Electromagnetic Field Theory 1
(fundamental relations and definitions)

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Ver. 2019/09/23
A specified distribution of elementary charges is in a state of arbitrary (but known) motion. At certain time we pick one of them and ask what is the force acting on it.

Rather difficult question – will not be fully answered
Elementary Charge

As far as we know, all charges in nature have values \( \pm Ne, N \in \mathbb{Z} \)

\[ e = 1.602176634 \times 10^{-19} \text{ C} \]
Amount of charge is conserved in every frame (even non-inertial).

Neutrality of atoms has been verified to 20 digits
Continuous approximation of charge distribution

Net charge \([C]\)

Volumetric density of charge \([C \cdot m^{-3}]\)

Surface density of charge \([C \cdot m^{-2}]\)

Line density of charge \([C \cdot m^{-1}]\)

Net charge allows for using powerful mathematics
Fundamental Question of Electrostatics

There exist a specified distribution of static elementary charges. We pick one of them and ask what is the force acting on it.

This will be answered in full details
Coulomb(’s) Law

\[ F(r) = \frac{qq'(r - r')}{4\pi\varepsilon_0 |r - r'|^3} \]

Measuring charge \([C]\)

Source charge \([C]\)

Radius vector of the measuring charge \([m]\)

Radius vector of the source charge \([m]\)

Permittivity of vacuum \(\varepsilon_0 = \frac{1}{\mu_0 c_0^2} = 8.8541878128 \times 10^{-12} \text{ F} \cdot \text{m}^{-1}\)

Speed of light \(c_0 = 299792458 \text{ m} \cdot \text{s}^{-1}\)

Permeability of vacuum \(\mu_0 = 1.25663706212 \times 10^{-6} \text{ H} \cdot \text{m}^{-1}\)

Farad

Henry

Measuring charge \([C]\)

Source charge \([C]\)

Radius vector of the measuring charge \([m]\)

Radius vector of the source charge \([m]\)

Force on measuring charge \([N]\)
Coulomb(’s) Law + Superposition Principle

\[ F(r) = \frac{q}{4\pi\varepsilon_0} \sum_n \frac{q'_n (r - r'_n)}{|r - r'_n|^3} \]

Entire electrostatics can be deduced from this formula
**Electric Field**

\[ F(r) = qE(r) \]

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \sum_{n} q_n \left( \frac{r - r_n'}{|r - r_n'|^3} \right) \]

Intensity of electric field

\[
\text{[V} \cdot \text{m}^{-1}]\]

*Force is represented by field – entity generated by charges and permeating the space*
Continuous Distribution of Charge

Continuous description of charge allows for using powerful mathematics.

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \sum_n q_n' \frac{(r - r_n')}{|r - r_n'|^3} \quad \rightarrow \quad E(r) = \frac{1}{4\pi \varepsilon_0} \int_{V'} \frac{\rho(r') (r - r')}{|r - r'|^3} \, dV' \]
Continuous Description of a Point Charge

Dirac's delta “function”

\[ \rho(r) = \sum_n q_n \delta(r - r_n) \]

Defining property of Dirac's delta “function”

\[ F(r_n) = \int_V F(r) \delta(r - r_n) \, dV \]
**Gauss(’) Law**

\[
\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}
\]

\[
\int_{S} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{V} \rho(\mathbf{r}) \, dV = \frac{Q}{\varepsilon_0}
\]

- **Differential law** (local)
- **Integral law** (global)
- Mind the orientation of the surface
- Total charge enclosed by the surface \([C']\)
Rotation of Electric Field

\[ \nabla \times \mathbf{E} = 0 \]

Differential law (local)

\[ \oint \mathbf{E} \cdot dl = 0 \]

Integral law (global)
Various Views on Electrostatics

Integral laws of electrostatics

\[ \oint_S E \cdot dS = \frac{Q}{\varepsilon_0} \]

\[ \oint_l E \cdot dl = 0 \]

Differential laws of electrostatics

\[ \nabla \cdot E(r) = \frac{\rho(r)}{\varepsilon_0} \]

\[ \nabla \times E = 0 \]

Coulomb’s law

\[ E(r) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(r')}{|r - r'|^3} dV' \]

The physics content is the same, the formalism is different.
Electric potential

\[ \nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = -\nabla \varphi(\mathbf{r}) \quad \Rightarrow \quad \varphi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' + K \]

Scalar description of electrostatic field

Defined up to arbitrary constant
Voltage

Potential difference is a unique number

Voltage \[ V \]

Work necessary to take charge \( q \) from point A to point B

\[
W = - \int_{A}^{B} \mathbf{F} \cdot d\mathbf{l} = qU
\]

\[
- \int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} = \phi(B) - \phi(A) = U
\]

