Electromagnetic Field Theory 1
(fundamental relations and definitions)

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Fundamental Question of Classical Electrodynamics

A specified distribution of elementary charges is in a state of arbitrary (but known) motion. At certain time we pick one of them and ask what is the force acting on it.

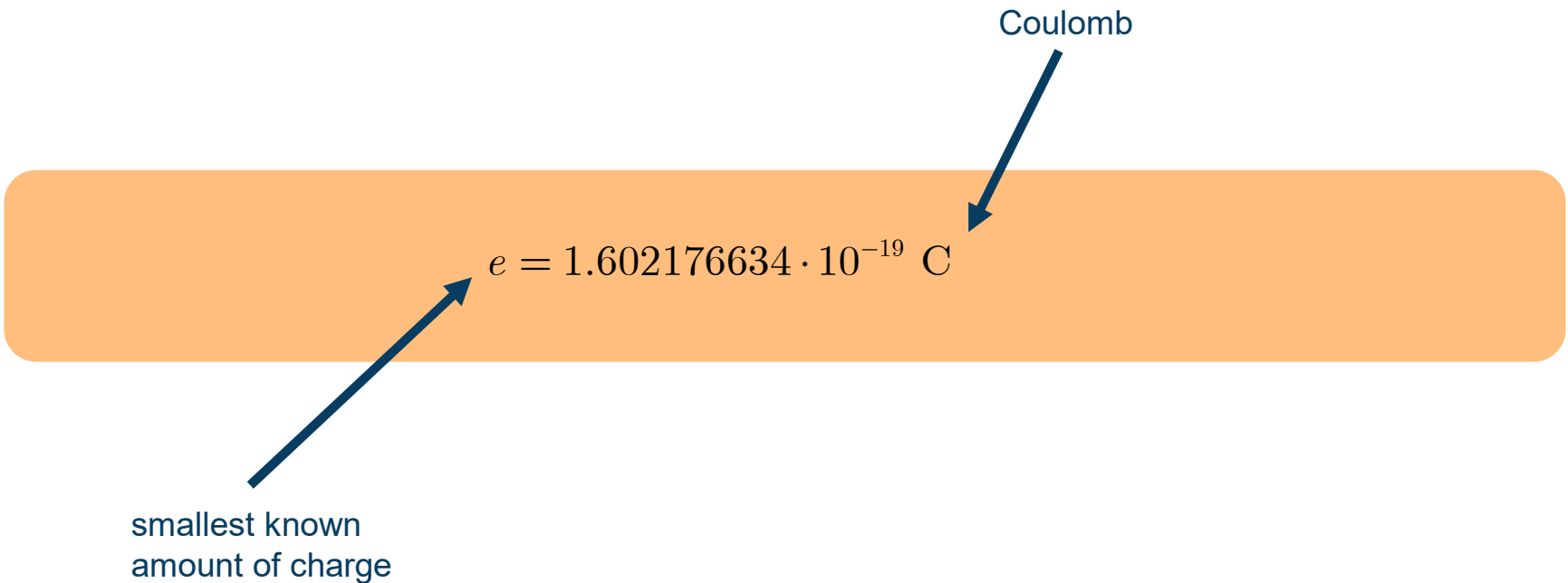*Rather difficult question – will not be fully answered*

Elementary Charge

Coulomb

$$e = 1.602176634 \cdot 10^{-19} \ \text{C}$$

smallest known
amount of charge

As far as we know, all charges in nature have values $\quad \pm Ne, N \in \mathbb{Z}$

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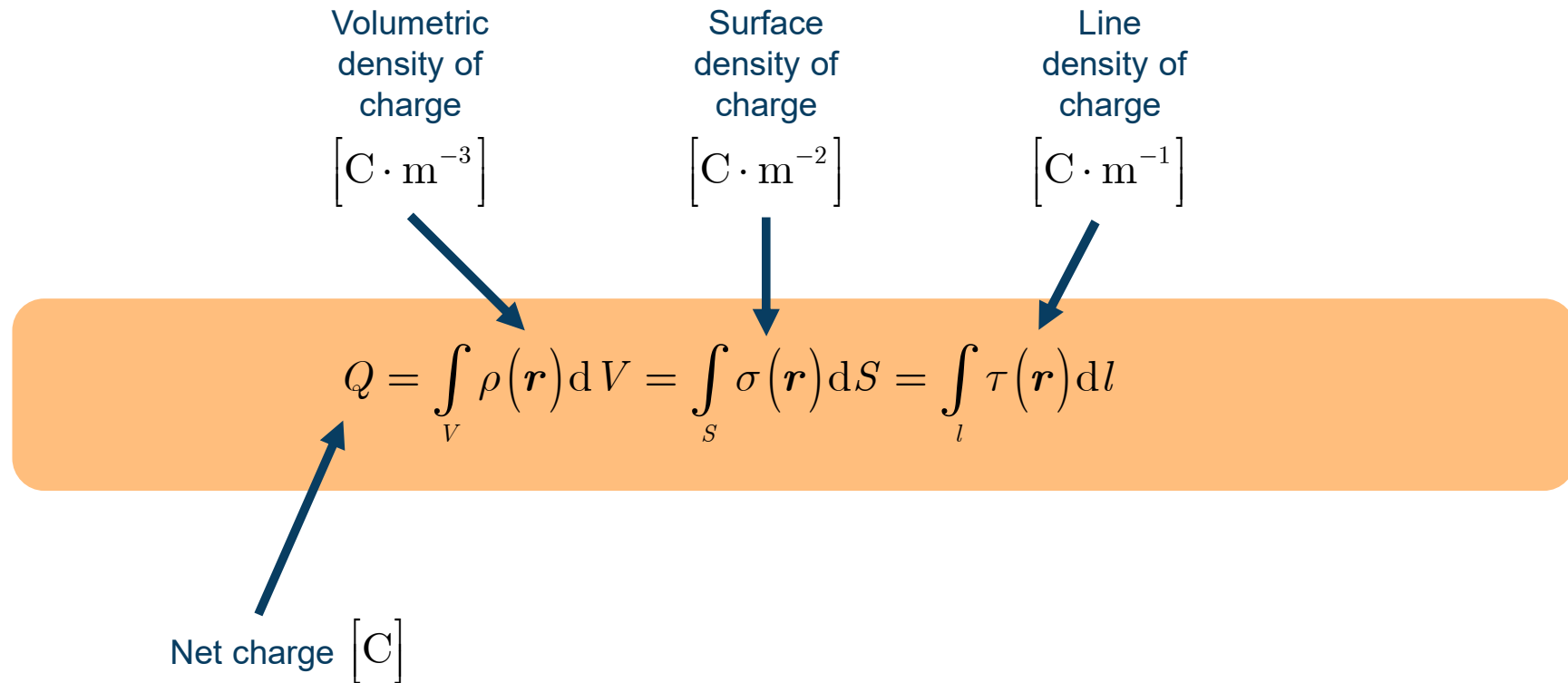CTU-FEE in Prague, Department of Electromagnetic Field

Charge conservation

Amount of charge is conserved in every frame (even non-inertial).

Neutrality of atoms has been verified to 20 digits

Continuous approximation of charge distribution

Volumetric density of charge
$$\left[\mathrm{C}\cdot\mathrm{m}^{-3}\right]$$

Surface density of charge
$$\left[\mathrm{C}\cdot\mathrm{m}^{-2}\right]$$

Line density of charge
$$\left[\mathrm{C}\cdot\mathrm{m}^{-1}\right]$$

$$Q = \int_V \rho\left(\boldsymbol{r}\right)\mathrm{d}V = \int_S \sigma\left(\boldsymbol{r}\right)\mathrm{d}S = \int_l \tau\left(\boldsymbol{r}\right)\mathrm{d}l$$

Net charge $\left[\mathrm{C}\right]$

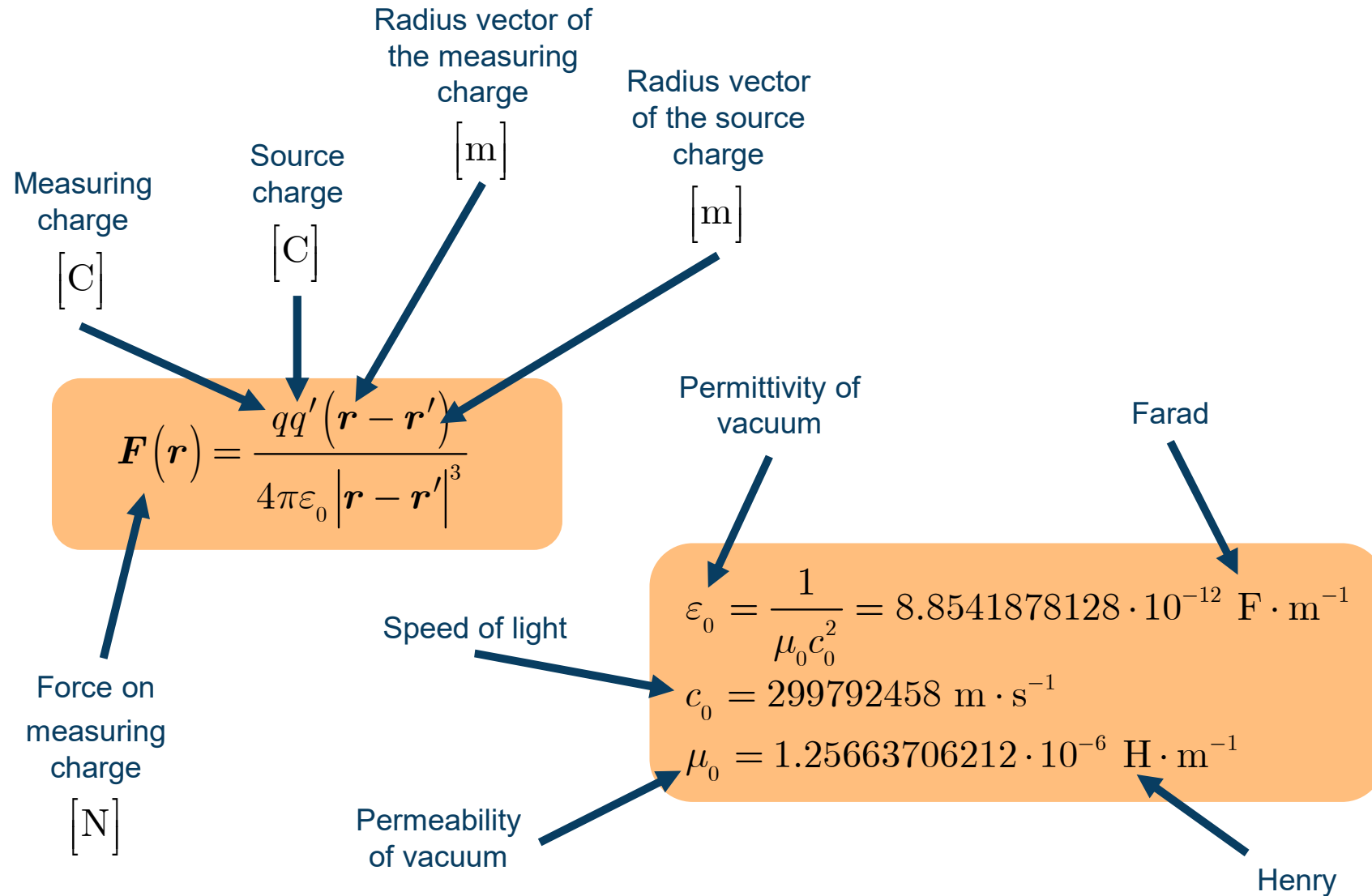*Continuous approximation allows for using powerful mathematics*

Fundamental Question of Electrostatics

There exist a specified distribution of static elementary charges. We pick one of them and ask what is the force acting on it.

