1.2 What is a Combinatorial Game? We now define the notion of a combinatorial game more precisely. It is a game that satisfies the following conditions.

(1) *There are two players.*

(2) *There is a set, usually finite, of possible positions of the game.*

(3) *The rules of the game specify for both players and each position which moves to other positions are legal moves. If the rules make no distinction between the players, that is if both players have the same options of moving from each position, the game is called* **impartial**; *otherwise, the game is called* **partizan**.

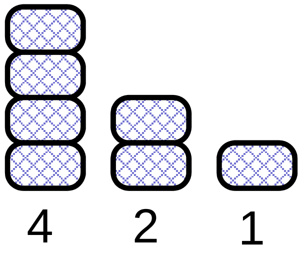
(4) *The players alternate moving.*

(5) *The game ends when a position is reached from which no moves are possible for the player whose turn it is to move. Under the* **normal play rule**, *the last player to move wins. Under the* **misère play rule** *the last player to move loses.*

If the game never ends, it is declared a draw. However, we shall nearly always add the following condition, called **the Ending Condition**. This eliminates the possibility of a draw.

(6) *The game ends in a finite number of moves no matter how it is played.*

Game example and analysis



Game rules

Three piles with 4, 2, 1, pegs(s).

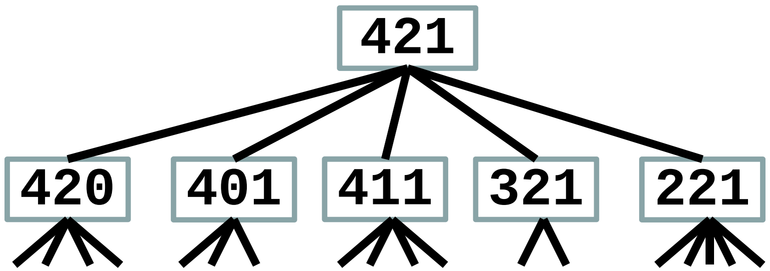
In one move, a player
can remove 1 or 2 pegs from any pile(s).

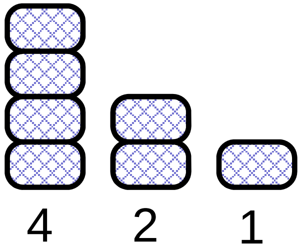
Player who removes the last peg wins.

Game representation

Represent the piles by a triple of integers, number of pegs in the piles, the initial state (position) is then [4,2,1].

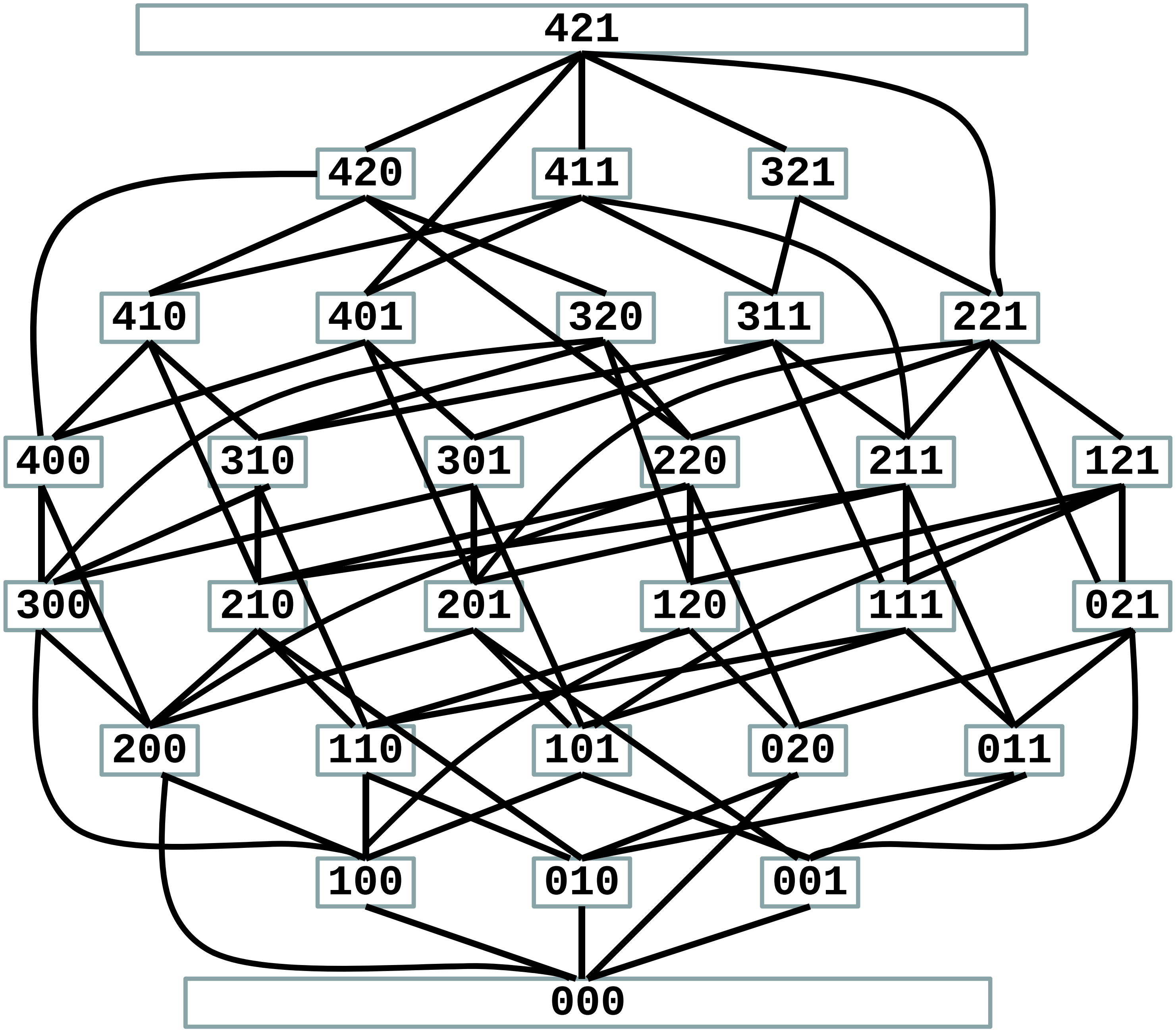
The states (positions) accesible in a single move are connected by (directed) edges.

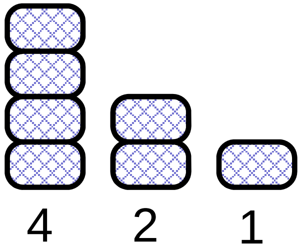




Each player can remove 1 or 2 pegs.

Player who removes the last peg wins.