Voltage represents connection of abstract field theory with experiments
Electrostatic Energy

Energy is carried by charges

\[ W = \frac{1}{2} \int_V \rho(r) \varphi(r) \, dV \]

Energy is carried by charges and fields

\[ W = \frac{1}{2} \varepsilon_0 \int_{V'} \left| E(r) \right|^2 \, dV' \]

Energy is carried by fields

\[ W = \frac{1}{2} \varepsilon_0 \int_{V'} \left| E(r) \right|^2 \, dV' \]

\[ W = \frac{1}{8\pi\varepsilon_0} \sum_{i,j} \frac{q_i q_j}{|r_i - r_j|} \]

\[ W = \frac{1}{8\pi\varepsilon_0} \int_V \int_{V'} \frac{\rho(r) \rho(r')}{|r - r'|} \, dV' dV \]

*Be careful with point charges (self-energy)*
Electrostatic Energy vs Force

Energy of a system of point charges

\[ W = \frac{1}{8 \pi \varepsilon_0} \sum_{i,j \neq i}^{i,j} \frac{q_i q_j}{|r_i - r_j|} \]

Coulomb’s law

\[ F\left(\mathbf{r}_\xi\right) = -\nabla_\xi W = \frac{q_\xi}{4 \pi \varepsilon_0} \sum_{j \neq \xi} q_j \frac{\left(\mathbf{r}_\xi - \mathbf{r}_j\right)}{||\mathbf{r}_\xi - \mathbf{r}_j||^3} \]

Electrostatic forces are always acting so as to minimize energy of the system
Electric Stress Tensor

Total electric force acting in a volume

Stress tensor

\[
\mathbf{F} = \int_V \rho(\mathbf{r}) \mathbf{E}(\mathbf{r}) \, dV = \varepsilon_0 \int_S \mathbf{T} \cdot d\mathbf{S}
\]

\[
\mathbf{T} = \mathbf{E} \mathbf{E} - \frac{1}{2} \mathbf{I} |\mathbf{E}|^2
\]

All the information on the volumetric Coulomb's force is contained at the boundary
Ideal Conductor – classical description

Ideal conductor contains unlimited amount of free charges which under action of external electric field rearrange so as to annihilate electric field inside the conductor.

In 3D, the free charge always resides on the external bounding surface of the conductor.

Generally, free charges in conductors move so as to minimize the energy.
Ideal Conductor – quantum description

In an ideal conductor, wave functions of electrons in outer shells perceive flat potential background. In reaction to an external electric field, these wave functions are slightly modified so as to provide zero average charge density inside the conductor. Due to flat potential background, there is no counter interaction.

*Long-range transport of charge does not truly happen in a solid conductor*
### Boundary Conditions on Ideal Conductor

- **Inside conductor**
  
  \[ \mathbf{E}(r) = 0 \quad \Leftrightarrow \quad \varphi(r) = \text{const.} \]

- **Just outside conductor**
  
  1. \[ \mathbf{n}(r) \times \mathbf{E}(r) = 0 \quad \Leftrightarrow \quad \varphi(r) = \text{const.} \]
  2. \[ \mathbf{n}(r) \cdot \mathbf{E}(r) = \frac{\sigma}{\varepsilon_0} \quad \Leftrightarrow \quad \frac{\partial \varphi(r)}{\partial n} = -\frac{\sigma}{\varepsilon_0} \]

- Potential is continuous across the boundary
- Surface charge residing on the outer surface of the conductor
- Outward normal to the conductor
- Normal derivative
Capacitance of a System of $N$ conductors

Electrostatic system is fully characterized by capacitances (we know the energy)

$$Q_i = \sum_j C_{ij} \varphi_j$$

$$W = \frac{1}{2} \sum_{i,j} C_{ij} \varphi_j \varphi_i$$

Self and mutual capacitances

Capacitances depend solely on geometry and position of conductors
Capacitance of a System of two conductors

Capacitance: $Q = CU$

Potential difference between conductors

Charge on positively charged conductor

$W = \frac{1}{2} CU^2$
Poisson(’s) equation

\[ \Delta \varphi (\mathbf{r}) = -\frac{\rho (\mathbf{r})}{\varepsilon_0} \]

The solution to Poisson’s equation is unique in a given volume once the potential is known on its bounding surface and the charge density is known throughout the volume.
The solution to Laplace’s equation is unique in a given volume once the potential is known on its bounding surface.
Mean Value Theorem

\[ \varphi \left( r_{\text{center}} \right) = \frac{1}{4\pi R^2} \int_\text{sphere} \varphi \left( r \right) \, dS \]

The solution to Laplace’s equation posses neither maxima nor minima inside the solved volume.
Mind that the solution to Laplace’s equation possess neither maxima nor minima inside the solved volume. This means that charged particle will always travel towards the boundary.
Image Method

When solving field generated by charges in the presence of conductors, it is sometimes possible to remove the conductor and mimic its boundary conditions by adding extra charges to the exterior of the solution volume. The uniqueness theorem claims that this is a correct solution.