This will be answered in full details

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Coulomb('s) Law

Radius vector of the measuring charge

$[\mathrm{m}]$

Radius vector of the source charge

$[\mathrm{m}]$

Source charge

$[\mathrm{C}]$

Measuring charge

$[\mathrm{C}]$

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = \frac{qq'\left(\boldsymbol{r}-\boldsymbol{r}'\right)}{4\pi\varepsilon_0\left|\boldsymbol{r}-\boldsymbol{r}'\right|^3}$$

Force on measuring charge

$[\mathrm{N}]$

Permittivity of vacuum

Farad

$$\varepsilon_0 = \frac{1}{\mu_0 c_0^2} = 8.8541878128 \cdot 10^{-12} \ \mathrm{F}\cdot\mathrm{m}^{-1}$$

Speed of light

$$c_0 = 299792458 \ \mathrm{m}\cdot\mathrm{s}^{-1}$$

$$\mu_0 = 1.25663706212 \cdot 10^{-6} \ \mathrm{H}\cdot\mathrm{m}^{-1}$$

Permeability of vacuum

Henry

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Coulomb('s) Law + Superposition Principle

$$F\left(r\right) = \frac{q}{4\pi\varepsilon_0} \sum_n \frac{q'_n\left(r - r'_n\right)}{\left|r - r'_n\right|^3}$$

Entire electrostatics can be deduced from this formula

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Electric Field

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = q\boldsymbol{E}\left(\boldsymbol{r}\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0}\sum_n \frac{q'_n\left(\boldsymbol{r} - \boldsymbol{r}'_n\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'_n\right|^3}$$

Intensity of
electric field

$$\left[\mathrm{V}\cdot\mathrm{m}^{-1}\right]$$

Force is represented by field – entity generated by charges and permeating the space

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Continuous Distribution of Charge

$$E(r) = \frac{1}{4\pi\varepsilon_0} \sum_n \frac{q'_n(r - r'_n)}{|r - r'_n|^3} \implies E(r) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(r')(r - r')}{|r - r'|^3} \, dV'$$

Continuous description of charge allows for using powerful mathematics

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Continuous Description of a Point Charge

Dirac's delta "function"

Defining property of Dirac's delta "function"

$$\rho\left(\boldsymbol{r}\right) = \sum_{n} q_n \delta\left(\boldsymbol{r} - \boldsymbol{r}_n\right)$$

$$\boldsymbol{F}\left(\boldsymbol{r}_n\right) = \int_{V} \boldsymbol{F}\left(\boldsymbol{r}\right) \delta\left(\boldsymbol{r} - \boldsymbol{r}_n\right) \mathrm{d}V$$

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Gauss(') Law

Mind the orientation of the surface

Total charge enclosed by the surface

$$\left[\mathrm{C}\right]$$

$$\nabla \cdot \boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{\rho\left(\boldsymbol{r}\right)}{\varepsilon_0} \qquad \Longleftrightarrow \qquad \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \int_V \rho\left(\boldsymbol{r}\right) \mathrm{d}V = \frac{Q}{\varepsilon_0}$$

Differential law (local)

Integral law (global)
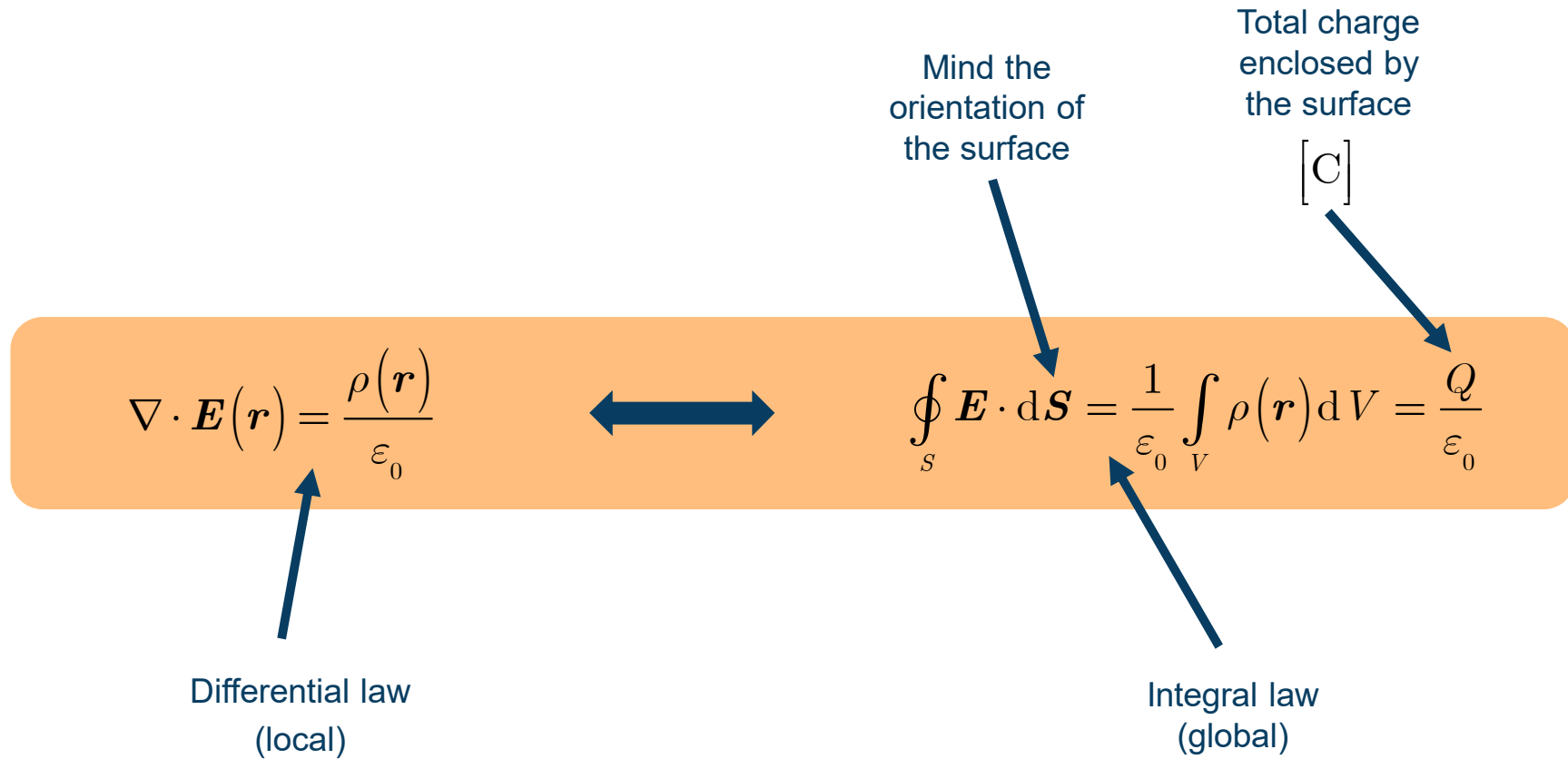
Rotation of Electric Field

$$\nabla \times \boldsymbol{E} = 0 \qquad \longleftrightarrow \qquad \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Differential law
(local)

Integral law
(global)

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Various Views on Electrostatics

Integral laws of electrostatics

Differential laws of electrostatics

Coulomb's law

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\varepsilon_0}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\boldsymbol{r}')(\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} \mathrm{d}V'$$

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

$$\nabla \times \boldsymbol{E} = 0$$

The physics content is the same, the formalism is different.

Electric potential

Electric potential

Defined up to arbitrary constant

$$\nabla \times \boldsymbol{E} = 0 \implies \boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r}) \implies \varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V' + K$$

Scalar description of electrostatic field

Voltage

Potential difference is a unique number

Voltage

$$[\mathrm{V}]$$

Work necessary to take charge q from point A to point B
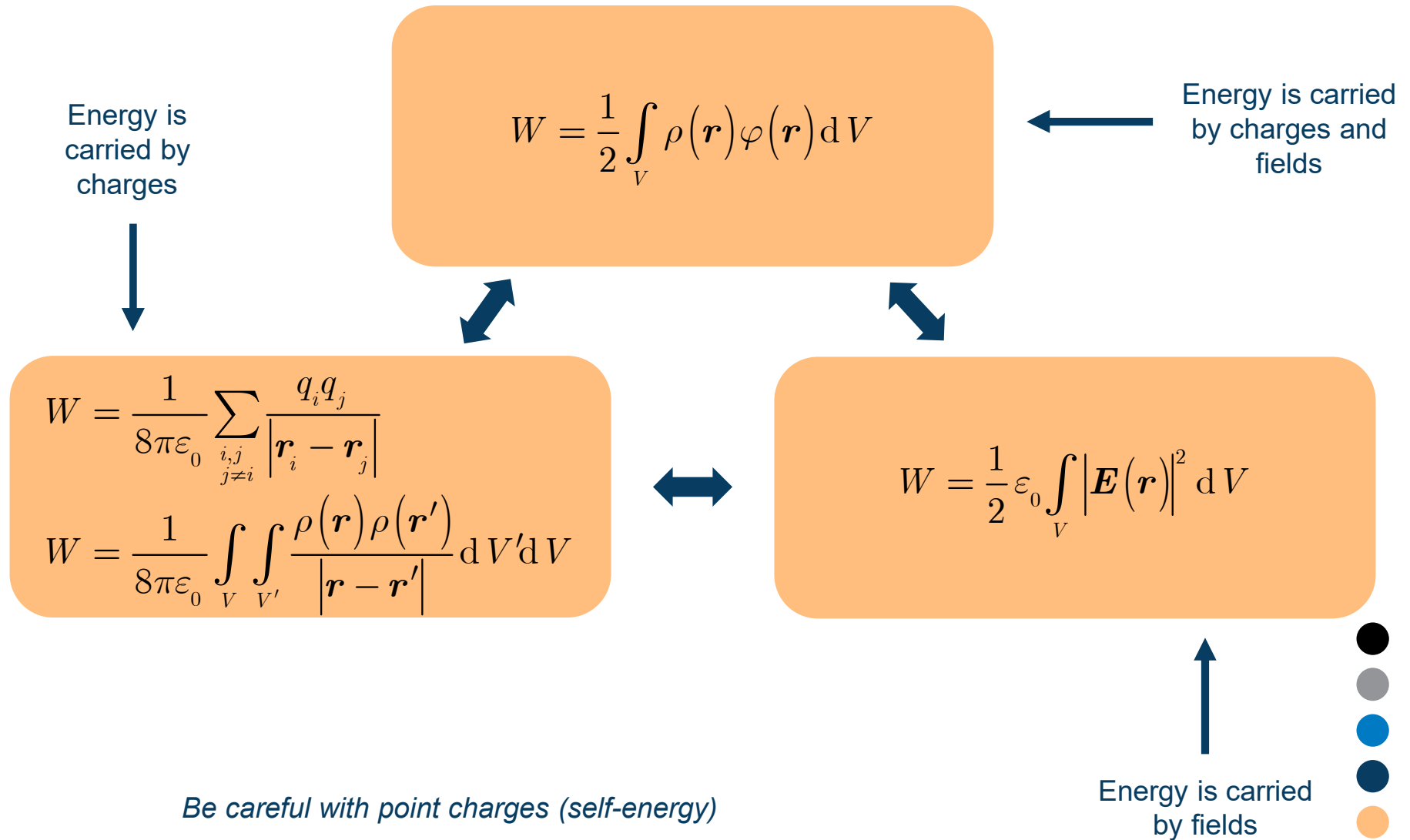
$$-\int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} = \varphi(B) - \varphi(A) = U$$

