsymbol{F} + \boldsymbol{F} \times (\nabla \times \boldsymbol{G}) + \boldsymbol{G} \times (\nabla \times \boldsymbol{F}) \\
&\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}) \\
&\nabla \times (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{F}\,(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}\,(\nabla \cdot \boldsymbol{F}) + (\boldsymbol{G} \cdot \nabla)\, \boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\, \boldsymbol{G} \\
&\nabla\,(f \circ g) = (f' \circ g)\, \nabla g \\
&\nabla \cdot (\boldsymbol{F} \circ g) = \big(\boldsymbol{F}' \circ g\big) \cdot \nabla g \\
&\nabla \times (\boldsymbol{F} \circ g) = -\big(\boldsymbol{F}' \circ g\big) \times \nabla g
\end{aligned} \tag{16}$$

0.5 Integration Identities

$$\begin{aligned}
&\int\limits_V (\nabla \cdot \boldsymbol{F})\,\mathrm{d}V = \oint\limits_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} \\
&\int\limits_V (\nabla f)\,\mathrm{d}V = \oint\limits_S f\,\mathrm{d}\boldsymbol{S} \\
&\int\limits_V (\nabla \times \boldsymbol{F})\,\mathrm{d}V = -\oint\limits_S \boldsymbol{F} \times \mathrm{d}\boldsymbol{S} \\
&\int\limits_V \left[f\left(\nabla^2 g\right)\right]\mathrm{d}V = \oint\limits_S f\,(\nabla g)\cdot \mathrm{d}\boldsymbol{S} - \int\limits_V \left[(\nabla f)\cdot(\nabla g)\right]\mathrm{d}V \\
&\int\limits_V \left[\boldsymbol{F}\cdot(\nabla\times\nabla\times\boldsymbol{G})\right]\mathrm{d}V = -\oint\limits_S \left[\boldsymbol{F}\times(\nabla\times\boldsymbol{G})\right]\cdot\mathrm{d}\boldsymbol{S} + \int\limits_V \left[(\nabla\times\boldsymbol{F})\cdot(\nabla\times\boldsymbol{G})\right]\mathrm{d}V \\
&\int\limits_V \left[f\left(\nabla^2 g\right) - g\left(\nabla^2 f\right)\right]\mathrm{d}V = \oint\limits_S \left[f\,(\nabla g) - g\,(\nabla f)\right]\cdot\mathrm{d}\boldsymbol{S} \\
&\int\limits_V \left[\boldsymbol{G}\cdot(\nabla\times\nabla\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\nabla\times\nabla\times\boldsymbol{G})\right]\mathrm{d}V = \oint\limits_S \left[\boldsymbol{F}\times(\nabla\times\boldsymbol{G}) - \boldsymbol{G}\times(\nabla\times\boldsymbol{F})\right]\cdot\mathrm{d}\boldsymbol{S} \\
&\int\limits_S (\nabla\times\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S} = \oint\limits_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l} \\
&\int\limits_S (\nabla f)\times\mathrm{d}\boldsymbol{S} = -\oint\limits_l f\,\mathrm{d}\boldsymbol{l}
\end{aligned} \tag{17}$$

0.6 3D Fourier's Transformation

Definition

$$\begin{aligned}
\mathrm{FT}\left\{f(\boldsymbol{r})\right\} = \tilde{f}(\boldsymbol{k}) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\mathrm{FT}^{-1}\left\{\tilde{f}(\boldsymbol{k})\right\} = f(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \tilde{f}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z
\end{aligned} \tag{18}$$

Convolution and its Fourier's transformation

$$\begin{aligned}
f(\boldsymbol{r}) * g(\boldsymbol{r}) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(\boldsymbol{r}-\boldsymbol{r'})\,g(\boldsymbol{r'})\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
\mathrm{FT}\left\{f(\boldsymbol{r}) * g(\boldsymbol{r})\right\} &= \tilde{f}(\boldsymbol{k})\,\tilde{g}(\boldsymbol{k})
\end{aligned} \tag{19}$$