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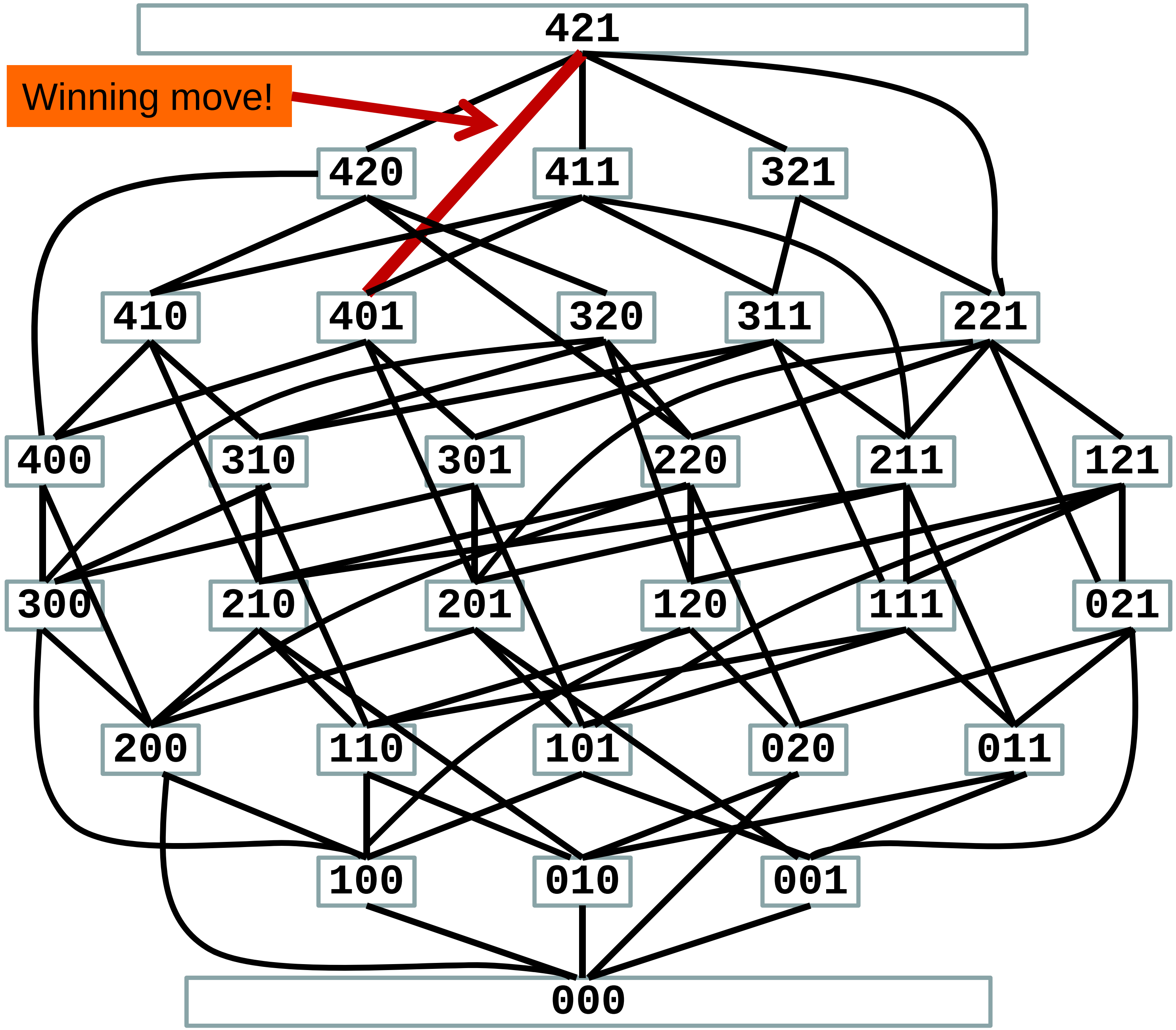
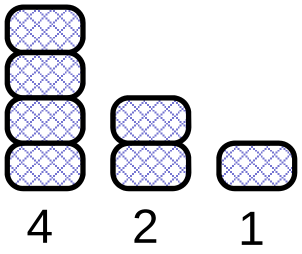


P positions are positions that are winning for the **P**revious player (the player who just moved to the position)

N positions are positions that are winning for the **N**ext player (the player who will move to some next position).



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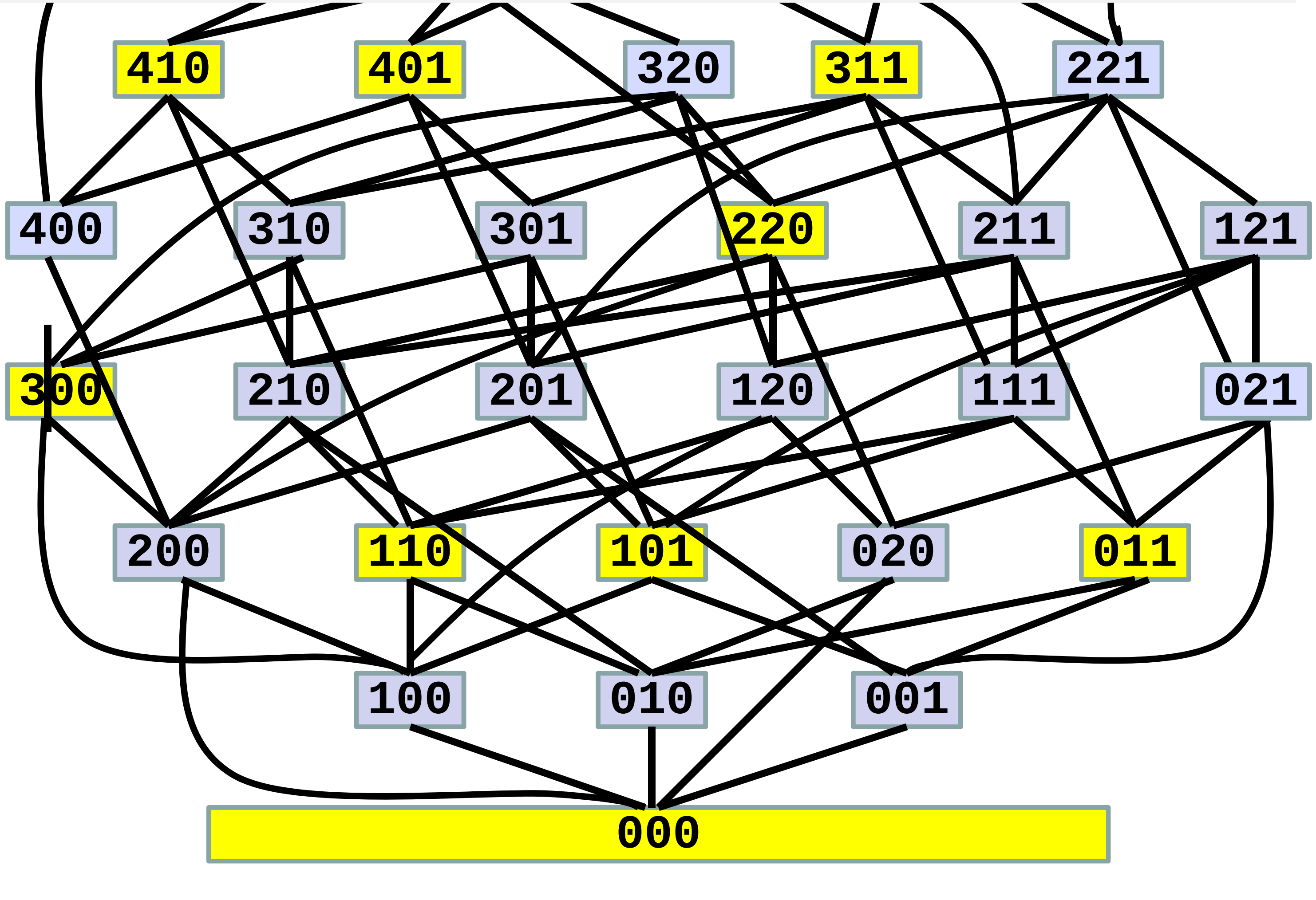
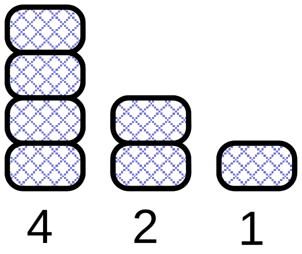


Characteristic Property. *P-positions and N-positions are defined recursively by the following three statements.*

(1) *All terminal positions are P-positions.*
(2) *From every N-position, there is at least one move to a P-position.*
(3) *From every P-position, every move is to an N-position.*



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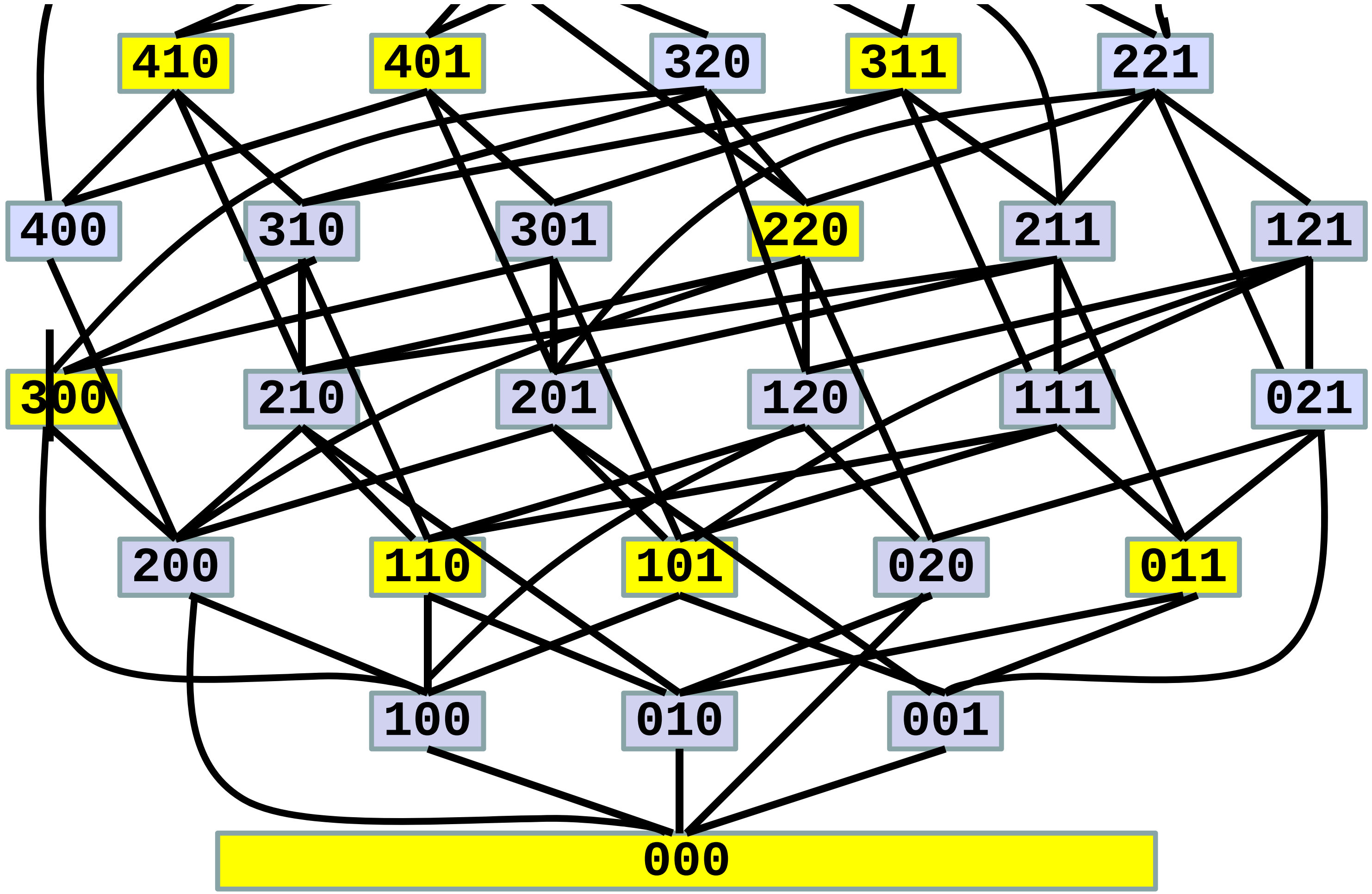