$$W = -\int_{A}^{B} \mathbf{F} \cdot d\mathbf{l} = qU$$

Voltage represents connection of abstract field theory with experiments

Electrostatic Energy

Energy is carried by charges

Energy is carried by charges and fields

$$W = \frac{1}{2} \int_V \rho(\boldsymbol{r}) \varphi(\boldsymbol{r}) \, \mathrm{d}V$$

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j \\ j \neq i}} \frac{q_i q_j}{\left| \boldsymbol{r}_i - \boldsymbol{r}_j \right|}$$

$$W = \frac{1}{8\pi\varepsilon_0} \int_V \int_{V'} \frac{\rho(\boldsymbol{r}) \rho(\boldsymbol{r}')}{\left| \boldsymbol{r} - \boldsymbol{r}' \right|} \, \mathrm{d}V' \mathrm{d}V$$

$$W = \frac{1}{2} \varepsilon_0 \int_V \left| \boldsymbol{E}(\boldsymbol{r}) \right|^2 \, \mathrm{d}V$$

Be careful with point charges (self-energy)

Energy is carried by fields

Electrostatic Energy vs Force

Energy of a
system of point
charges

Coulomb's law

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j \\ j\neq i}} \frac{q_i q_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|}$$

$\Longrightarrow$

$$\boldsymbol{F}\left(\boldsymbol{r}_\xi\right) = -\nabla_\xi W = \frac{q_\xi}{4\pi\varepsilon_0} \sum_{\substack{j \\ j\neq\xi}} q_j \frac{\left(\boldsymbol{r}_\xi - \boldsymbol{r}_j\right)}{\left|\boldsymbol{r}_\xi - \boldsymbol{r}_j\right|^3}$$

Electrostatic forces are always acting so as to minimize energy of the system

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Electric Stress Tensor

Total electric force acting in a volume

Stress tensor

$$\boldsymbol{F} = \int_V \rho(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) \mathrm{d}V = \varepsilon_0 \oint_S \overline{\overline{\boldsymbol{T}}} \cdot \mathrm{d}\boldsymbol{S} \qquad \Longrightarrow \qquad \overline{\overline{\boldsymbol{T}}} = \boldsymbol{E}\boldsymbol{E} - \frac{1}{2}\overline{\overline{\boldsymbol{I}}}\left|\boldsymbol{E}\right|^2$$

All the information on the volumetric Coulomb's force is contained at the boundary

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Ideal Conductor

Ideal conductor contains unlimited amount of free charges which under action of external electric field rearrange so as to annihilate electric field inside the conductor.

In 3D, the free charge always resides on the external bounding surface of the conductor.

In 1D and 2D
it is not so

Generally, free charges in conductors move so as to minimize the energy

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Boundary Conditions on Ideal Conductor

- Inside conductor

  - $$\boldsymbol{E}\big(\boldsymbol{r}\big) = 0 \quad \Leftrightarrow \quad \varphi\big(\boldsymbol{r}\big) = \text{const.}$$

    Potential is continuous across the boundary

- Just outside conductor

  - $$\boldsymbol{n}\big(\boldsymbol{r}\big) \times \boldsymbol{E}\big(\boldsymbol{r}\big) = 0 \quad \Leftrightarrow \quad \varphi\big(\boldsymbol{r}\big) = \text{const.}$$

  - $$\boldsymbol{n}\big(\boldsymbol{r}\big) \cdot \boldsymbol{E}\big(\boldsymbol{r}\big) = \frac{\sigma}{\varepsilon_0} \quad \Leftrightarrow \quad \frac{\partial \varphi\big(\boldsymbol{r}\big)}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

    Surface charge residing on the outer surface of the conductor

    Outward normal to the conductor

    Normal derivative

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Capacitance of a System of *N* conductors

Self and mutual capacitances

$$[\mathrm{F}]$$

Electrostatic energy

Capacitances depend solely on geometry and position of conductors

$$Q_i = \sum_j C_{ij}\varphi_j \qquad \Longrightarrow \qquad W = \frac{1}{2}\sum_{i,j} C_{ij}\varphi_j\varphi_i$$
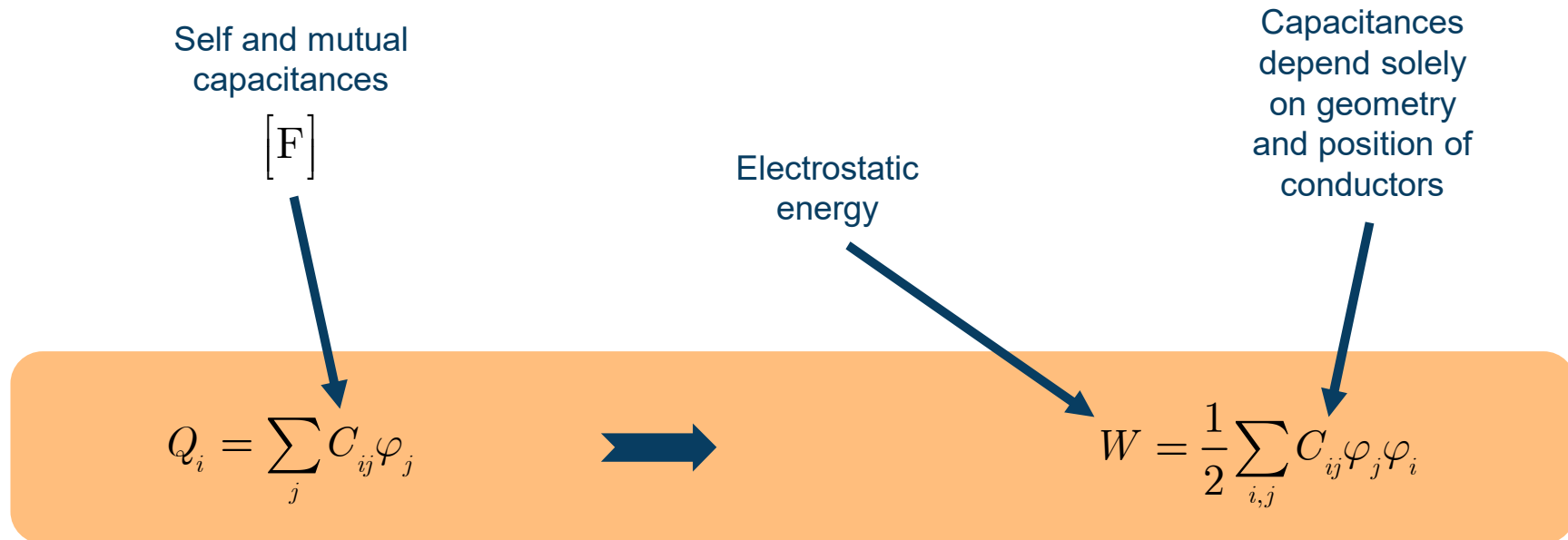
Electrostatic system is fully characterized by capacitances (we know the energy)

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Capacitance of a System of two conductors

Capacitance

Potential difference between conductors

$$Q = CU$$

Charge on positively charged conductor

$$W = \frac{1}{2}CU^2$$
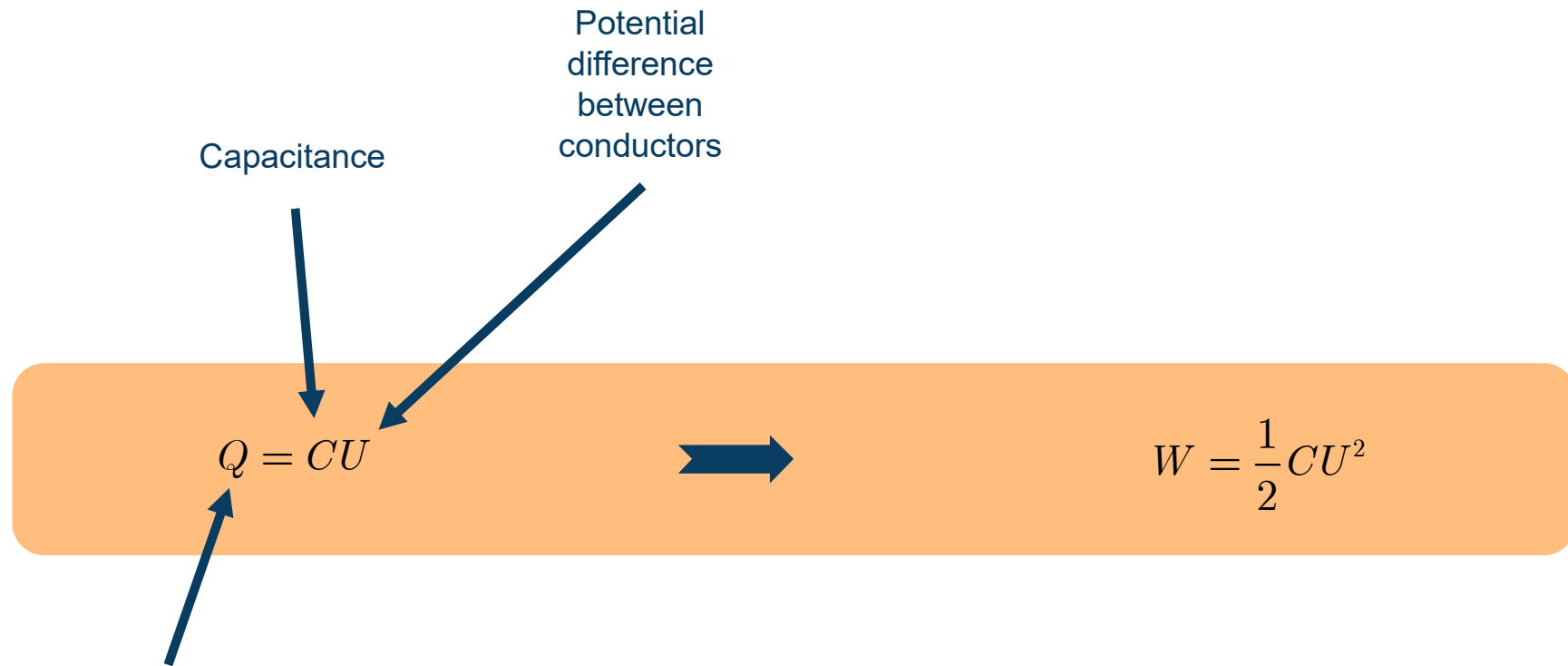
Poisson('s) equation

$$\Delta\varphi\left(\boldsymbol{r}\right) = -\frac{\rho\left(\boldsymbol{r}\right)}{\varepsilon_0}$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the charge density is known throughout the volume.

