TRIANGULATIONS

PETR FELKEL

FEL CTU PRAGUE

felkel@fel.cvut.cz

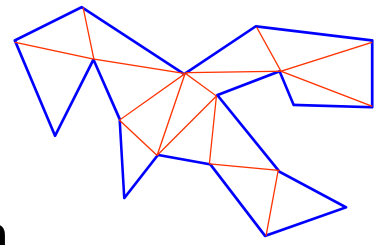
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

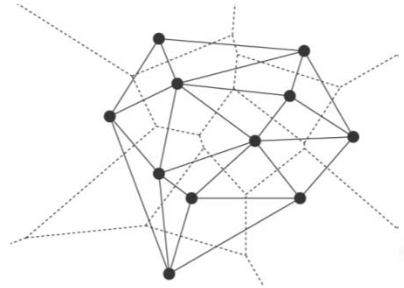
Based on [Berg] and [Mount]

Version from 20.11.2014

Talk overview

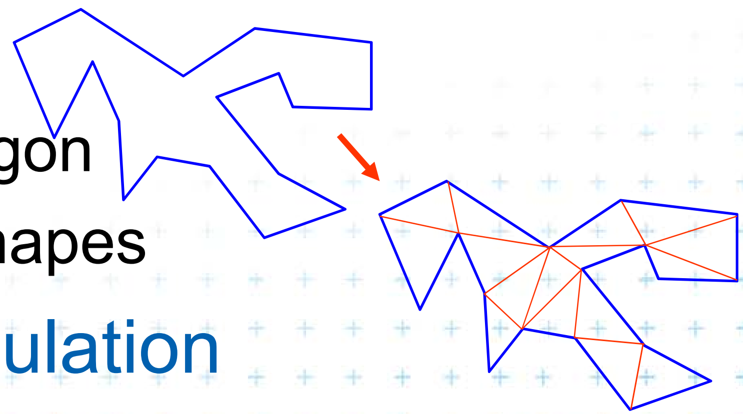
- Polygon triangulation
  - Monotone polygon triangulation
  - Monotonization of non-monotone polygon
- Delaunay triangulation (DT) of points
  - Input: set of 2D points
  - Properties
  - Incremental Algorithm
  - Relation of DT in 2D and lower envelope (CH) in 3D and
    relation of VD in 2D to upper envelope in 3D





Polygon triangulation problem

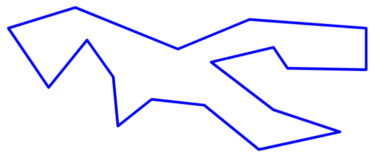
- Triangulation (in general)
  = subdividing a spatial domain into simplices
- Application
  – decomposition of complex shapes into simpler shapes
  – art gallery problem (how many cameras and where)
- We will discuss
  – Triangulation of a simple polygon
  – without demand on triangle shapes



- Complexity of polygon triangulation
  – O(n) alg. exists [Chazelle91], but it is too complicated
  – practical algorithms run in O(n log n)

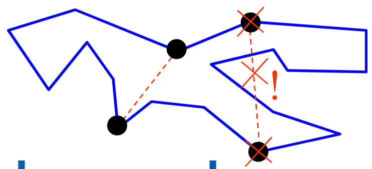
Terminology

Simple polygon



= region enclosed by a closed polygonal chain that does not intersect itself

Visible points



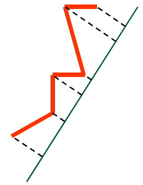
= two points on the boundary are visible if the interior of the line segment joining them lies entirely in the interior of the polygon

Diagonal

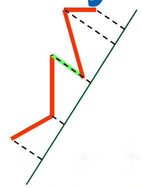
= line segment joining any pair of visible vertices

Terminology

- A polygonal chain C is strictly monotone with respect to line L, if any line orthogonal to L intersects C in at most one *point*



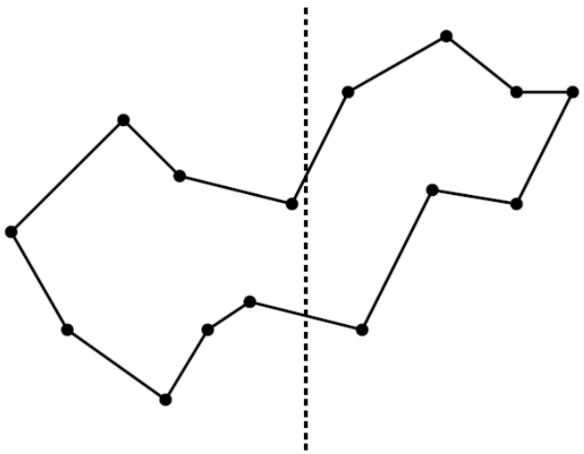
- A chain C is monotone with respect to line L, if any line orthogonal to L intersects C in at most one *connected component* (point, line segment,...)