*Image method always works with planes and spheres.*
Separation of Variables

\[ \Delta \varphi(r) = 0 \quad \rightarrow \quad \varphi_{ijk}(r) = X_i(x)Y_j(y)Z_k(z) \quad \rightarrow \quad \varphi(r) = \sum_{ijk} C_{ijk} \varphi_{ijk}(r) \]

Semi-analytical method for canonical problems

Constants determined by boundary conditions
Finite Differences

\[ \varphi(x + h, y, z) \rightarrow \varphi_{(i+1)jk} \]

\[
\Delta \varphi(r) \approx \frac{\varphi_{(i+1)jk} - 2\varphi_{ijk} + \varphi_{(i-1)jk}}{h^2} + \frac{\varphi_{(j+1)k} - 2\varphi_{ijk} + \varphi_{(j-1)k}}{h^2} + \frac{\varphi_{(k+1)} - 2\varphi_{ijk} + \varphi_{ijk(k-1)}}{h^2}
\]

\[ \Delta \varphi(r) = 0 \quad \Rightarrow \quad \varphi_{ijk} = \frac{\varphi_{(i+1)jk} + \varphi_{(i-1)jk} + \varphi_{(j+1)k} + \varphi_{(j-1)k} + \varphi_{ij(k+1)} + \varphi_{ij(k-1)}}{6} \]

Approximation by a system of linear algebraic equations

Mind the mean value theorem

Powerful numerical method for closed problems
Method of Moments

\[ \varphi(r) = \frac{1}{4\pi\varepsilon_0} \int V \frac{\rho(r')}{|r - r'|} \, dV' \]

\[ \rho(r) \approx \sum_n \alpha_n \rho_n(r) \]

\[ \int_V \rho_m(r) \varphi(r) \, dV = \sum_n \alpha_n \frac{1}{4\pi\varepsilon_0} \int_V \int_{V'} \rho_m(r) \rho_n(r') \frac{1}{|r - r'|} \, dV' \, dV \]

Assumed to be known in volume where the charge resides

Distribution of charge is unknown

Simple functions for which the potential integral can be easily evaluated

Known

Simple functions for which the potential integral can be easily evaluated

Known

Approximation by a system of linear algebraic equations

Powerful numerical method for open problems
Dielectrics

- Material in which charges cannot move freely
- Charges are forming clusters (atoms, molecules)
- Under influence of electric field the clusters change shape or rotate
- Electric field induces electric dipoles with density $P(r)$ $[C \cdot m^{-2}]$
Electric Field of a Dipole

Two opposite charges very close to each other

\[ \mathbf{r} - \mathbf{r}_{\text{center}} \gg |\mathbf{r}_1 - \mathbf{r}_2| \]

General formula

\[
\varphi(r) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{|\mathbf{r} - \mathbf{r}_1|} - \frac{q}{|\mathbf{r} - \mathbf{r}_2|} \right) \approx \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_{\text{center}})}{|\mathbf{r} - \mathbf{r}_{\text{center}}|^3}
\]

Formula for two opposite charges

\[ \mathbf{p} = q (\mathbf{r}_1 - \mathbf{r}_2) = \int_V \mathbf{r} \rho(\mathbf{r}) \, dV \]

Electric dipole moment

\[ [C \cdot m] \]
Field Produced by Polarized Matter

\[ \varphi(r) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{P(r') \cdot (r - r')}{|r - r'|^3} \, dV' = \frac{1}{4\pi\varepsilon_0} \oint_{S'} \frac{P(r')}{|r - r'|} \cdot dS' - \frac{1}{4\pi\varepsilon_0} \int_{V'} \nabla' \cdot \frac{P(r')}{|r - r'|} \, dV' \]

Only apply at infinitely sharp boundary (unrealistic)
Potential of volumetric charge density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside.
**Electric Displacement**

\[ \nabla \cdot D(r) = \rho(r) \]

\[ D(r) = \varepsilon_0 E(r) + P(r) \]

\[ \oint_S D \cdot dS = \int_V \rho(r) \, dV = Q \]

Electric displacement \([C \cdot m^{-2}]\)

Only free charge
(compare to divergence
of electric field)
Linear Isotropic Dielectrics

Relative permittivity

\[
\varepsilon_r(r) = 1 + \chi_e(r)
\]

Electric susceptibility

\[
P(r) = \varepsilon_0 \chi_e(r) E(r)
\]

Permittivity

\[
D(r) = \varepsilon_0 \varepsilon_r(r) E(r) = \varepsilon(r) E(r)
\]

All the complicated structure of matter reduces to a simple scalar quantity
Fields in Presence of Dielectrics 1/2

Analogy with electric field in vacuum can only be used when entire space is homogeneously filled with dielectric.

\[ \nabla \times D(r) = \nabla \times [\varepsilon(r)E(r)] \neq 0 \]

Inequality is due to boundaries.