Laplace('s) equation

$$\Delta \varphi \left(\boldsymbol{r} \right) = 0$$

The solution to Laplace's equation is unique in a given volume once the potential is known on its bounding surface.

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Mean Value Theorem

Center of the sphere

Only for spheres containing no charge

$$\varphi\left(\boldsymbol{r}_{\mathrm{center}}\right) = \frac{1}{4\pi R^2} \oint\limits_{\mathrm{sphere}} \varphi\left(\boldsymbol{r}\right)\mathrm{d}S$$

Radius of the sphere

The solution to Laplace's equation posses neither maxima nor minima inside the solved volume.

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Earnshaw('s) Theorem

Consequence of
mean value theorem
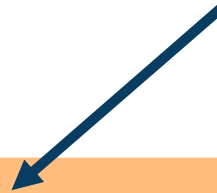
A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

Mind that the solution to Laplace's equation posses neither maxima nor minima inside the solved volume. This means that charged particle will always travel towards the boundary.

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Image Method

When solving field generated by charges in the presence of conductors, it is sometimes possible to remove the conductor and mimic its boundary conditions by adding extra charges to the exterior of the solution volume. The uniqueness theorem claims that this is a correct solution.

Image method always works with planes and spheres.

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Separation of Variables

Constants determined by boundary conditions

$$\Delta\varphi\left(\boldsymbol{r}\right) = 0 \quad\Longrightarrow\quad \varphi_{ijk}\left(\boldsymbol{r}\right) = X_i\left(x\right)Y_j\left(y\right)Z_k\left(z\right) \quad\Longrightarrow\quad \varphi\left(\boldsymbol{r}\right) = \sum_{ijk} C_{ijk}\varphi_{ijk}\left(\boldsymbol{r}\right)$$

Semi-analytical method for canonical problems

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Finite Differences

$$\varphi\left(x + h, y, z\right) \rightarrow \varphi_{(i+1)jk}$$

$$\Delta\varphi\left(\boldsymbol{r}\right) \approx \frac{\varphi_{(i+1)jk} - 2\varphi_{ijk} + \varphi_{(i-1)jk}}{h^2} + \frac{\varphi_{i(j+1)k} - 2\varphi_{ijk} + \varphi_{i(j-1)k}}{h^2} + \frac{\varphi_{ij(k+1)} - 2\varphi_{ijk} + \varphi_{ij(k-1)}}{h^2}$$

$$\Delta\varphi\left(\boldsymbol{r}\right) = 0 \qquad \varphi_{ijk} = \frac{\varphi_{(i+1)jk} + \varphi_{(i-1)jk} + \varphi_{i(j+1)k} + \varphi_{i(j-1)k} + \varphi_{ij(k+1)} + \varphi_{ij(k-1)}}{6}$$

Approximation by a system of linear algebraic equations

Mind the mean value theorem

Powerful numerical method for closed problems

Method of Moments

Assumed to be known in volume where the charge resides

Distribution of charge is unknown

Simple functions for which the potential integral can be easily evaluated

$$\varphi\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0} \int\limits_{V'} \frac{\rho\left(\boldsymbol{r}'\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \, \mathrm{d}V'$$

$$\rho\left(\boldsymbol{r}\right) \approx \sum_n \alpha_n \rho_n\left(\boldsymbol{r}\right)$$

$$\int\limits_{V} \rho_m\left(\boldsymbol{r}\right)\varphi\left(\boldsymbol{r}\right) \mathrm{d}V = \sum_n \alpha_n \frac{1}{4\pi\varepsilon_0} \int\limits_{V}\int\limits_{V'} \frac{\rho_m\left(\boldsymbol{r}\right)\rho_n\left(\boldsymbol{r}'\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \, \mathrm{d}V' \mathrm{d}V$$

Known

Approximation by a system of linear algebraic equations

Known

Powerful numerical method for open problems

Dielectrics

- Material in which charges cannot move freely

- Charges are forming clusters (atoms, molecules)

Clusters are electrically neutral

- Under influence of electric field the clusters change shape or rotate

- Electric field induces electric dipoles with density $P(r)$ $\left[\mathrm{C} \cdot \mathrm{m}^{-2}\right]$

Number of dipoles in unitary volume
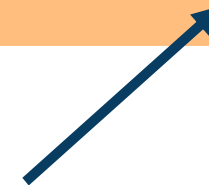
Electric Field of a Dipole

Two opposite charges very close to each other

$$\left|\boldsymbol{r} - \boldsymbol{r}_{\text{center}}\right| \gg \left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|$$

$$\varphi\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\left|\boldsymbol{r} - \boldsymbol{r}_1\right|} - \frac{1}{4\pi\varepsilon_0} \frac{q}{\left|\boldsymbol{r} - \boldsymbol{r}_2\right|} \approx \frac{1}{4\pi\varepsilon_0} \frac{\boldsymbol{p} \cdot \left(\boldsymbol{r} - \boldsymbol{r}_{\text{center}}\right)}{\left|\boldsymbol{r} - \boldsymbol{r}_{\text{center}}\right|^3}$$

$$\boldsymbol{p} = q\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right) = \int_V \boldsymbol{r}\rho\left(\boldsymbol{r}\right)\mathrm{d}V$$

Electric dipole moment

$$\left[\mathrm{C} \cdot \mathrm{m}\right]$$

General formula

Formula for two opposite charges

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Field Produced by Polarized Matter

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\boldsymbol{P}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} \, \mathrm{d}V' = \frac{1}{4\pi\varepsilon_0} \oint_{S'} \frac{\boldsymbol{P}(\boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \cdot \mathrm{d}\boldsymbol{S}' - \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\nabla' \cdot \boldsymbol{P}(\boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \, \mathrm{d}V'$$

Only apply at infinitely sharp boundary (unrealistic)

Potential of volumetric charge density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside

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Electric Displacement

Electric displacement $\left[\mathrm{C}\cdot\mathrm{m}^{-2}\right]$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r})$$

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r})$$

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho(\boldsymbol{r}) \mathrm{d}V = Q$$

Only free charge
(compare to divergence
of electric field)

Linear Isotropic Dielectrics
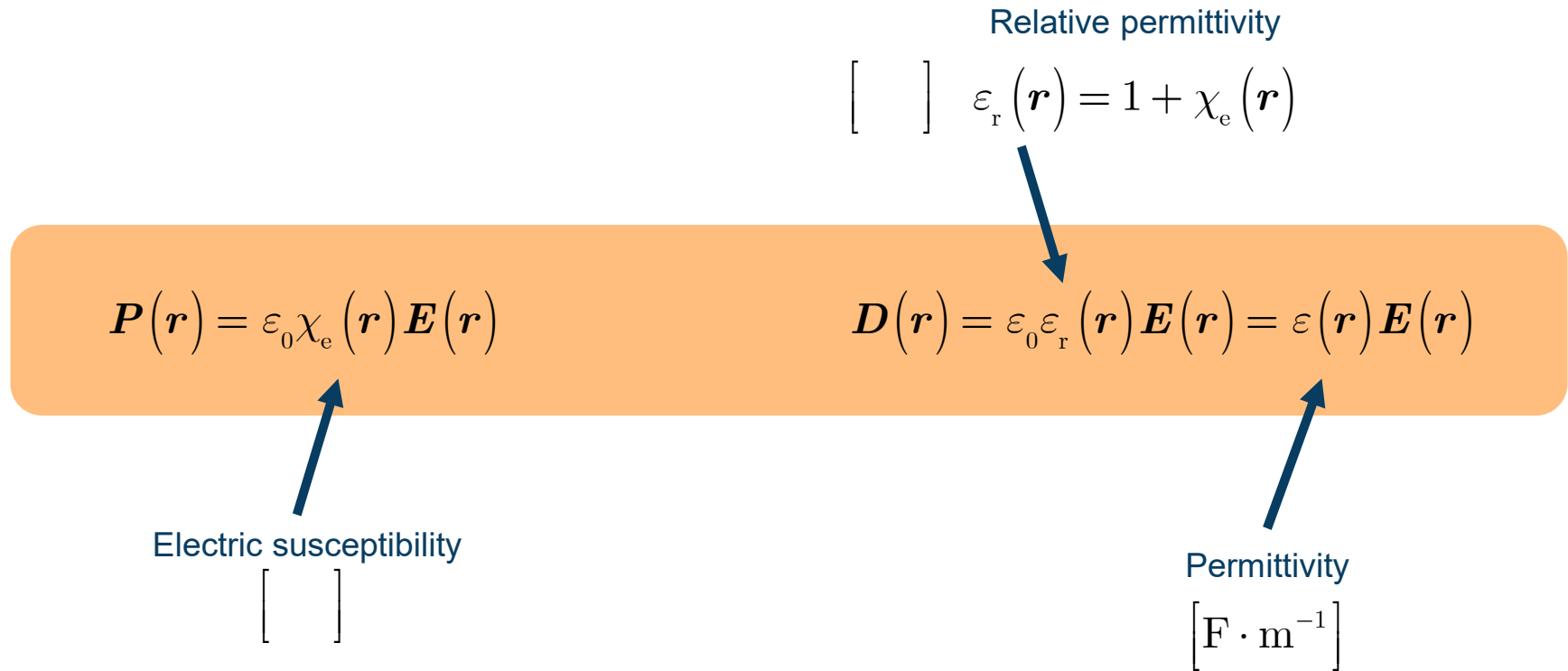
Relative permittivity

$$\varepsilon_{\mathrm{r}}\left(\boldsymbol{r}\right) = 1 + \chi_{\mathrm{e}}\left(\boldsymbol{r}\right)$$

$$\left[\quad\right]$$

$$\boldsymbol{P}\left(\boldsymbol{r}\right) = \varepsilon_{0}\chi_{\mathrm{e}}\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right) \qquad \boldsymbol{D}\left(\boldsymbol{r}\right) = \varepsilon_{0}\varepsilon_{\mathrm{r}}\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right) = \varepsilon\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right)$$