Determining P and N positions

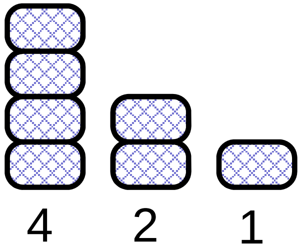
Step 1: Label every terminal position as a P-position.

Step 2: Label every position that can reach a labelled P-position in one move as an N-position.

Step 3: Find those positions whose only moves are to labelled N-positions; label such positions as P-positions.

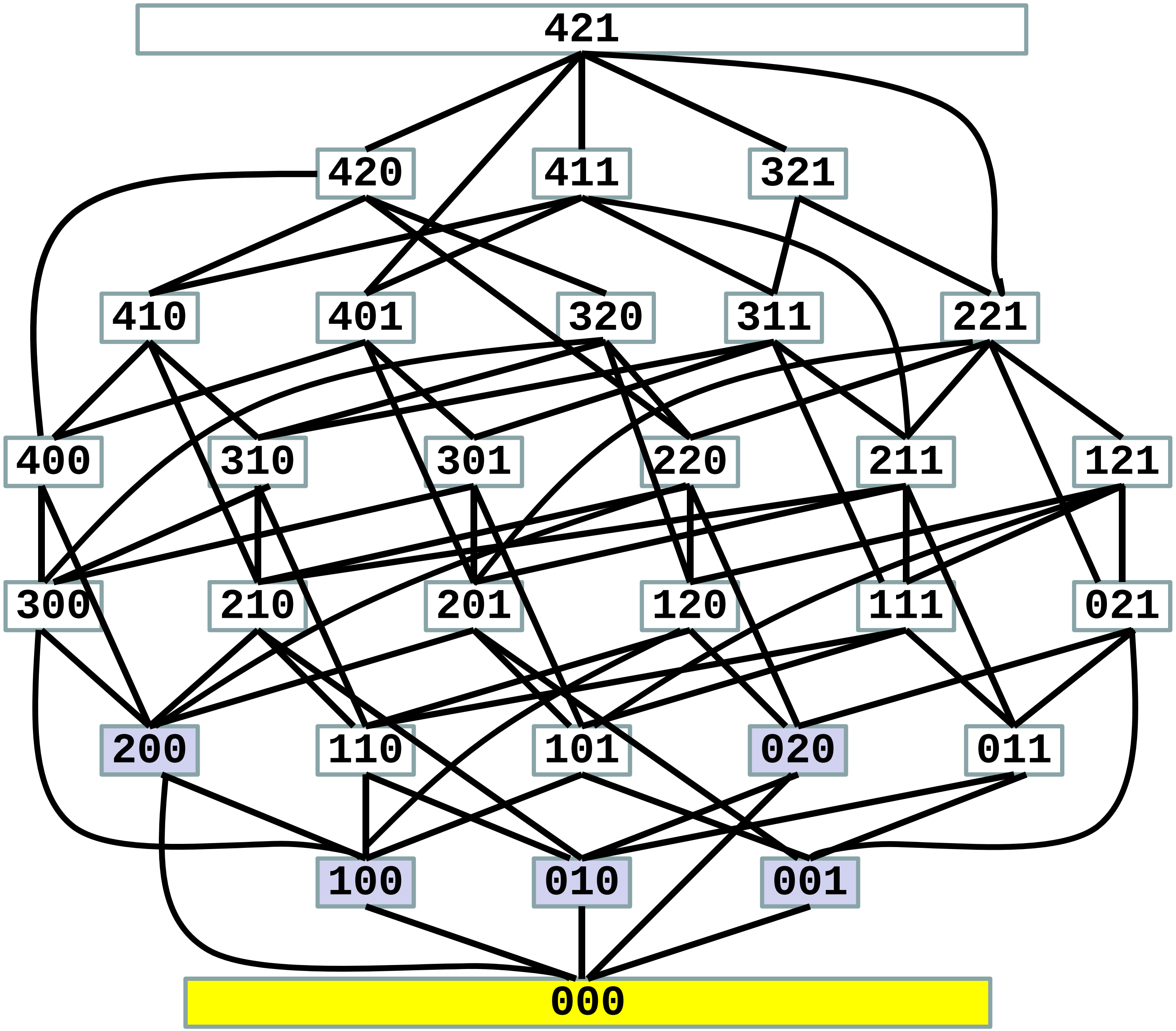
Step 4: If no new P-positions were found in step 3, stop; otherwise return to step 2.

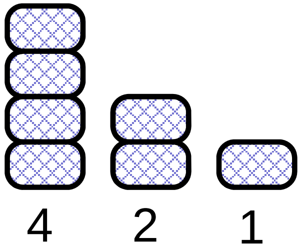
It is easy to see that the strategy of moving to P-positions wins. From a P-position, your opponent can move only to an N-position (3). Then you may move back to a P-position (2). Eventually the game ends at a terminal position and since this is a P-position, you win (1).



Each player can remove 1 or 2 pegs.

Player who removes the last peg wins.