Differential operators and their Fourier's transformation

$$\begin{aligned}
\mathrm{FT}\left\{\nabla\cdot\boldsymbol{F}(\boldsymbol{r})\right\} &= \mathrm{j}\boldsymbol{k}\cdot\tilde{\boldsymbol{F}}(\boldsymbol{k}) \\
\mathrm{FT}\left\{\nabla\times\boldsymbol{F}(\boldsymbol{r})\right\} &= \mathrm{j}\boldsymbol{k}\times\tilde{\boldsymbol{F}}(\boldsymbol{k})
\end{aligned} \tag{20}$$

0.7 Trigonometric identities

$$\begin{aligned}
\sin^2(\alpha) &= \frac{1-\cos(2\alpha)}{2} \\
\cos^2(\alpha) &= \frac{1+\cos(2\alpha)}{2} \\
\\
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) \\
\\
\sin(\alpha)+\sin(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
\sin(\alpha)-\sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\
\cos(\alpha)+\cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
\cos(\alpha)-\cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned} \tag{21}$$

0.8 Identities involving separation vector

$$\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r'}, \quad R = |\boldsymbol{R}|, \quad \boldsymbol{R}_0 = \frac{\boldsymbol{R}}{R}$$

$$\begin{aligned}
&\nabla R = \boldsymbol{R}_0 \\
&\nabla R^n = nR^{n-1}\nabla R = nR^{n-1}\boldsymbol{R}_0 \\
&\nabla \cdot \boldsymbol{R} = 3 \\
&\nabla \times \boldsymbol{R} = 0 \\
\\
&\nabla \cdot \boldsymbol{R}_0 = \frac{2}{R} \\
&\nabla\left(\mathrm{e}^{-\mathrm{j}kR}\right) = -\mathrm{j}k\mathrm{e}^{-\mathrm{j}kR}\boldsymbol{R}_0 \\
\\
&\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\boldsymbol{R}) \\
&\left(\nabla^2+k^2\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{R} = -4\pi\delta(\boldsymbol{R})
\end{aligned} \tag{22}$$

0.9 Useful integrals

$$
\begin{aligned}
&\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \\
&\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin(x) \\
&\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \ln\left(x+\sqrt{x^2-1}\right) \\
&\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \\
&\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} \\
&\int x^n \ln\left(\frac{a}{x}\right)\mathrm{d}x = \frac{x^{n+1}}{(n+1)^2} + \frac{x^{n+1}}{n+1}\ln\left(\frac{a}{x}\right) \\
&\int x\mathrm{e}^{ax}\mathrm{d}x = \frac{ax-1}{a^2}\mathrm{e}^{ax} \\
&\int x^2\mathrm{e}^{ax}\mathrm{d}x = \frac{a^2x^2-2ax+2}{a^3}\mathrm{e}^{ax} \\
&\int \sin^3(x)\,\mathrm{d}x = -\cos(x) + \frac{\cos^3(x)}{3} \\
&\int \cos^3(x)\,\mathrm{d}x = \sin(x) - \frac{\sin^3(x)}{3} \\
&\int \frac{\mathrm{d}x}{\cos(x)} = \ln\left(\frac{1+\sin(x)}{\cos(x)}\right) \\
&\int \frac{\mathrm{d}x}{\sin(x)} = \ln\left(\frac{1-\cos(x)}{\sin(x)}\right) \\
&\int \frac{\cos(x)\,\mathrm{d}x}{(1-a^2\cos^2(x))^{3/2}} = \frac{\sin(x)}{(1-a^2)\sqrt{1-a^2\cos^2(x)}} \\
&\int \frac{\sin(x)\,\mathrm{d}x}{\left(1-a^2\sin^2(x)\right)^{3/2}} = \frac{-\cos(x)}{(1-a^2)\sqrt{1-a^2\sin^2(x)}}
\end{aligned}
\tag{23}
$$