Electric susceptibility

$$\left[\quad\right]$$

Permittivity

$$\left[\mathrm{F}\cdot\mathrm{m}^{-1}\right]$$

All the complicated structure of matter reduces to a simple scalar quantity

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Fields in Presence of Dielectrics 1/2

Analogy with electric field in vacuum can only be used when entire space is homogeneously filled with dielectric.

$$\nabla \times \boldsymbol{D}(\boldsymbol{r}) = \nabla \times \left[\varepsilon(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) \right] \neq 0$$

Inequality is due to boundaries

Analogy with vacuum can only be used when space is homogeneously filled with dielectric

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Fields in Presence of Dielectrics 2/2

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \Leftrightarrow \boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r}) \quad \Longrightarrow \quad \nabla \cdot \left[\varepsilon(\boldsymbol{r}) \nabla \varphi(\boldsymbol{r})\right] = -\rho(\boldsymbol{r})$$

$$\Delta \varphi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon}$$

Not a function of coordinates

Poisson's equation holds only when permittivity does not depend on coordinates

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Dielectric Boundaries

$$\boldsymbol{n}\left(\boldsymbol{r}\right)\times\left[\boldsymbol{E}_{1}\left(\boldsymbol{r}\right)-\boldsymbol{E}_{2}\left(\boldsymbol{r}\right)\right]=0 \quad\Leftrightarrow\quad \varphi_{1}\left(\boldsymbol{r}\right)-\varphi_{2}\left(\boldsymbol{r}\right)=0$$

$$\boldsymbol{n}\left(\boldsymbol{r}\right)\cdot\left[\varepsilon_{1}\boldsymbol{E}_{1}\left(\boldsymbol{r}\right)-\varepsilon_{2}\boldsymbol{E}_{2}\left(\boldsymbol{r}\right)\right]=\sigma\left(\boldsymbol{r}\right) \quad\Leftrightarrow\quad \varepsilon_{1}\frac{\partial\varphi_{1}\left(\boldsymbol{r}\right)}{\partial n}-\varepsilon_{2}\frac{\partial\varphi_{2}\left(\boldsymbol{r}\right)}{\partial n}=-\sigma\left(\boldsymbol{r}\right)$$

Normal
pointing to
region (1)

Both conditions are needed for unique solution

Electrostatic Energy in Dielectrics

$$W = \frac{1}{2}\varepsilon_0 \int_V \left| \boldsymbol{E}\left(\boldsymbol{r}\right) \right|^2 \mathrm{d}V \qquad \Longrightarrow \qquad W = \frac{1}{2}\int_V \boldsymbol{E}\left(\boldsymbol{r}\right) \cdot \boldsymbol{D}\left(\boldsymbol{r}\right) \mathrm{d}V$$

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Forces on Dielectrics

This only holds when charge is held constant

$$W = \frac{1}{2}CU^2 = \frac{1}{2}\frac{Q^2}{C}$$

$$W = \frac{1}{2}\int_V \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{D}(\boldsymbol{r}) \mathrm{d}V$$

$$\boldsymbol{F}(\boldsymbol{r}_\xi) = -\nabla_\xi W$$

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Electric Current

Current density

$$\left[\mathrm{A} \cdot \mathrm{m}^{-2}\right]$$

Charge

$$\left[\mathrm{C}\right]$$

Velocity of charge

$$\left[\mathrm{m} \cdot \mathrm{s}^{-1}\right]$$

$$\boldsymbol{J}\left(\boldsymbol{r}, t\right) = \sum_{k=1}^{N} q_k \delta\left(\boldsymbol{r} - \boldsymbol{r}_k\left(t\right)\right) \boldsymbol{v}_k\left(t\right)$$

Volumetric density represented by Dirac delta

$$\left[\mathrm{m}^{-3}\right]$$

Charges in motion are represented by current density

Local Charge Conservation

$$\nabla \cdot \boldsymbol{J}\left(\boldsymbol{r},t\right) = -\frac{\partial}{\partial t}\sum_{k=1}^{N} q_k \delta\left(\boldsymbol{r}-\boldsymbol{r}_k\left(t\right)\right) = -\frac{\partial\rho\left(\boldsymbol{r},t\right)}{\partial t}$$

Charge is conserved locally at every space-time point

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Global Charge Conservation

When charge leaves a given volume, it is always accompanied by a current through the bounding envelope

$$\oint_S \boldsymbol{J}\left(\boldsymbol{r},t\right) \cdot d\boldsymbol{S} = -\frac{\partial Q\left(t\right)}{\partial t}$$

Charge can neither be created nor destroyed. It can only be displaced.

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Stationary Current

When charge enters a volume, another must leave it without any delay

$$\nabla \cdot \boldsymbol{J}\big(\boldsymbol{r}\big) = 0 \qquad \longleftrightarrow \qquad \oint\limits_{S} \boldsymbol{J}\big(\boldsymbol{r}\big) \cdot d\boldsymbol{S} = 0$$

There is no charge accumulation in stationary flow

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Ohm('s) Law

Conductivity

$$\left[\mathrm{S} \cdot \mathrm{m}^{-1} \right]$$

$$\boldsymbol{J}(\boldsymbol{r}) = \sigma(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r})$$

This simple linear relation holds for enormous interval of electric field strengths

Electromotive Force

Stationary flow of charges cannot be caused by electrostatic field. The motion forces are non-conservative, are called electromotive forces, and are commonly of chemical, magnetic or photoelectric origin.

$$\oint_l \boldsymbol{E}\left(\boldsymbol{r}\right) \cdot d\boldsymbol{l} \neq 0 \qquad\qquad \oint_l \boldsymbol{E}\left(\boldsymbol{r}\right) \cdot d\boldsymbol{l} = 0$$

For curves passing through sources of electromotive force

For curves not crossing sources of electromotive force

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Boundary Conditions for Stationary Current

$$\boldsymbol{n}\left(\boldsymbol{r}\right)\times\left[\boldsymbol{E}_1\left(\boldsymbol{r}\right)-\boldsymbol{E}_2\left(\boldsymbol{r}\right)\right]=0 \quad \Leftrightarrow \quad \varphi_1\left(\boldsymbol{r}\right)-\varphi_2\left(\boldsymbol{r}\right)=0$$

$$\boldsymbol{n}\left(\boldsymbol{r}\right)\cdot\left[\varepsilon_1\boldsymbol{E}_1\left(\boldsymbol{r}\right)-\varepsilon_2\boldsymbol{E}_2\left(\boldsymbol{r}\right)\right]=\sigma\left(\boldsymbol{r}\right) \quad \Leftrightarrow \quad \varepsilon_1\frac{\partial\varphi_1\left(\boldsymbol{r}\right)}{\partial n}-\varepsilon_2\frac{\partial\varphi_2\left(\boldsymbol{r}\right)}{\partial n}=-\sigma\left(\boldsymbol{r}\right)$$

$$\boldsymbol{n}\left(\boldsymbol{r}\right)\cdot\left[\sigma_1\boldsymbol{E}_1\left(\boldsymbol{r}\right)-\sigma_2\boldsymbol{E}_2\left(\boldsymbol{r}\right)\right]=0 \quad \Leftrightarrow \quad \sigma_1\frac{\partial\varphi_1\left(\boldsymbol{r}\right)}{\partial n}-\sigma_2\frac{\partial\varphi_2\left(\boldsymbol{r}\right)}{\partial n}=0$$

Charge conservation forces the continuity of current across the boundary

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Electric Current

Current

$$[\mathrm{A}]$$

$$I = \int_S \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{S}$$

Cross-section of current path

$$\left[\mathrm{m}^2\right]$$

Existence of high contrast in conductivity between conductors and dielectrics allows for well defined current paths.

Resistance (Conductance)

Potential difference (voltage)

$$[\mathrm{V}]$$

Resistance

$$[\Omega]$$

Current

$$[\mathrm{A}]$$

Conductance

$$[\mathrm{S}]$$

$$U = RI \qquad \longleftrightarrow \qquad I = GU$$

$$R = \frac{1}{G} = \frac{L}{\sigma S}$$

Resistance of a cylinder homogeneous cylinder of conductive material

Cross-section of current path

$$[\mathrm{m}^2]$$

Length along current path

$$[\mathrm{m}]$$
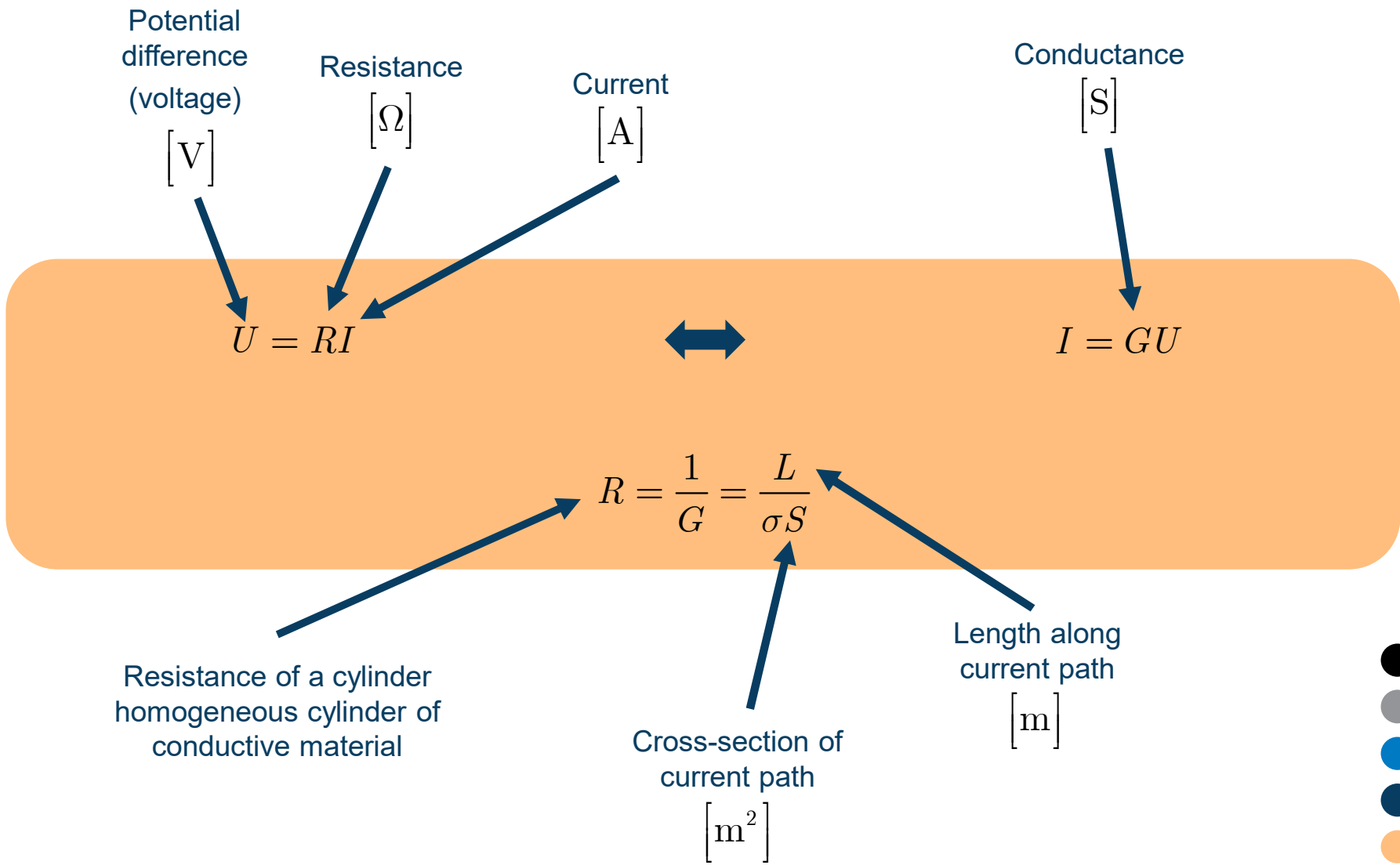
Resistive Circuits and Kirchhoff('s) Laws