Analogy with vacuum can only be used when space is homogeneously filled with dielectric.
ELECTROSTATICS

Fields in Presence of Dielectrics 2/2

\[ \nabla \times \mathbf{E}(\mathbf{r}) = 0 \iff \mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r}) \quad \Rightarrow \quad \nabla \cdot \left[ \varepsilon(\mathbf{r}) \nabla \phi(\mathbf{r}) \right] = -\rho(\mathbf{r}) \]

\[ \Delta \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon} \]

Not a function of coordinates

Poisson’s equation holds only when permittivity does not depend on coordinates
Dielectric Boundaries

\[ \mathbf{n}(\mathbf{r}) \times \left[ \mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r}) \right] = 0 \quad \Leftrightarrow \quad \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r}) = 0 \]

\[ \mathbf{n}(\mathbf{r}) \cdot \left[ \epsilon_1 \mathbf{E}_1(\mathbf{r}) - \epsilon_2 \mathbf{E}_2(\mathbf{r}) \right] = \sigma(\mathbf{r}) \quad \Leftrightarrow \quad \epsilon_1 \frac{\partial \varphi_1(\mathbf{r})}{\partial n} - \epsilon_2 \frac{\partial \varphi_2(\mathbf{r})}{\partial n} = -\sigma(\mathbf{r}) \]

Both conditions are needed for unique solution.
Electrostatic Energy in Dielectrics

\[ W = \frac{1}{2} \varepsilon_0 \int_V |\mathbf{E}(\mathbf{r})|^2 \, dV \]  \quad \Rightarrow \quad  \[ W = \frac{1}{2} \int_V \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) \, dV \]
Forces on Dielectrics

\[ W = \frac{1}{2} CU^2 = \frac{1}{2} \frac{Q^2}{C} \]

\[ W = \frac{1}{2} \int_V E(r) \cdot D(r) \, dV \]

\[ F(r_\xi) = -\nabla_\xi W \]

This only holds when charge is held constant.
Electric Current

Current density
\[ \text{A} \cdot \text{m}^{-2} \]

Charge
\[ \text{C} \]

Velocity of charge
\[ \text{m} \cdot \text{s}^{-1} \]

\[ J(r,t) = \sum_{k=1}^{N} q_k \delta(r - r_k(t)) v_k(t) \]

Volumetric density represented by Dirac delta
\[ \text{m}^{-3} \]

Charges in motion are represented by current density
Local Charge Conservation

\[ \nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \sum_{k=1}^{N} q_k \delta(\mathbf{r} - \mathbf{r}_k(t)) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t} \]

Charge is conserved locally at every space-time point.
Global Charge Conservation

When charge leaves a given volume, it is always accompanied by a current through the bounding envelope

\[ \oint_{\partial V} J(\mathbf{r}, t) \cdot d\mathbf{S} = -\frac{\partial Q(t)}{\partial t} \]

Charge can neither be created nor destroyed. It can only be displaced.
Stationary Current

There is no charge accumulation in stationary flow

\[ \nabla \cdot \mathbf{J}(\mathbf{r}) = 0 \quad \leftrightarrow \quad \oint_S \mathbf{J}(\mathbf{r}) \cdot d\mathbf{S} = 0 \]

When charge enters a volume, another must leave it without any delay.
Ohm(’s) Law

This simple linear relation holds for enormous interval of electric field strengths

\[ J(r) = \sigma(r) E(r) \]

Conductivity

\[ \left[ \text{S} \cdot \text{m}^{-1} \right] \]
Stationary flow of charges cannot be caused by electrostatic field. The motion forces are non-conservative, are called electromotive forces, and are commonly of chemical, magnetic or photoelectric origin.

\[ \oint E(\mathbf{r}) \cdot d\mathbf{l} \neq 0 \]

For curves passing through sources of electromotive force

\[ \oint E(\mathbf{r}) \cdot d\mathbf{l} = 0 \]