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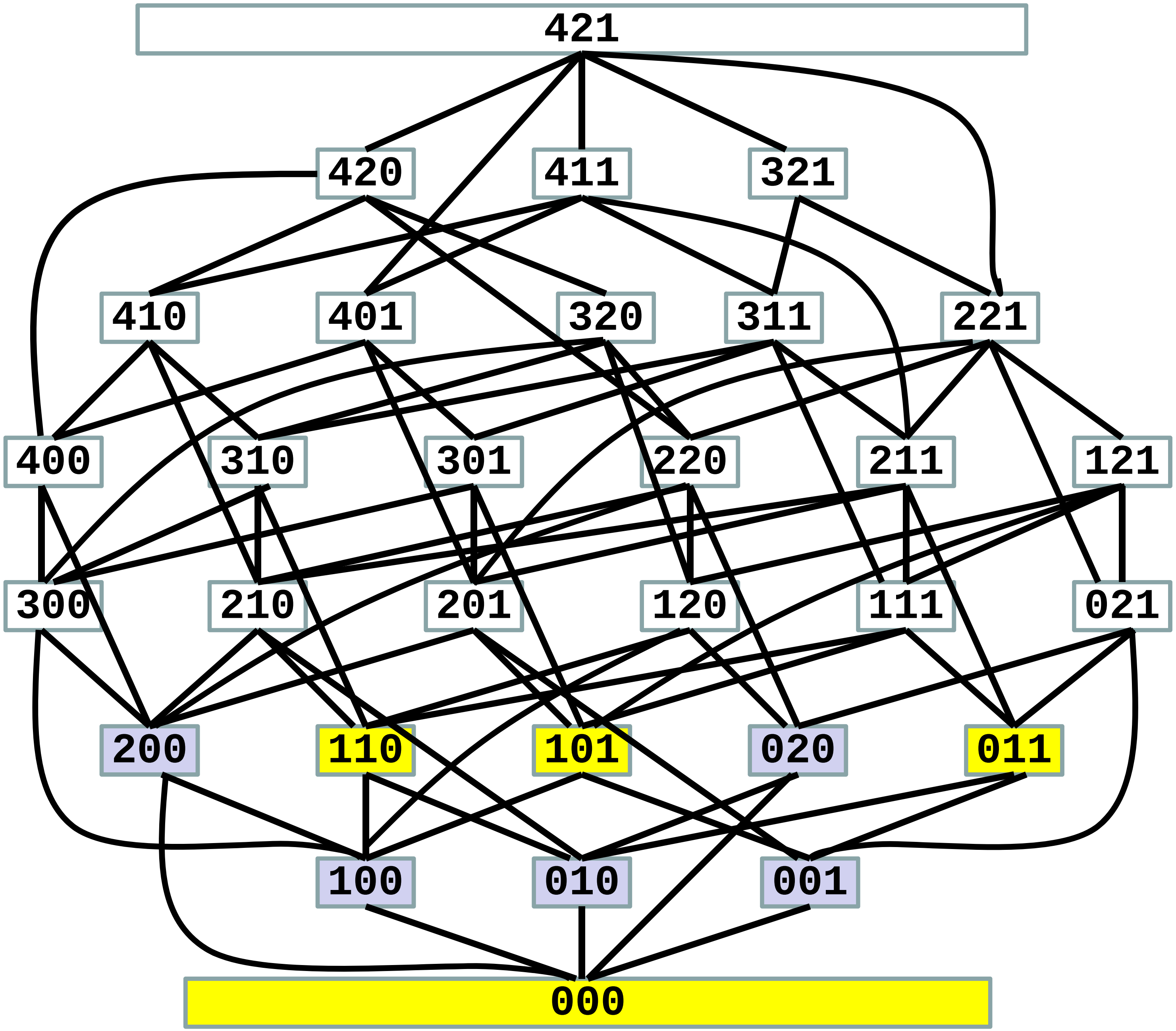

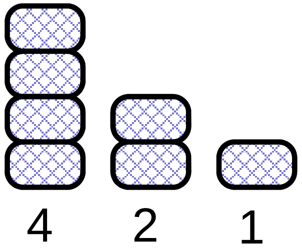

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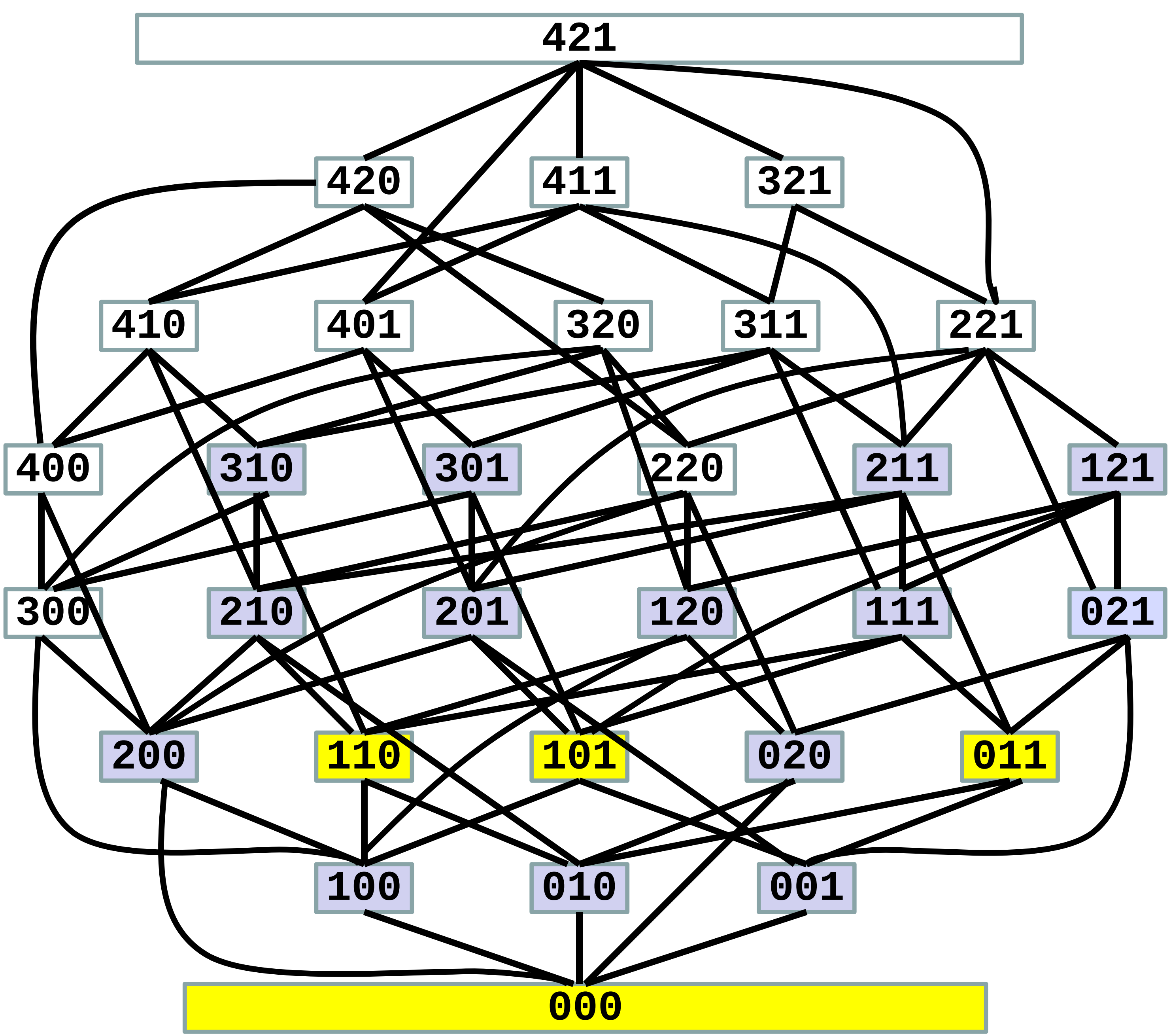

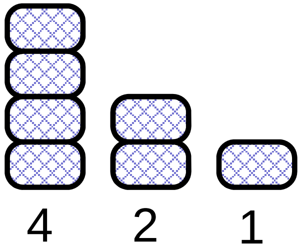

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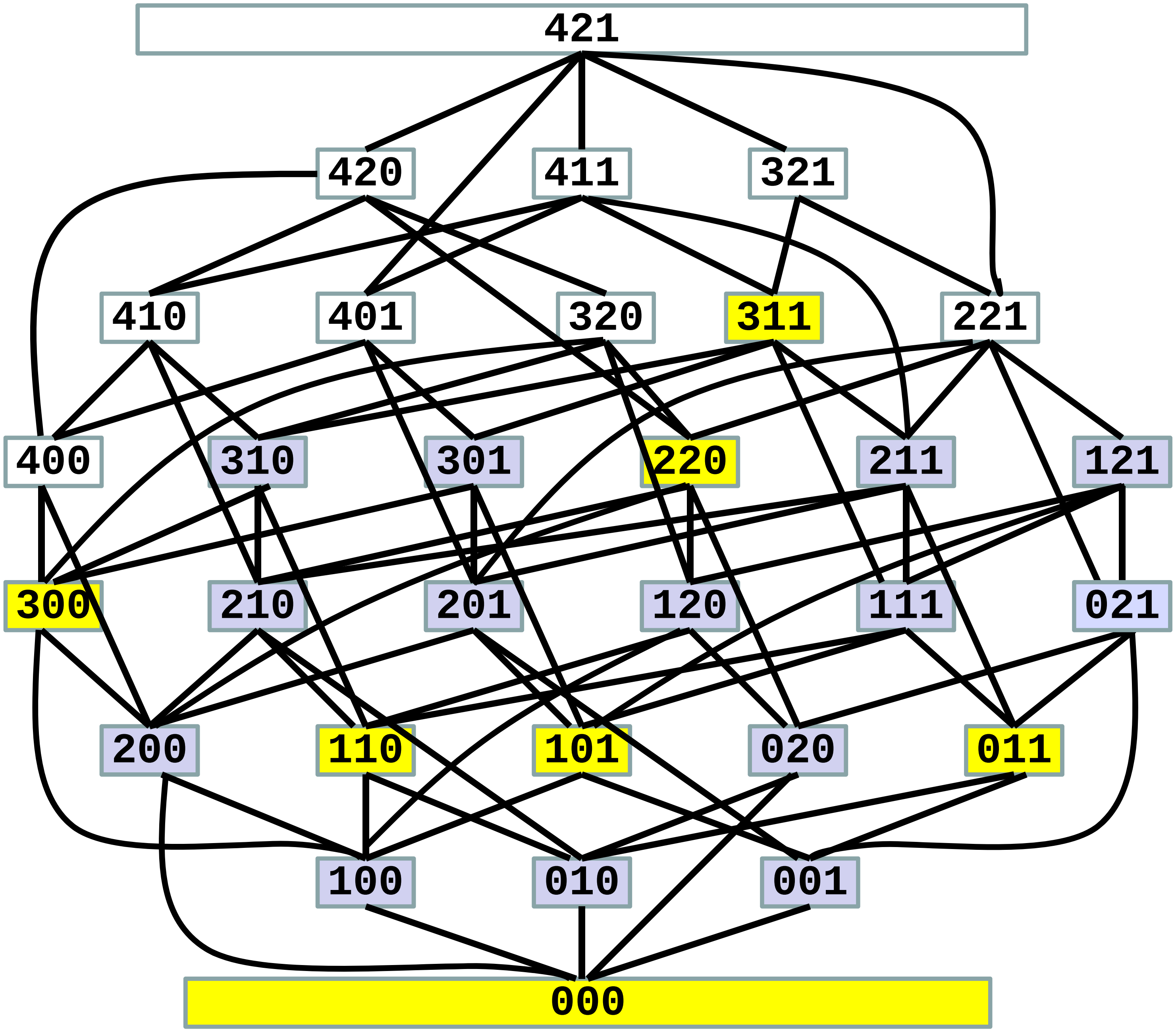