In a loop

On a resistor

At a junction

$$\sum_i U_i = U_{\text{electromotive}}$$

$$U_i = R_i I_i$$

$$\sum_i I_i = 0$$

Kirchhoff's laws are a consequence of electrostatics and law's of stationary current flow

Joule('s) Heat

Power lost via conduction

$$[\mathrm{W}]$$

Power lost on resistor

$$[\mathrm{W}]$$

$$P = \int\limits_V \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}V = \int\limits_V \sigma(\boldsymbol{r})\big|\boldsymbol{E}(\boldsymbol{r})\big|^2\,\mathrm{d}V \qquad \longleftrightarrow \qquad P = UI = RI^2 = \frac{U^2}{R}$$

Electric field within conducting material produce heat

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Fundamental Question of Magnetostatics

There exist a specified distribution of stationary current. We pick a differential volume of it and ask what is the force acting on it.

Biot-Savart('s) Law

Permeability of vacuum

$$\mu_0 = 4\pi \cdot 10^{-7} \ \mathrm{H} \cdot \mathrm{m}^{-1}$$

Measuring current element

$$\big[\mathrm{A}\big]$$

Source current element

$$\big[\mathrm{A}\big]$$

$$\boldsymbol{F}\big(\boldsymbol{r}\big) = \frac{\mu_0}{4\pi} \cdot \frac{\boldsymbol{J}\big(\boldsymbol{r}\big)\,\mathrm{d}V \times \big[\boldsymbol{J}\big(\boldsymbol{r}'\big)\,\mathrm{d}V' \times \big(\boldsymbol{r} - \boldsymbol{r}'\big)\big]}{\big|\boldsymbol{r} - \boldsymbol{r}'\big|^3}$$

Force on measuring current

$$\big[\mathrm{N}\big]$$

Radius vector of the measuring current

$$\big[\mathrm{m}\big]$$

Radius vector of the source current

$$\big[\mathrm{m}\big]$$

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Biot-Savart('s) Law + Superposition Principle

$$\boldsymbol{F}\big(\boldsymbol{r}\big) = \boldsymbol{J}\big(\boldsymbol{r}\big)\,\mathrm{d}V \times \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}\big(\boldsymbol{r}'\big) \times \big(\boldsymbol{r} - \boldsymbol{r}'\big)}{\big|\boldsymbol{r} - \boldsymbol{r}'\big|^3}\,\mathrm{d}V'$$

Entire magnetostatics can be deduced from this formula

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Magnetic Field

$$F(r) = J(r)\,\mathrm{d}V \times B(r) \qquad\longleftrightarrow\qquad B(r) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J(r') \times (r - r')}{|r - r'|^3}\,\mathrm{d}V'$$

Magnetic field
(Magnetic induction)

$$\big[\mathrm{T}\big]$$

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Divergence of Magnetic Field

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \qquad \longleftrightarrow \qquad \oint_S \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = 0$$

There are no point sources of magnetostatic field

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Curl of Magnetic Field – Ampere('s) Law

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \qquad \longleftrightarrow \qquad \oint_l \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I$$

Total current captured within the curve

$$[\mathrm{A}]$$

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Magnetic Vector Potential

Magnetic vector potential

Defined up to arbitrary scalar function

$$\nabla \cdot \boldsymbol{B} = 0 \implies \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \implies \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \mathrm{d}V' + \nabla \psi(\boldsymbol{r})$$

Reduced description of magnetostatic field

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Poisson('s) equation

$$\Delta \boldsymbol{A}\left(\boldsymbol{r}\right) = -\mu_0 \boldsymbol{J}\left(\boldsymbol{r}\right)$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the current density is known through out the volume.

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Boundary Conditions

Surface current on the boundary

$$n\left(r\right) \times \left[B_1\left(r\right) - B_2\left(r\right)\right] = \mu_0 K\left(r\right)$$

$$n\left(r\right) \cdot \left[B_1\left(r\right) - B_2\left(r\right)\right] = 0$$

$$A_1\left(r\right) - A_2\left(r\right) = 0$$

Normal pointing to region (1)

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Magnetostatic Energy

$$W = \frac{1}{2} \int_V \boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r}) \, \mathrm{d}V \qquad \Longleftrightarrow \qquad W = \frac{1}{2\mu_0} \int_V \left| \boldsymbol{B}(\boldsymbol{r}) \right|^2 \mathrm{d}V$$

For now it is just a formula that works – it must be derived with the help of time varying fields

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Magnetostatic Energy – Current Circuits

$$M_{ij} = M_{ji} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_j} \int_{V_i'} \frac{\boldsymbol{J}_j\left(\boldsymbol{r}_j\right) \cdot \boldsymbol{J}_i\left(\boldsymbol{r}_i'\right)}{\left|\boldsymbol{r}_j - \boldsymbol{r}_i'\right|} \mathrm{d}V_i' \mathrm{d}V_j$$

Mutual-Inductance $\left[\mathrm{H}\right]$

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2}\sum_{i \neq j} M_{ij} I_i I_j$$

Self-Inductance $\left[\mathrm{H}\right]$

$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} \int_{V_i'} \frac{\boldsymbol{J}_i\left(\boldsymbol{r}_i\right) \cdot \boldsymbol{J}_i\left(\boldsymbol{r}_i'\right)}{\left|\boldsymbol{r}_i - \boldsymbol{r}_i'\right|} \mathrm{d}V_i' \mathrm{d}V_i$$

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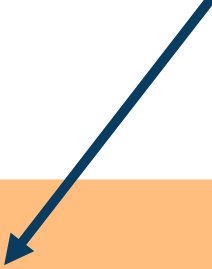
Mutual Inductance – Thin Current Loop

$$\Phi_{ji} = \int\limits_{S_j} \boldsymbol{B}_i\left(\boldsymbol{r}_j\right) \cdot \mathrm{d}\boldsymbol{S}_j$$

Magnetic flux induced by *i*-th current through *j*-th current

$$\left[\mathrm{Wb}\right]$$

$$M_{ij} = \frac{\Phi_{ji}}{I_i}$$

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Magnetic Materials

- Material response is due to magnetic dipole moments

- Magnetic moment comes from spin or orbital motion of an electron

- Magnetic field tends to align magnetic moments

- Magnetic field induces magnetic dipoles with density $\boldsymbol{M}\left(\boldsymbol{r}\right)$ $\left[\mathrm{A} \cdot \mathrm{m}^{-1}\right]$

Number of dipoles
in unitary volume

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Magnetic Field of a Dipole

Dipole is assumed at the origin

$r \neq 0$

$$A\left(r\right) = \frac{\mu_0}{4\pi} \cdot \frac{m \times r}{r^3} \qquad \Longleftrightarrow \qquad B\left(r\right) = \frac{\mu_0}{4\pi}\left[\frac{3r\left(r \cdot m\right)}{r^5} - \frac{m}{r^3}\right]$$

$$m = \frac{1}{2}\int\limits_V r \times J\left(r\right)\mathrm{d}V$$

Magnetic dipole moment

$$\left[\mathrm{A} \cdot \mathrm{m}^2\right]$$

Magnetic dipole approximates infinitesimally small current loop

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Field Produced by Magnetized Matter

$$A\left(r\right)=\frac{\mu_0}{4\pi}\int_{V'}\frac{M\left(r'\right)\times\left(r-r'\right)}{\left|r-r'\right|^3}\,\mathrm{d}V'=\frac{\mu_0}{4\pi}\oint_{S'}\frac{M\left(r'\right)\times\mathrm{d}S'}{\left|r-r'\right|}+\frac{\mu_0}{4\pi}\int_{V'}\frac{\nabla'\times M\left(r'\right)}{\left|r-r'\right|}\,\mathrm{d}V'$$

Only applies at infinitely
sharp boundary
(unrealistic)

Potential of volumetric
current density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside

Magnetic Intensity

Magnetic Intensity $\left[\mathrm{A}\cdot\mathrm{m}^{-1}\right]$

$$\nabla\times\boldsymbol{H}\left(\boldsymbol{r}\right)=\boldsymbol{J}\left(\boldsymbol{r}\right)$$

$$\boldsymbol{B}\left(\boldsymbol{r}\right)=\mu_{0}\left(\boldsymbol{H}\left(\boldsymbol{r}\right)+\boldsymbol{M}\left(\boldsymbol{r}\right)\right)$$

$$\oint_{l}\boldsymbol{H}\left(\boldsymbol{r}\right)\cdot\mathrm{d}\boldsymbol{l}=I$$

Only free current

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Linear Isotropic Magnetic Materials

Relative permeability

$$\left[\quad\right] \quad \mu_{\mathrm{r}}\left(\boldsymbol{r}\right) = 1 + \chi_{\mathrm{m}}\left(\boldsymbol{r}\right)$$

$$\boldsymbol{M}\left(\boldsymbol{r}\right) = \chi_{\mathrm{m}}\left(\boldsymbol{r}\right)\boldsymbol{H}\left(\boldsymbol{r}\right) \qquad \boldsymbol{B}\left(\boldsymbol{r}\right) = \mu_{0}\mu_{\mathrm{r}}\left(\boldsymbol{r}\right)\boldsymbol{H}\left(\boldsymbol{r}\right) = \mu\left(\boldsymbol{r}\right)\boldsymbol{H}\left(\boldsymbol{r}\right)$$