For curves not crossing sources of electromotive force
Boundary Conditions for Stationary Current

\[ n(r) \times [E_1(r) - E_2(r)] = 0 \quad \Leftrightarrow \quad \varphi_1(r) - \varphi_2(r) = 0 \]

\[ n(r) \cdot [\varepsilon_1 E_1(r) - \varepsilon_2 E_2(r)] = \sigma(r) \quad \Leftrightarrow \quad \varepsilon_1 \frac{\partial \varphi_1(r)}{\partial n} - \varepsilon_2 \frac{\partial \varphi_2(r)}{\partial n} = -\sigma(r) \]

\[ n(r) \cdot [\sigma_1 E_1(r) - \sigma_2 E_2(r)] = 0 \quad \Leftrightarrow \quad \sigma_1 \frac{\partial \varphi_1(r)}{\partial n} - \sigma_2 \frac{\partial \varphi_2(r)}{\partial n} = 0 \]

Charge conservation forces the continuity of current across the boundary.
**Electric Current**

Existence of high contrast in conductivity between conductors and dielectrics allows for well defined current paths.

\[
I = \int_{S} J(r) \cdot dS
\]

Current [A]

Cross-section of current path \([m^2]\)
Resistance (Conductance)

Potential difference (voltage)

\[ [V] \]

Resistance

\[ \left[ \Omega \right] \]

Current

\[ [A] \]

Conductance

\[ [S] \]

\[ U = RI \]

\[ I = GU \]

\[ R = \frac{1}{G} = \frac{L}{\sigma S} \]

Resistance of a cylinder homogeneous cylinder of conductive material

Length along current path

\[ [m] \]

Cross-section of current path

\[ [m^2] \]
Resistive Circuits and Kirchhoff(’s) Laws

Kirchhoff’s laws are a consequence of electrostatics and law’s of stationary current flow

In a loop
\[ \sum U_i = U_{\text{electromotive}} \]

On a resistor
\[ U_i = R_i I_i \]

At a junction
\[ \sum I_i = 0 \]
Joule’s Heat

Power lost via conduction

\[ P = \int V \cdot J \, dV = \int \sigma \left| E(r) \right|^2 \, dV \]

Power lost on resistor

\[ P = UI = RI^2 = \frac{U^2}{R} \]

Electric field within conducting material produce heat.
There exist a specified distribution of stationary current. We pick a differential volume of it and ask what is the force acting on it.
Biot-Savart(‘s) Law

\[ \mathbf{F}(r) = \frac{\mu_0}{4\pi} \int \mathbf{J}(r) \, dV \times \left[ \mathbf{J}(r') \, dV' \times (\mathbf{r} - \mathbf{r}') \right] \left/ |\mathbf{r} - \mathbf{r}'|^3 \right. \]

- \( \mu_0 = 4\pi \cdot 10^{-7} \text{ H} \cdot \text{m}^{-1} \)
- Measuring current element \([\mathbf{A}]\)
- Source current element \([\mathbf{A}]\)
- Permeability of vacuum
- Force on measuring current \([\mathbf{N}]\)
- Radius vector of the measuring current \([\mathbf{m}]\)
- Radius vector of the source current \([\mathbf{m}]\)
- Measuring current element
- Source current element
Biot-Savart’s Law + Superposition Principle

\[ F(\mathbf{r}) = J(\mathbf{r}) \, dV \times \frac{\mu_0}{4\pi} \int_{V'} \frac{J(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \]

Entire magnetostatics can be deduced from this formula
Magnetic Field

\[ F(\mathbf{r}) = J(\mathbf{r}) \, dV \times B(\mathbf{r}) \]

\[ B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \]

Magnetic field (Magnetic induction) \[ [T] \]
Divergence of Magnetic Field

\[ \nabla \cdot B(r) = 0 \quad \iff \quad \oint_S B(r) \cdot dS = 0 \]

There are no point sources of magnetostatic field
Curl of Magnetic Field – Ampere’s Law

\[ \nabla \times B(r) = \mu_0 J(r) \]

Total current captured within the curve
\[ \oint_C B(r) \cdot dl = \mu_0 I \]
Magnetic Vector Potential

\[ \nabla \cdot B = 0 \quad \Rightarrow \quad B(r) = \nabla \times A(r) \quad \Rightarrow \quad A(r) = \frac{\mu_0}{4\pi} \int_V \frac{J(r')}{|r - r'|} \, dV' + \nabla \psi(r) \]

Reduction description of magnetostatic field

Magnetic vector potential

Defined up to arbitrary scalar function
Poisson’s equation

\[ \Delta A(r) = -\mu_0 J(r) \]

The solution to Poisson’s equation is unique in a given volume once the potential is known on its bounding surface and the current density is known throughout the volume.
Boundary Conditions

\[ n(r) \times [B_1(r) - B_2(r)] = \mu_0 K(r) \]

\[ n(r) \cdot [B_1(r) - B_2(r)] = 0 \]

\[ A_1(r) - A_2(r) = 0 \]
Magnetostatic Energy

\[ W = \frac{1}{2} \int_V \mathbf{A}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \, dV \]

\[ W = \frac{1}{2\mu_0} \int_V |\mathbf{B}(\mathbf{r})|^2 \, dV \]