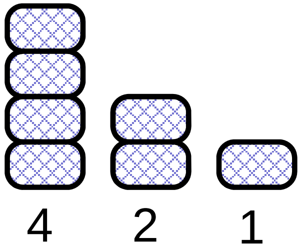

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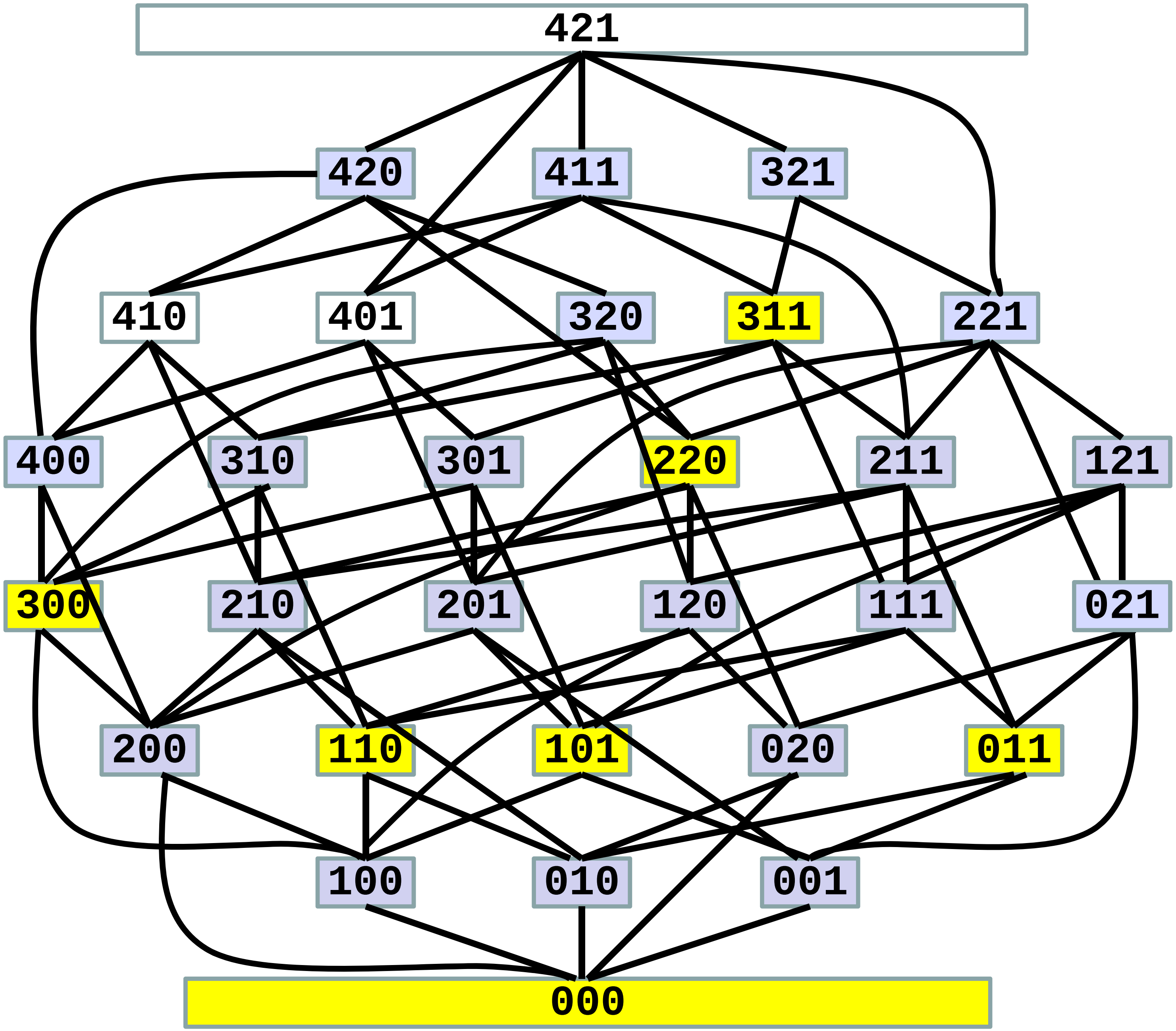

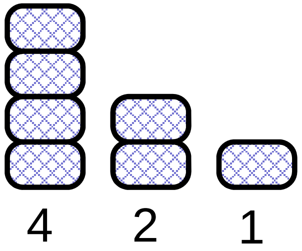

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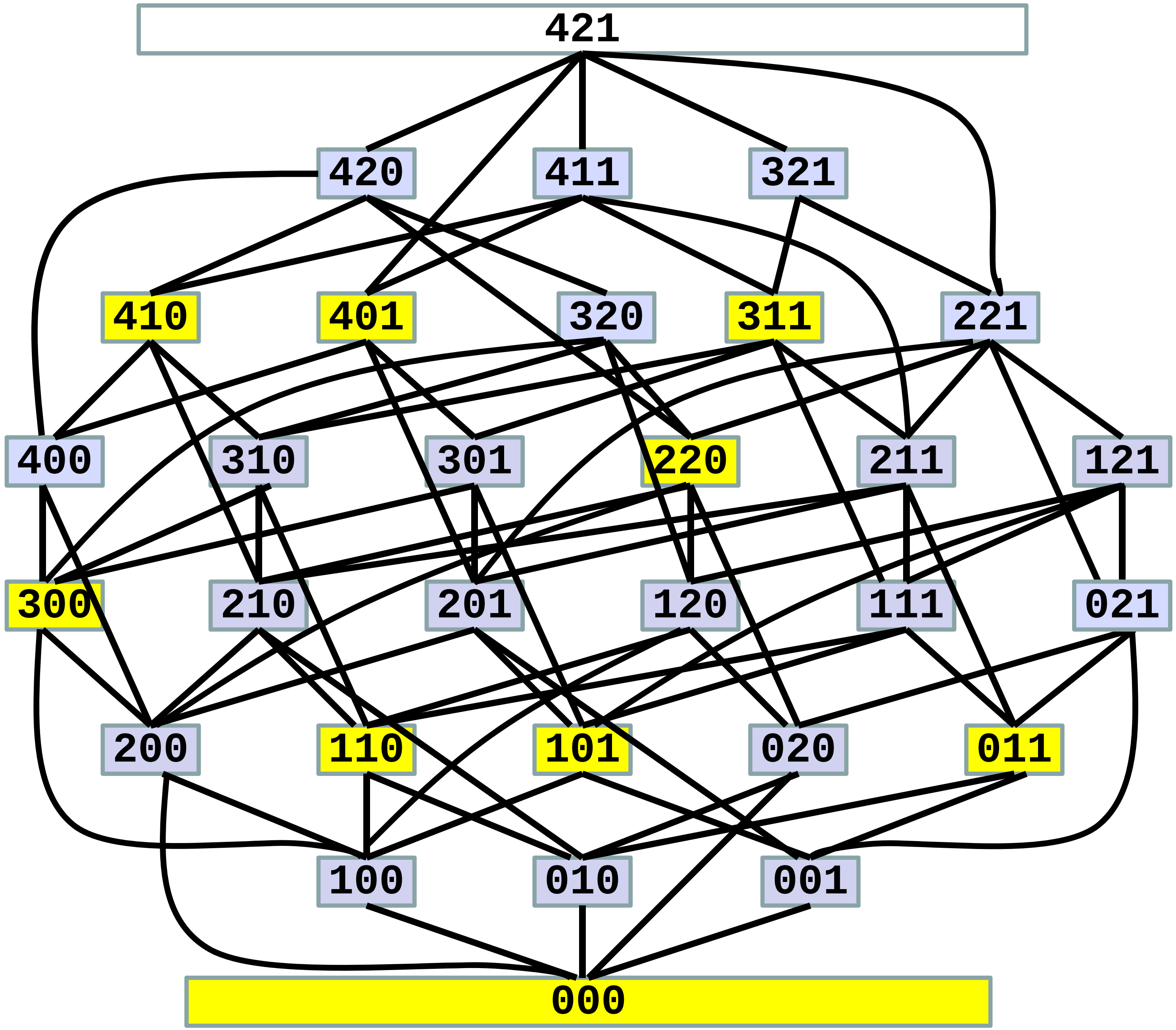