Magnetic susceptibility

$$\left[\quad\right]$$

Permeability

$$\left[\mathrm{H}\cdot\mathrm{m}^{-1}\right]$$
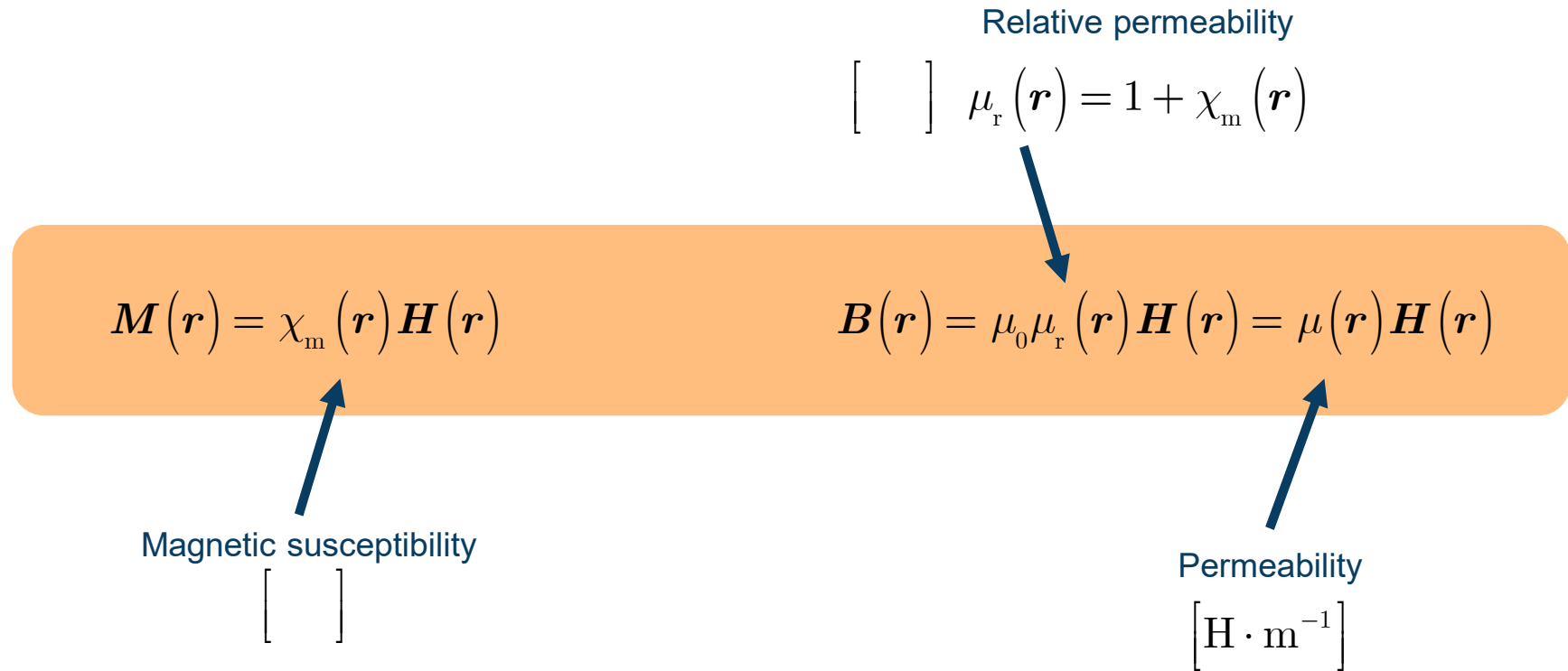
All the complicated structure of matter reduces to a simple scalar quantity

Fields in Presence of Magnetic Material

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \Leftrightarrow \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \quad \Longrightarrow \quad \nabla \times \left[\frac{1}{\mu(\boldsymbol{r})} \nabla \times \boldsymbol{A}(\boldsymbol{r}) \right] = \boldsymbol{J}(\boldsymbol{r})$$

$$\Delta \boldsymbol{A}(\boldsymbol{r}) = -\mu \boldsymbol{J}(\boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$$

Coulomb('s) gauge

Not a function of coordinates

Poisson's equation holds only when permittivity does not depend on coordinates

Magnetic Material Boundaries

$$n\left(r\right)\times\left[H_1\left(r\right)-H_2\left(r\right)\right]=K\left(r\right)$$

$$n\left(r\right)\cdot\left[\mu_1 H_1\left(r\right)-\mu_2 H_2\left(r\right)\right]=0$$

Normal
pointing to
region (1)

Both conditions are needed for unique solution

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Magnetostatic Energy in Magnetic Material

$$W = \frac{1}{2\mu_0} \int_V \left| \boldsymbol{B}\left(\boldsymbol{r}\right) \right|^2 \mathrm{d}V \qquad \Longrightarrow \qquad W = \frac{1}{2} \int_V \boldsymbol{H}\left(\boldsymbol{r}\right) \cdot \boldsymbol{B}\left(\boldsymbol{r}\right) \mathrm{d}V$$

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Magnetic Materials

- Paramagnetic – small positive susceptibility
  (small attraction – linear)

- Diamagnetic – small negative susceptibility
  (small repulsion – linear)

- Ferromagnetic – "*large positive susceptibility*"
  (large attraction – nonlinear)

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Ferromagnetic Materials

- Spins are ordered within domains

- Magnetization is a non-linear function of field intensity

- Magnetization curve – Hysteresis, Remanence

- Susceptibility can only be defined as local approximation

- Above Curie('s) temperature ferromagnetism disappears

Exact calculations are very difficult – use simplified models (soft material, permanent magnet)

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Faraday('s) Law

Minus sign is called
Lenz('s) law

$-\dfrac{\partial \Phi}{\partial t}$ Time variation of
magnetic flux

$$\oint\limits_{l} \boldsymbol{E}\big(\boldsymbol{r},t\big)\cdot \mathrm{d}\boldsymbol{l} = -\frac{\partial}{\partial t}\int\limits_{S} \boldsymbol{B}\big(\boldsymbol{r},t\big)\cdot \mathrm{d}\boldsymbol{S} \qquad \Longleftrightarrow \qquad \nabla \times \boldsymbol{E}\big(\boldsymbol{r},t\big) = -\frac{\partial \boldsymbol{B}\big(\boldsymbol{r},t\big)}{\partial t}$$

Time variation in magnetic field produces electric field that tries to counter the change in magnetic flux (electromotive force)
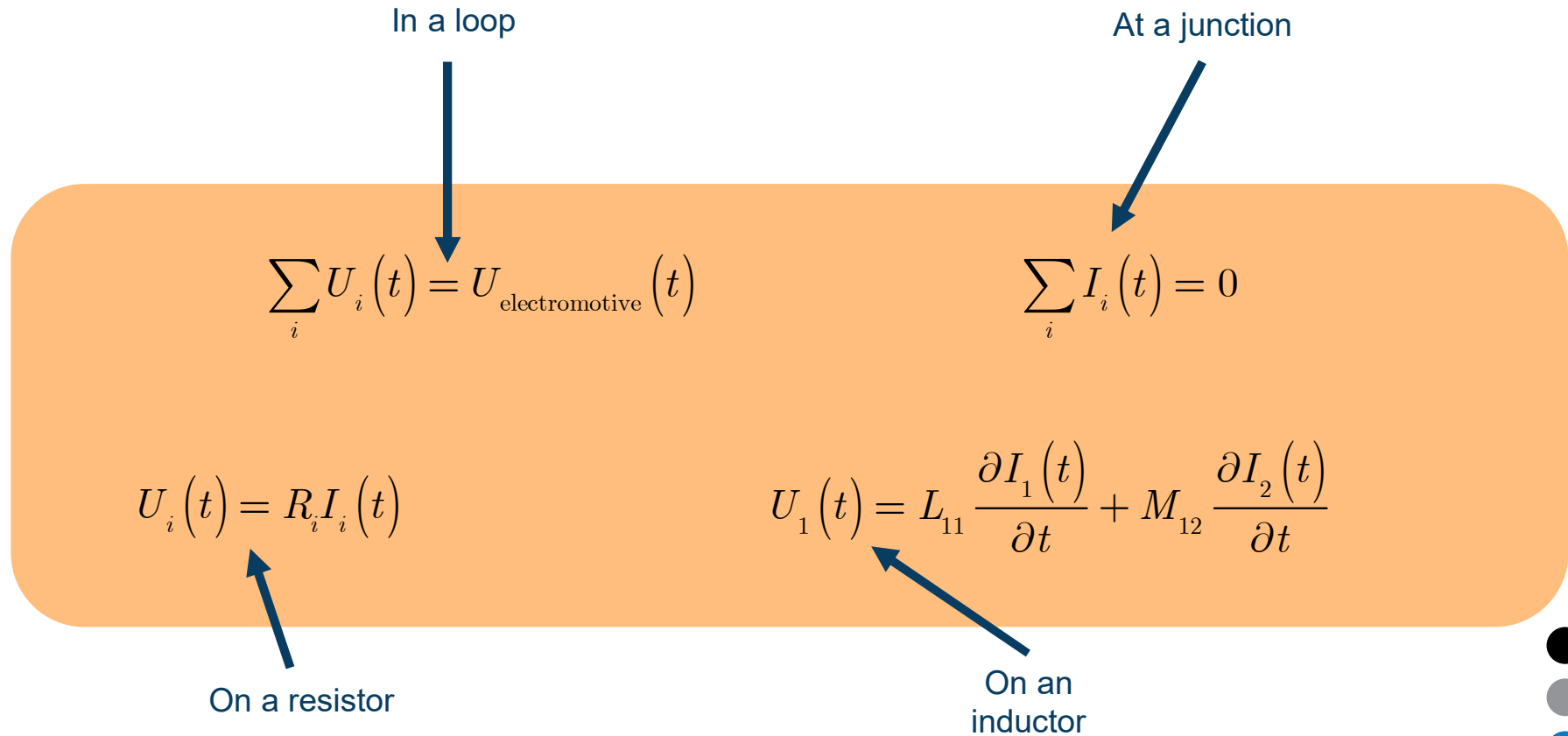
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Lenz('s) Law

The current created by time variation of magnetic flux is directed so as to oppose the flux creating it.