For now it is just a formula that works – it must be derived with the help of time varying fields.
Magnetostatic Energy – Current Circuits

\[ M_{ij} = M_{ji} = \frac{\mu_0}{4\pi} \int_{V_j} \int_{V_j'} \frac{J_j(r_j) \cdot J_i(r_i')}{|r_j - r_i'|} \, dV_i' \, dV_j \]

Mutual-Inductance \([H]\)

\[ W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j \]

Self-Inductance \([H]\)

\[ L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} \int_{V_i'} \frac{J_i(r_i) \cdot J_i(r_i')}{|r_i - r_i'|} \, dV_i' \, dV_i \]
Mutual Inductance – Thin Current Loop

\[ \Phi_{ji} = \int_{S_j} B_i(r_j) \cdot dS_j \]

Magnetic flux induced by \( i \)-th current through \( j \)-th current

\[ M_{ij} = \frac{\Phi_{ji}}{I_i} \]
Magnetic Materials

- Material response is due to magnetic dipole moments.
- Magnetic moment comes from spin or orbital motion of an electron.
- Magnetic field tends to align magnetic moments.
- Magnetic field induces magnetic dipoles with density $M(r)$ [A · m$^{-1}$].

Number of dipoles in unitary volume
Magnetic Field of a Dipole

Magnetic dipole approximates infinitesimally small current loop

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}
\]

\[
B(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\mathbf{r} (\mathbf{r} \cdot \mathbf{m})}{r^5} - \frac{\mathbf{m}}{r^3} \right]
\]

\[
m = \frac{1}{2} \int_V \mathbf{r} \times \mathbf{J}(\mathbf{r}) \, dV
\]

Dipole is assumed at the origin

\[ r \neq 0 \]
Field Produced by Magnetized Matter

\[ A(r) = \frac{\mu_0}{4\pi} \int_{V'} \frac{M(r') \times (r - r')}{|r - r'|^3} \, dV' = \frac{\mu_0}{4\pi} \oint_{S'} \frac{M(r') \times dS'}{|r - r'|} + \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times M(r')}{|r - r'|} \, dV' \]

Only applies at infinitely sharp boundary (unrealistic)

Potential of volumetric current density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside
Magnetic Intensity

\[ \nabla \times H(r) = J(r) \]

\[ B(r) = \mu_0 \left( H(r) + M(r) \right) \]

\[ \oint_{C} H(r) \cdot dl = I \]

Only free current

Magnetic Intensity \([\text{A} \cdot \text{m}^{-1}]\)
Linear Isotropic Magnetic Materials

\[ M(r) = \chi_m(r) H(r) \]

Relative permeability

\[ \mu_r(r) = 1 + \chi_m(r) \]

\[ B(r) = \mu_0 \mu_r(r) H(r) = \mu(r) H(r) \]

Magnetic susceptibility

Permeability

\[ [H \cdot m^{-1}] \]

All the complicated structure of matter reduces to a simple scalar quantity
**Fields in Presence of Magnetic Material**

\[ \nabla \cdot B(r) = 0 \Leftrightarrow B(r) = \nabla \times A(r) \quad \Rightarrow \quad \nabla \times \left[ \frac{1}{\mu(r)} \nabla \times A(r) \right] = J(r) \]

\[ \Delta A(r) = -\mu J(r) \]

\[ \nabla \cdot A(r) = 0 \quad \text{Coulomb’s gauge} \\
\text{Not a function of coordinates} \]

*Poisson’s equation holds only when permittivity does not depend on coordinates*
Magnetic Material Boundaries

\[ n(r) \times [H_1(r) - H_2(r)] = K(r) \]

\[ n(r) \cdot [\mu_1 H_1(r) - \mu_2 H_2(r)] = 0 \]

Both conditions are needed for unique solution
Magnetostatic Energy in Magnetic Material

\[
W = \frac{1}{2\mu_0} \int_V \left| B(r) \right|^2 \, dV 
\]

\[
W = \frac{1}{2} \int_V H(r) \cdot B(r) \, dV
\]
Magnetic Materials

- **Paramagnetic** – small positive susceptibility (small attraction – linear)
- **Diamagnetic** – small negative susceptibility (small repulsion – linear)
- **Ferromagnetic** – “large positive susceptibility” (large attraction – nonlinear)
Ferromagnetic Materials

- **Spins** are ordered within **domains**
- **Magnetization** is a **non-linear** function of field intensity
- Magnetization curve – **Hysteresis, Remanence**
- **Susceptibility** can only be defined as **local approximation**
- Above **Curie(‘s) temperature** ferromagnetism disappears