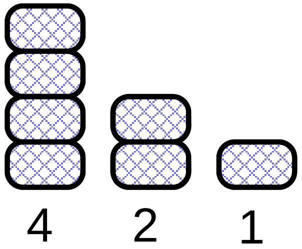

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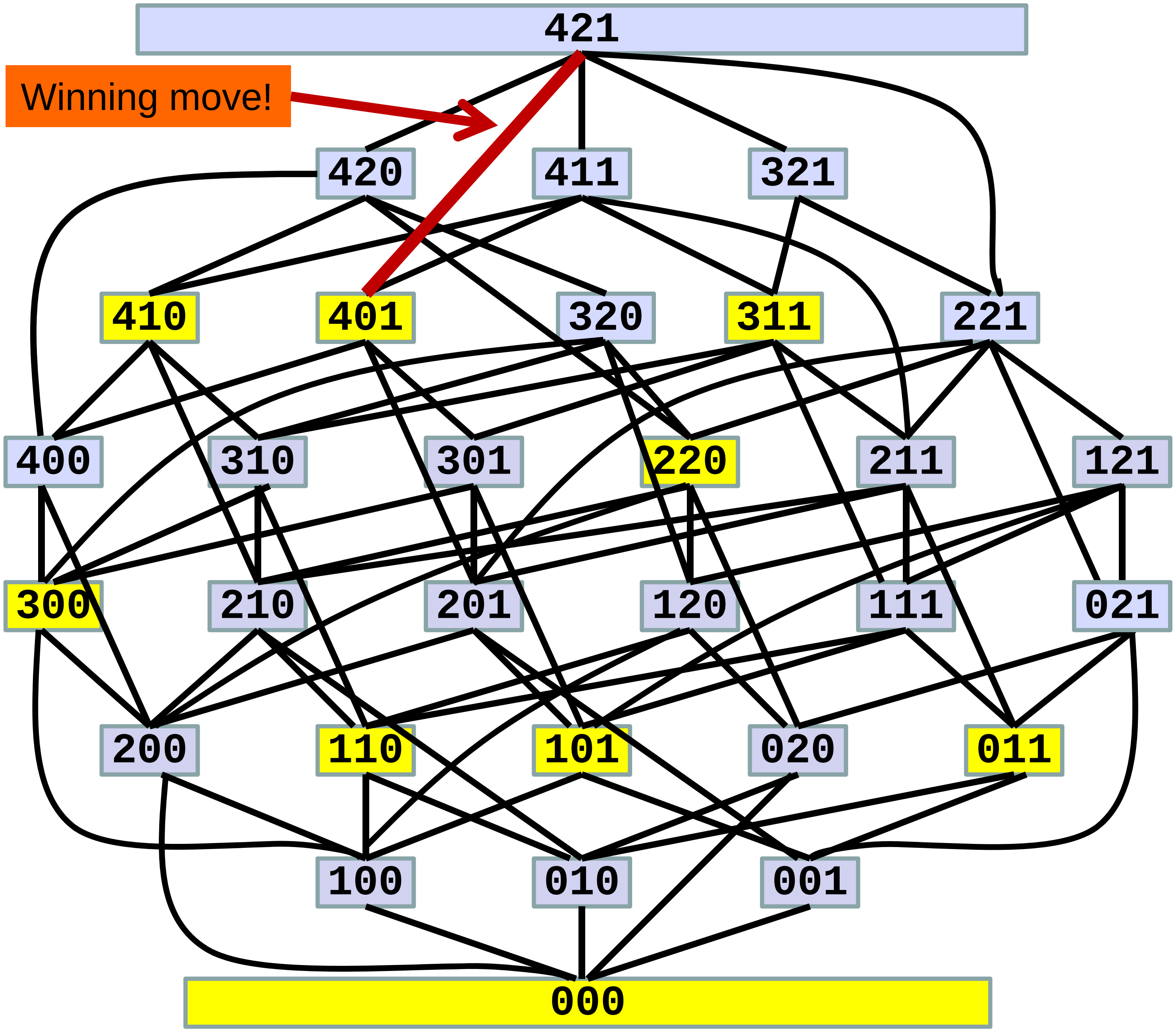
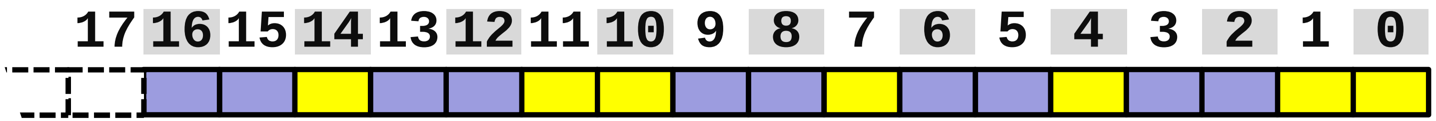


Subtraction Games

Let *S* be a set of positive integers.
The subtraction game with subtraction set *S* is played as follows.
From a pile with a large number, say *n*, of chips, two players alternate moves. A move consists of removing *s* chips from the pile where $s \in S$.
Last player to move wins.

Example



P- and N- positions in subtraction game with subtraction set {2, 5, 8}.

From state K there is a transition to the states K–2, K–5 and K–8
(if those are non-negative).

P- and N-

positions in subtraction game with subtraction set {2, 5, 8}.