Time Varying RL Circuits

In a loop

At a junction

$$\sum_i U_i\left(t\right) = U_{\text{electromotive}}\left(t\right)$$

$$\sum_i I_i\left(t\right) = 0$$

$$U_i\left(t\right) = R_i I_i\left(t\right)$$

$$U_1\left(t\right) = L_{11}\frac{\partial I_1\left(t\right)}{\partial t} + M_{12}\frac{\partial I_2\left(t\right)}{\partial t}$$

On a resistor

On an inductor

Circuit laws are valid as long as time variations are not too fast

Time Varying Potentials

$$\nabla \cdot \boldsymbol{A}\big(\boldsymbol{r},t\big) = -\sigma\mu\varphi\big(\boldsymbol{r},t\big)$$

$$\boldsymbol{B}\big(\boldsymbol{r},t\big) = \nabla \times \boldsymbol{A}\big(\boldsymbol{r},t\big)$$

$$\boldsymbol{E}\big(\boldsymbol{r},t\big) = -\nabla\varphi\big(\boldsymbol{r},t\big) - \frac{\partial \boldsymbol{A}\big(\boldsymbol{r},t\big)}{\partial t}$$

In time varying fields scalar potential becomes redundant

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Source and Induced Currents

Those are fixed, not reacting to fields

$$\nabla \times \boldsymbol{H}\left(\boldsymbol{r},t\right) = \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right) + \boldsymbol{J}_{\text{induced}}\left(\boldsymbol{r},t\right) = \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right) + \sigma\boldsymbol{E}\left(\boldsymbol{r},t\right)$$

Diffusion Equation

$$\Delta \boldsymbol{A}(\boldsymbol{r},t) - \sigma\mu\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = -\mu\boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)$$

$$\Delta \boldsymbol{H}(\boldsymbol{r},t) - \sigma\mu\frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t} = -\nabla \times \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)$$

$$\Delta \boldsymbol{E}(\boldsymbol{r},t) - \sigma\mu\frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} = \frac{1}{\varepsilon}\nabla\rho_{\text{source}}(\boldsymbol{r},t) + \mu\frac{\partial \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)}{\partial t}$$

Material parameters are assumed
independent of coordinates

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Maxwell('s)-Lorentz('s) Equations

$$\nabla \times \boldsymbol{H}\left(\boldsymbol{r},t\right) = \boldsymbol{J}\left(\boldsymbol{r},t\right) + \frac{\partial \boldsymbol{D}\left(\boldsymbol{r},t\right)}{\partial t}$$

Equations of motion for fields

$$\nabla \times \boldsymbol{E}\left(\boldsymbol{r},t\right) = -\frac{\partial \boldsymbol{B}\left(\boldsymbol{r},t\right)}{\partial t}$$

$$\nabla \cdot \boldsymbol{B}\left(\boldsymbol{r},t\right) = 0$$

$$\nabla \cdot \boldsymbol{D}\left(\boldsymbol{r},t\right) = \rho\left(\boldsymbol{r},t\right)$$

Equation of motion for particles

$$\boldsymbol{f}\left(\boldsymbol{r},t\right) = \rho\left(\boldsymbol{r},t\right)\boldsymbol{E}\left(\boldsymbol{r},t\right) + \boldsymbol{J}\left(\boldsymbol{r},t\right) \times \boldsymbol{B}\left(\boldsymbol{r},t\right)$$

Interaction with materials

$$\boldsymbol{D}\left(\boldsymbol{r},t\right) = \varepsilon_0 \boldsymbol{E}\left(\boldsymbol{r},t\right) + \boldsymbol{P}\left(\boldsymbol{r},t\right)$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right) = \mu_0 \left(\boldsymbol{H}\left(\boldsymbol{r},t\right) + \boldsymbol{M}\left(\boldsymbol{r},t\right)\right)$$

Absolute majority of things happening around you is described by these equations

Boundary Conditions

$$n(r) \times \left[E_1(r,t) - E_2(r,t) \right] = 0$$

$$n(r) \times \left[H_1(r,t) - H_2(r,t) \right] = K(r,t)$$

Normal pointing to region (1)

$$n(r) \cdot \left[B_1(r,t) - B_2(r,t) \right] = 0$$

$$n(r) \cdot \left[D_1(r,t) - D_2(r,t) \right] = \sigma(r,t)$$

Electromagnetic Potentials

Lorentz('s) calibration

$$\nabla \cdot \boldsymbol{A}\big(\boldsymbol{r},t\big) = -\sigma\mu\varphi\big(\boldsymbol{r},t\big) - \varepsilon\mu\frac{\partial\varphi\big(\boldsymbol{r},t\big)}{\partial t}$$

$$\boldsymbol{B}\big(\boldsymbol{r},t\big) = \nabla \times \boldsymbol{A}\big(\boldsymbol{r},t\big)$$

$$\boldsymbol{E}\big(\boldsymbol{r},t\big) = -\nabla\varphi\big(\boldsymbol{r},t\big) - \frac{\partial\boldsymbol{A}\big(\boldsymbol{r},t\big)}{\partial t}$$

Wave Equation

$$\Delta \boldsymbol{A}\left(\boldsymbol{r},t\right) - \sigma\mu\frac{\partial \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t} - \varepsilon\mu\frac{\partial^2 \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t^2} = -\mu\boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)$$

Material parameters are assumed independent of coordinates

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Poynting('s)-Umov('s) Theorem

Power passing the
bounding envelope

Energy storage

$$-\int_V \boldsymbol{E} \cdot \boldsymbol{J}_{\text{source}} \, \mathrm{d}V = \oint_S \left(\boldsymbol{E} \times \boldsymbol{H}\right) \cdot \mathrm{d}\boldsymbol{S} + \int_V \sigma \left|\boldsymbol{E}\right|^2 \mathrm{d}V + \frac{1}{2}\frac{\partial}{\partial t} \int_V \left(\varepsilon \left|\boldsymbol{E}\right|^2 + \mu \left|\boldsymbol{H}\right|^2\right) \mathrm{d}V$$

Power supplied
by sources

Heat losses

Energy balance in an electromagnetic system

Linear Momentum Carried by Fields

Volume integration considerably change
the meaning of Poynting('s) vector

$$\boldsymbol{p} = \frac{1}{c_0^2} \int_V \left(\boldsymbol{E} \times \boldsymbol{H} \right) \mathrm{d}\,V$$

This formula is only valid in vacuum. In material media things are more tricky.

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Angular Momentum Carried by Fields

$$\boldsymbol{L} = \frac{1}{c_0^2} \int_V \boldsymbol{r} \times \left(\boldsymbol{E} \times \boldsymbol{H} \right) \mathrm{d}V$$

This formula is only valid in vacuum. In material media things are more tricky.

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Frequency Domain

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) \in \mathbb{R} \qquad\qquad\qquad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \in \mathbb{C}$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \mathrm{e}^{\mathrm{j}\omega t}\,d\omega \qquad \longleftrightarrow \qquad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) = \int\limits_{-\infty}^{\infty} \boldsymbol{F}\left(\boldsymbol{r},t\right) \mathrm{e}^{-\mathrm{j}\omega t}\,dt$$

$$\frac{\partial \boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial t} \leftrightarrow \mathrm{j}\omega \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)$$

Spatial derivatives are untouched

Time derivatives reduce to algebraic multiplication

$$\frac{\partial \boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial r_{\xi}} \leftrightarrow \frac{\partial \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)}{\partial r_{\xi}}$$

Frequency domain helps us to remove explicit time derivatives

Phasors

$$\hat{\boldsymbol{F}}\left(\boldsymbol{r},-\omega\right) = \hat{\boldsymbol{F}}^{*}\left(\boldsymbol{r},\omega\right) \qquad \Longrightarrow \qquad \boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{\pi}\int\limits_{0}^{\infty}\mathrm{Re}\!\left[\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\right]d\omega$$

Reduced frequency domain representation

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Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right) + \mathrm{j}\omega\varepsilon\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \times \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = -\mathrm{j}\omega\mu\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \cdot \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = 0$$

$$\nabla \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \frac{\hat{\rho}\left(\boldsymbol{r},\omega\right)}{\varepsilon}$$

We assume linearity of material relations

Wave Equation – Frequency Domain

$$\Delta \hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) - \mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)\hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) = -\mu\hat{\boldsymbol{J}}_{\mathrm{source}}\left(\boldsymbol{r},\omega\right)$$

Helmholtz('s) equation

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Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \left\langle \boldsymbol{E} \cdot \boldsymbol{J}_{\text{source}} \right\rangle \mathrm{d}V = \oint_S \left\langle \boldsymbol{E} \times \boldsymbol{H} \right\rangle \cdot \mathrm{d}\boldsymbol{S} + \int_V \left\langle \sigma \left|\boldsymbol{E}\right|^2 \right\rangle \mathrm{d}V$$

$$-\frac{1}{2}\int_V \mathrm{Re}\!\left[\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{J}}^*_{\text{source}}\right] \mathrm{d}V = \frac{1}{2}\oint_S \mathrm{Re}\!\left[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*\right] \cdot \mathrm{d}\boldsymbol{S} + \frac{1}{2}\int_V \sigma \left|\hat{\boldsymbol{E}}\right|^2 \mathrm{d}V$$

Valid for time-harmonic steady state

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Plane Wave

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_0\left(\omega\right)\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \frac{k}{\omega\mu}\left[\boldsymbol{n}\times\boldsymbol{E}_0\left(\omega\right)\right]\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\boldsymbol{n}\cdot\boldsymbol{E}_0\left(\omega\right) = 0$$

$$\boldsymbol{n}\cdot\boldsymbol{H}_0\left(\omega\right) = 0$$

$$k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$$

Wave-number

The simplest wave solution of Maxwell('s) equations

Plane Wave Characteristics

$$k = \sqrt{-\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)}$$

$$\mathrm{Re}\big[k\big] > 0; \ \mathrm{Im}\big[k\big] < 0$$

$$\lambda = \frac{2\pi}{\mathrm{Re}\big[k\big]}$$

$$v_{\mathrm{f}} = \frac{\omega}{\mathrm{Re}\big[k\big]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\mathrm{Im}\big[k\big]}$$

Vacuum

General isotropic material

$$k = \frac{\omega}{c_0}$$

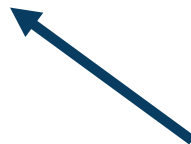$$\mathrm{Re}\big[k\big] > 0; \ \mathrm{Im}\big[k\big] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_{\mathrm{f}} = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \ \Omega$$

$$\delta \to \infty$$

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Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\left\langle \boldsymbol{E}\left(\boldsymbol{r},t\right) \times \boldsymbol{H}\left(\boldsymbol{r},t\right) \right\rangle = \frac{1}{2} \frac{\mathrm{Re}\left[k\right]}{\omega\mu} \left| \boldsymbol{E}_0\left(\omega\right) \right|^2 \mathrm{e}^{2\,\mathrm{Im}\left[k\right]\boldsymbol{n}\cdot\boldsymbol{r}} \, \boldsymbol{n}$$

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