*Exact calculations are very difficult – use simplified models (soft material, permanent magnet)*
Faraday(‘s) Law

Time variation in magnetic field produces electric field that tries to counter the change in magnetic flux (electromotive force)

\[ \oint E(r,t) \cdot dl = -\frac{\partial}{\partial t} \int B(r,t) \cdot dS \]

\[ \nabla \times E(r,t) = -\frac{\partial B(r,t)}{\partial t} \]
Lenz’s Law

The current created by time variation of magnetic flux is directed so as to oppose the flux creating it.
Time Varying RL Circuits

In a loop:

\[ \sum U_i(t) = U_{\text{electromotive}}(t) \]

At a junction:

\[ \sum I_i(t) = 0 \]

On a resistor:

\[ U_i(t) = R_i I_i(t) \]

On an inductor:

\[ U_1(t) = L_{11} \frac{\partial I_1(t)}{\partial t} + M_{12} \frac{\partial I_2(t)}{\partial t} \]

Circuit laws are valid as long as time variations are not too fast.
Time Varying Potentials

Potential calibration

$$\nabla \cdot A(r,t) = -\sigma \mu \varphi(r,t)$$

$$B(r,t) = \nabla \times A(r,t)$$

$$E(r,t) = -\nabla \varphi(r,t) - \frac{\partial A(r,t)}{\partial t}$$

In time varying fields scalar potential becomes redundant
Source and Induced Currents

\[ \nabla \times \mathbf{H}(r,t) = J_{\text{source}}(r,t) + J_{\text{induced}}(r,t) = J_{\text{source}}(r,t) + \sigma \mathbf{E}(r,t) \]

Those are fixed, not reacting to fields
Diffusion Equation

\[ \Delta A(r,t) - \sigma \mu \frac{\partial A(r,t)}{\partial t} = -\mu J_{\text{source}}(r,t) \]

\[ \Delta H(r,t) - \sigma \mu \frac{\partial H(r,t)}{\partial t} = -\nabla \times J_{\text{source}}(r,t) \]

\[ \Delta E(r,t) - \sigma \mu \frac{\partial E(r,t)}{\partial t} = \frac{1}{\varepsilon} \nabla \rho_{\text{source}}(r,t) + \mu \frac{\partial J_{\text{source}}(r,t)}{\partial t} \]

Material parameters are assumed independent of coordinates
Maxwell’s-Lorentz’s Equations

\[ \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \]

\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \]

\[ \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \]

\[ \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \]

\[ \mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \]

\[ \mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) \]

\[ \mathbf{B}(\mathbf{r}, t) = \mu_0 \left( \mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t) \right) \]

Absolute majority of things happening around you is described by these equations.
Boundary Conditions

\[
\mathbf{n}(\mathbf{r}) \times \left[ \mathbf{E}_1(\mathbf{r}, t) - \mathbf{E}_2(\mathbf{r}, t) \right] = 0
\]

\[
\mathbf{n}(\mathbf{r}) \times \left[ \mathbf{H}_1(\mathbf{r}, t) - \mathbf{H}_2(\mathbf{r}, t) \right] = \mathbf{K}(\mathbf{r}, t)
\]

\[
\mathbf{n}(\mathbf{r}) \cdot \left[ \mathbf{B}_1(\mathbf{r}, t) - \mathbf{B}_2(\mathbf{r}, t) \right] = 0
\]

\[
\mathbf{n}(\mathbf{r}) \cdot \left[ \mathbf{D}_1(\mathbf{r}, t) - \mathbf{D}_2(\mathbf{r}, t) \right] = \sigma(\mathbf{r}, t)
\]
Electromagnetic Potentials

\[ \nabla \cdot \mathbf{A}(r, t) = -\sigma \mu \phi(r, t) - \varepsilon \mu \frac{\partial \phi(r, t)}{\partial t} \]

\[ B(r, t) = \nabla \times \mathbf{A}(r, t) \]

\[ E(r, t) = -\nabla \phi(r, t) - \frac{\partial \mathbf{A}(r, t)}{\partial t} \]
Wave Equation

\[ \Delta A(r, t) - \sigma \mu \frac{\partial A(r, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 A(r, t)}{\partial t^2} = -\mu J_{\text{source}}(r, t) \]

Material parameters are assumed independent of coordinates.
Poynting(’s)-Umov(’s) Theorem

\[-\int_V E \cdot J_{source} \, dV = \oint_S (E \times H) \cdot dS + \int_V \sigma |E|^2 \, dV + \frac{1}{2} \frac{\partial}{\partial t} \int_V \left( \varepsilon |E|^2 + \mu |H|^2 \right) \, dV\]