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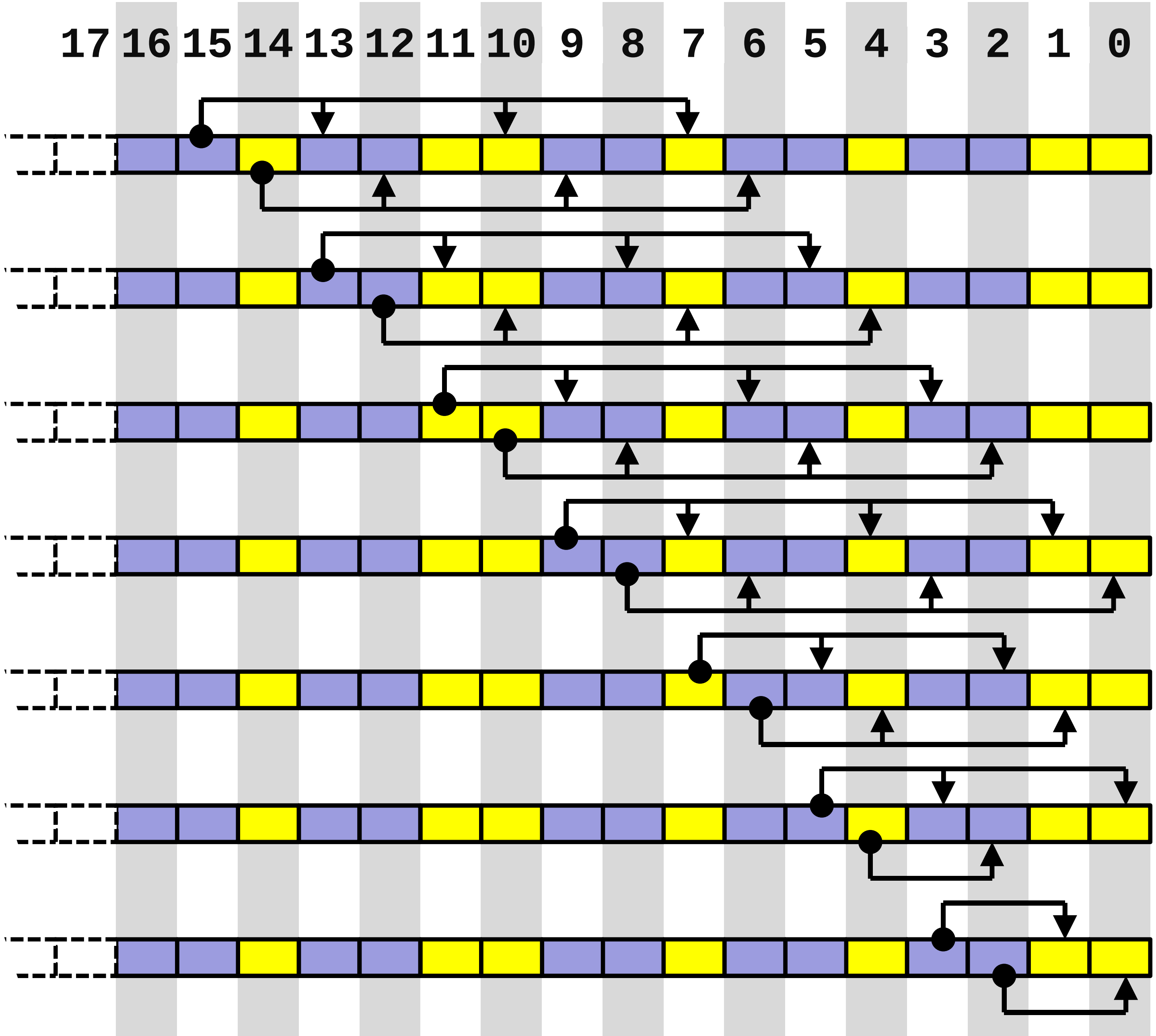
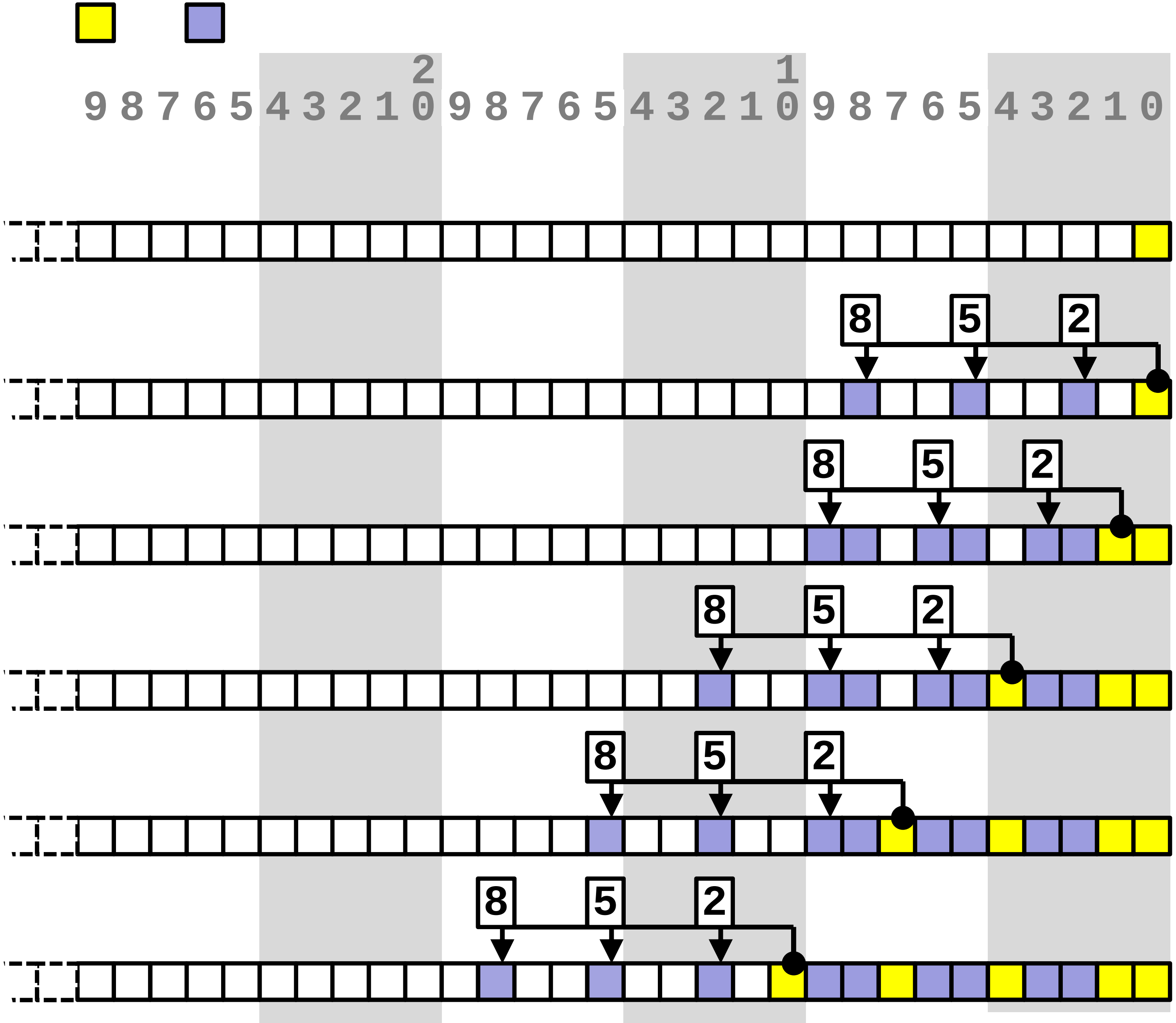


Generate P and N positions in the subtraction game with subtraction set {2, 5, 8}.



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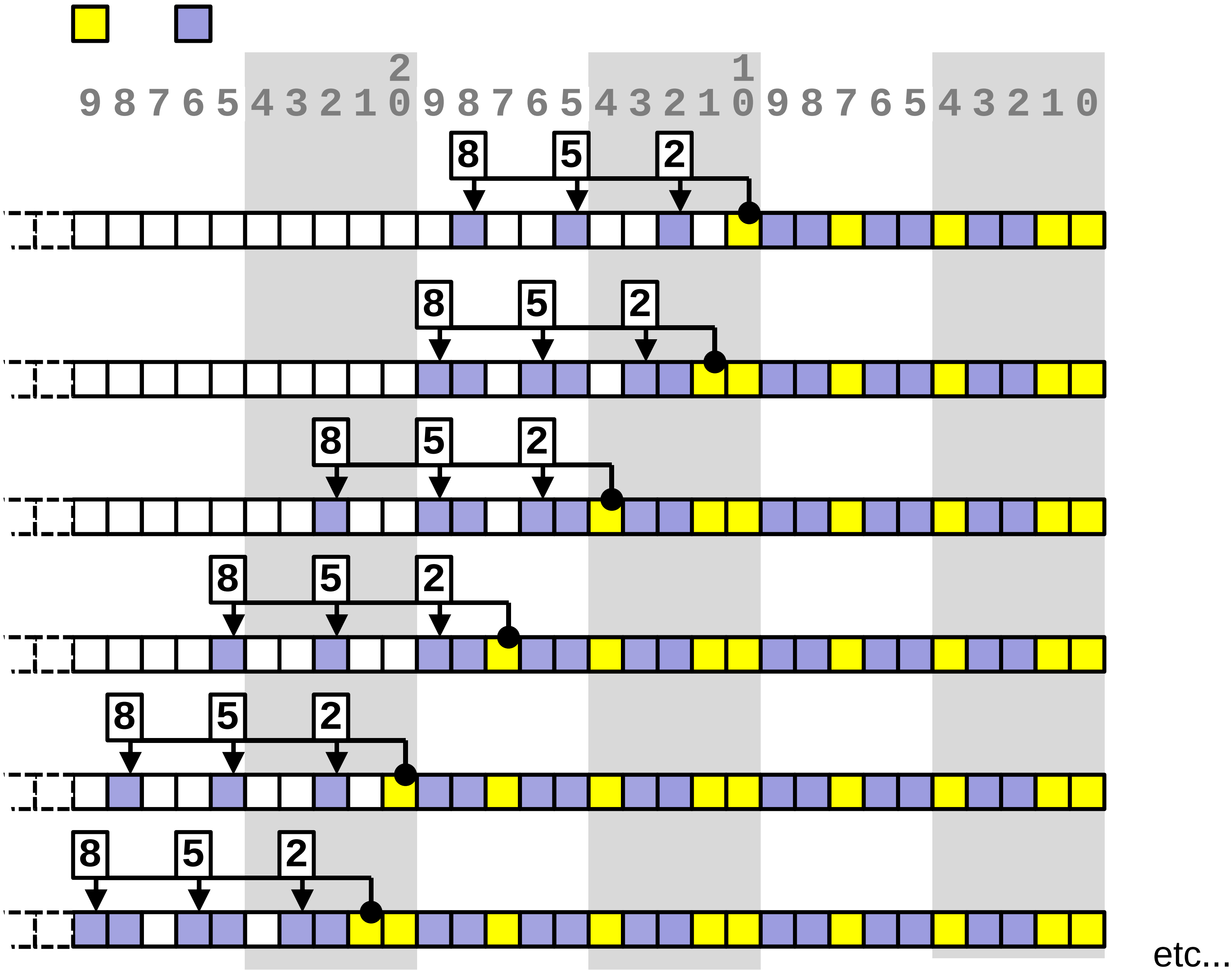

Binary representation of positive integers in Fibonacci base

N =	... 34	21	13	8	5	3	2	1
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	1	0	0
4	0	0	0	0	0	1	0	1
5	0	0	0	0	1	0	0	0
6	0	0	0	0	1	0	0	1
7	0	0	0	0	1	0	1	0
8	0	0	0	1	0	0	0	0
9	0	0	0	1	0	0	0	1
10	0	0	0	1	0	0	1	0
11	0	0	0	1	0	1	0	0
12	0	0	0	1	0	1	0	1
13	0	0	1	0	0	0	0	0
14	0	0	1	0	0	0	0	1
15	0	0	1	0	0	0	1	0
16	0	0	1	0	0	1	0	0
17	0	0	1	0	0	1	0	1
18	0	0	1	0	1	0	0	0
19	0	0	1	0	1	0	0	1
20	0	0	1	0	1	0	1	0
21	0	1	0	0	0	0	0	0
22	0	1	0	0	0	0	0	1
23	0	1	0	0	0	0	1	0
24	0	1	0	0	0	1	0	0

N =	... 55	34	21	13	8	5	3	2	1
25	0	0	1	0	0	0	1	0	1
26	0	0	1	0	0	1	0	1	0
27	0	0	1	0	0	1	1	0	1
28	0	0	1	0	0	1	0	1	0
29	0	0	1	0	1	0	0	0	0
30	0	0	1	0	1	0	0	0	1
31	0	0	1	0	1	0	0	1	0
32	0	0	1	0	1	0	1	0	0
33	0	0	1	0	1	0	1	0	1
34	0	1	0	0	0	0	0	0	0
35	0	1	0	0	0	0	0	0	1
36	0	1	0	0	0	0	0	1	0
37	0	1	0	0	0	0	1	0	0
38	0	1	0	0	0	0	1	0	1
39	0	1	0	0	0	1	0	0	0
40	0	1	0	0	0	1	0	0	1
41	0	1	0	0	0	0	0	1	0
42	0	1	0	0	1	0	0	0	0
43	0	1	0	0	1	0	0	0	1
44	0	1	0	0	1	0	0	1	0
45	0	1	0	0	1	0	1	0	0
46	0	1	0	0	1	0	1	0	1
47	0	1	0	1	0	0	0	0	0
48	0	1	0	1	0	0	0	0	1

Fibonacci Nim:

One pile of tokens.
First player can remove 1 or more tokens but not all of them.
Next, each player than can remove at most twice the number of tokens removed in his oponent's last move.
Player who removes last token wins.

Let:

- **N** be the current number of tokens.
- **RemLim** be maximum tokens which can be currently removed.
- **Fmin** be the rightmost base element present in **N** (marked by the rightmost **1**).

Then:

if **RemLim** < **Fmin** then P-position
if **RemLim** >= **Fmin** then N-position

Rule:
In N-position remove **Fmin** tokens.

base:

N =	... 55	34	21	13	8	5	3	2	1
25	0	0	1	0	0	0	1	0	1
26	0	0	1	0	0	1	0	1	0
27	0	0	1	0	0	1	1	0	1
28	0	0	1	0	0	1	0	1	0
29	0	0	1	0	1	0	0	0	0
30	0	0	1	0	1	0	0	0	1
31	0	0	1	0	1	0	0	1	0
32	0	0	1	0	1	0	1	0	0
33	0	0	1	0	1	0	1	0	1
34	0	1	0	0	0	0	0	0	0
35	0	1	0	0	0	0	0	0	1
36	0	1	0	0	0	0	0	1	0
37	0	1	0	0	0	0	1	0	0
38	0	1	0	0	0	0	1	0	1
39	0	1	0	0	0	1	0	0	0
40	0	1	0	0	0	1	0	0	1
41	0	1	0	0	0	0	0	1	0
42	0	1	0	0	1	0	0	0	0
43	0	1	0	0	1	0	0	0	1
44	0	1	0	0	1	0	0	1	0
45	0	1	0	0	1	0	1	0	0
46	0	1	0	0	1	0	1	0	1
47	0	1	0	1	0	0	0	0	0
48	0	1	0	1	0	0	0	0	1

RemLim < Fmin .... P-position
RemLim >= Fmin ....N-position

Example:
Pile with 45 tokens.

First move:
N = 45, RemLim = 44, Fmin = 3.
RemLim >= Fmin .... N-position
Remove Fmin: **N** = 45–3 = 42

Next move:
N = 42, RemLim = 6, Fmin = 8.
RemLim < Fmin .... P-position

The opponent can remove 1 to 6 tokens, that is, he can set the pile to 41, 40, 39, 38, 37, 36 tokens.

All these are N-positions, because RemLim = 6, Fmin <= 5.

base:

N =	... 55	34	21	13	8	5	3	2	1
25	0	0	1	0	0	0	1	0	1
26	0	0	1	0	0	1	0	1	0
27	0	0	1	0	0	1	1	0	1
28	0	0	1	0	0	1	0	1	0
29	0	0	1	0	1	0	0	0	0
30	0	0	1	0	1	0	0	0	1
31	0	0	1	0	1	0	0	1	0
32	0	0	1	0	1	0	1	0	0
33	0	0	1	0	1	0	1	0	1
34	0	1	0	0	0	0	0	0	0
35	0	1	0	0	0	0	0	0	1
36	0	1	0	0	0	0	0	1	0
37	0	1	0	0	0	0	1	0	0
38	0	1	0	0	0	0	1	0	1
39	0	1	0	0	0	1	0	0	0
40	0	1	0	0	0	1	0	0	1
41	0	1	0	0	0	0	0	1	0
42	0	1	0	0	1	0	0	0	0
43	0	1	0	0	1	0	0	0	1
44	0	1	0	0	1	0	0	1	0
45	0	1	0	0	1	0	1	0	0
46	0	1	0	0	1	0	1	0	1
47	0	1	0	1	0	0	0	0	0
48	0	1	0	1	0	0	0	0	1

RemLim < Fmin .... P-position
RemLim >= Fmin ....N-position

Example continues:
Opponent took 4.
Pile with 38 tokens.

Next move:
N = 38, RemLim = 8, Fmin = 1.
RemLim >= Fmin .... N-position
Remove Fmin: **N** = 38 –1 = 37

Next move:
The opponent can remove 1 or 2 tokens, that is, he can set the pile to 36 or 35 tokens.

All these are N-positions, because RemLim = 2, Fmin <= 3.

base:

N =	... 55	34	21	13	8	5	3	2	1
25	0	0	1	0	0	0	1	0	1
26	0	0	1	0	0	1	0	1	0
27	0	0	1	0	0	1	1	0	1
28	0	0	1	0	0	1	0	1	0
29	0	0	1	0	1	0	0	0	0
30	0	0	1	0	1	0	0	0	1
31	0	0	1	0	1	0	0	1	0
32	0	0	1	0	1	0	1	0	0
33	0	0	1	0	1	0	1	0	1
34	0	1	0	0	0	0	0	0	0
35	0	1	0	0	0	0	0	0	1
36	0	1	0	0	0	0	0	1	0
37	0	1	0	0	0	0	1	0	0
38	0	1	0	0	0	0	1	0	1
39	0	1	0	0	0	1	0	0	0
40	0	1	0	0	0	1	0	0	1
41	0	1	0	0	0	0	0	1	0
42	0	1	0	0	1	0	0	0	0
43	0	1	0	0	1	0	0	0	1
44	0	1	0	0	1	0	0	1	0
45	0	1	0	0	1	0	1	0	0
46	0	1	0	0	1	0	1	0	1
47	0	1	0	1	0	0	0	0	0
48	0	1	0	1	0	0	0	0	1