Energy balance in an electromagnetic system
Linear Momentum Carried by Fields

Volume integration considerably change the meaning of Poynting’s vector

\[ p = \frac{1}{c^2} \int_V (E \times H) \, dV \]

This formula is only valid in vacuum. In material media things are more tricky.
Angular Momentum Carried by Fields

\[ L = \frac{1}{c^2} \int_V r \times (E \times H) \, dV \]

This formula is only valid in vacuum. In material media things are more tricky.
Frequency Domain

\[
F(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(r, \omega) e^{j\omega t} d\omega
\]

\[
\hat{F}(r, \omega) = \int_{-\infty}^{\infty} F(r, t) e^{-j\omega t} dt
\]

Time derivatives reduce to algebraic multiplication

Spatial derivatives are untouched

Frequency domain helps us to remove explicit time derivatives
Phasors

\[ \hat{F}(r, -\omega) = \hat{F}^*(r, \omega) \quad \rightarrow \quad F(r, t) = \frac{1}{\pi} \int_0^\infty \Re \left[ \hat{F}(r, \omega) e^{j\omega t} \right] d\omega \]
Maxwell(’s) Equations – Frequency Domain

\[ \nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{J}}(\mathbf{r}, \omega) + j\omega\varepsilon\hat{\mathbf{E}}(\mathbf{r}, \omega) \]

\[ \nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\mu\hat{\mathbf{H}}(\mathbf{r}, \omega) \]

\[ \nabla \cdot \hat{\mathbf{H}}(\mathbf{r}, \omega) = 0 \]

\[ \nabla \cdot \hat{\mathbf{E}}(\mathbf{r}, \omega) = \frac{\hat{\rho}(\mathbf{r}, \omega)}{\varepsilon} \]

*We assume linearity of material relations*
Wave Equation – Frequency Domain

\[ \Delta \hat{A}(r, \omega) - j \omega \mu (\sigma + j \omega \varepsilon) \hat{A}(r, \omega) = -\mu \hat{J}_{\text{source}}(r, \omega) \]

Helmholtz('s) equation
Heat Balance in Time-Harmonic Steady State

\[- \int_V \langle \mathbf{E} \cdot \mathbf{J}_{\text{source}} \rangle \, dV = \oint_S \langle \mathbf{E} \times \mathbf{H} \rangle \cdot dS + \int_V \langle |\mathbf{E}|^2 \rangle \, dV \]

\[= \frac{1}{2} \int_V \text{Re} \left[ \hat{\mathbf{E}} \cdot \hat{\mathbf{J}}^*_{\text{source}} \right] \, dV = \frac{1}{2} \oint_S \text{Re} \left[ \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \right] \cdot dS + \frac{1}{2} \int_V \sigma |\hat{\mathbf{E}}|^2 \, dV \]

Valid for general periodic steady state

Cycle mean

Valid for time-harmonic steady state
Plane Wave

The simplest wave solution of Maxwell’s equations

\[
\hat{E}(r, \omega) = E_0(\omega) e^{-jkr}\]

\[
\hat{H}(r, \omega) = \frac{k}{\omega \mu} \left[ n \times E_0(\omega) \right] e^{-jkr}\]

\[
n \cdot E_0(\omega) = 0\]

\[
n \cdot H_0(\omega) = 0\]

\[
k^2 = -j\omega \mu(\sigma + j\omega \varepsilon)\]

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

Wave-number

The simplest wave solution of Maxwell(‘s) equations
Plane Wave Characteristics

For general isotropic material:

\[ k = \sqrt{-j\omega \mu (\sigma + j\omega \varepsilon)} \]

- \( \text{Re}[k] > 0; \text{Im}[k] < 0 \)

- \( \lambda = \frac{2\pi}{\text{Re}[k]} \)

- \( v_t = \frac{\omega}{\text{Re}[k]} \)

- \( Z = \frac{\omega \mu}{k} \)

- \( \delta = -\frac{1}{\text{Im}[k]} \)

For vacuum:

\[ k = \frac{\omega}{c_0} \]

- \( \text{Re}[k] > 0; \text{Im}[k] = 0 \)

- \( \lambda = \frac{c_0}{f} \)

- \( v_t = c_0 \)

- \( Z = c_0 \mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \, \Omega \)

- \( \delta \to \infty \)
Cycle Mean Power Density of a Plane Wave

\[
\langle \mathbf{E}(r, t) \times \mathbf{H}(r, t) \rangle = \frac{1}{2} \frac{\text{Re}[k]}{\omega \mu} |E_0(\omega)|^2 e^{2\text{Im}[k] \cdot \mathbf{n} \cdot \mathbf{r}}
\]

Power propagation coincides with phase propagation.