- Polygon P is monotone with respect to line L, if its boundary (bnd(P), ∂P) can be split into two chains, each of which is monotone with respect to L

Terminology

- Horizontally monotone polygon
  = monotone with respect to *x*-axis
  - Can be tested in $O(n)$
  - Find leftmost and rightmost point in $O(n)$
  - Split boundary to upper and lower chain
  - Walk left to right, verifying that x-coord are non-decreasing



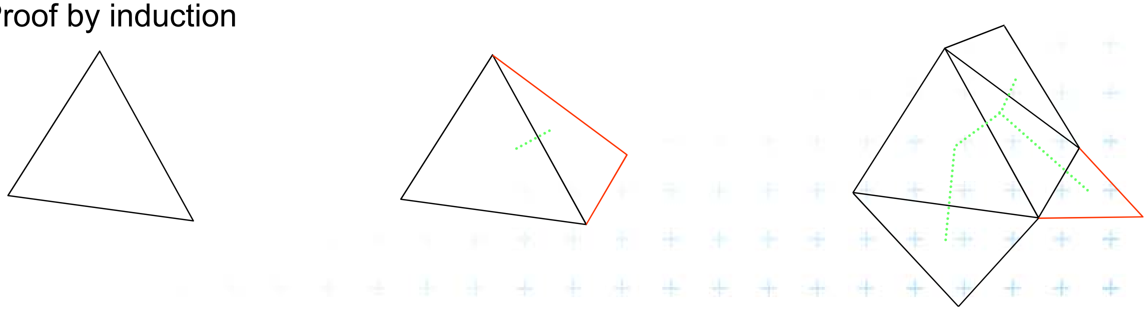
x–monotone polygon

[Mount]

Terminology

- Every simple polygon can be triangulated
- Simple polygon with *n* vertices consists of
  - exactly n-2 triangles
  - exactly n-3 diagonals
  - Each diagonal is added once
    => O(n) sweep line algorithm exist

Proof by induction



n = 3 => 0 diagonal

n = 4 => 1 diagonal

n := n+1 => n + 1 – 3 diagonals
n + 1 = 7 => 4 diagonals)

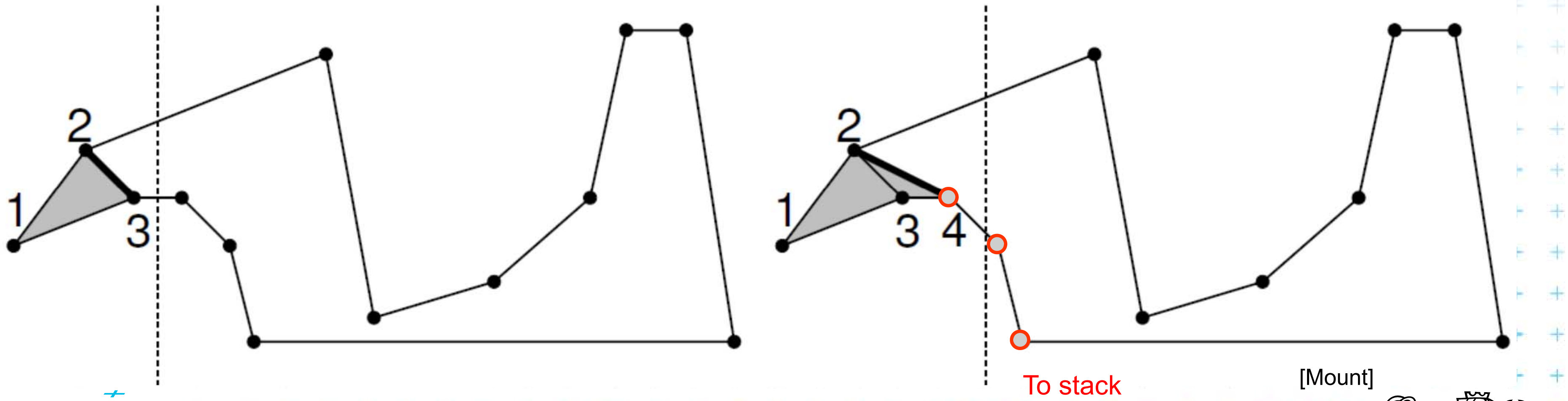
Simple polygon triangulation

- Simple polygon can be triangulated in 2 steps:
  1. Partition the polygon into x-monotone pieces
  2. Triangulate all monotone pieces

(we will discuss the steps in the reversed order)

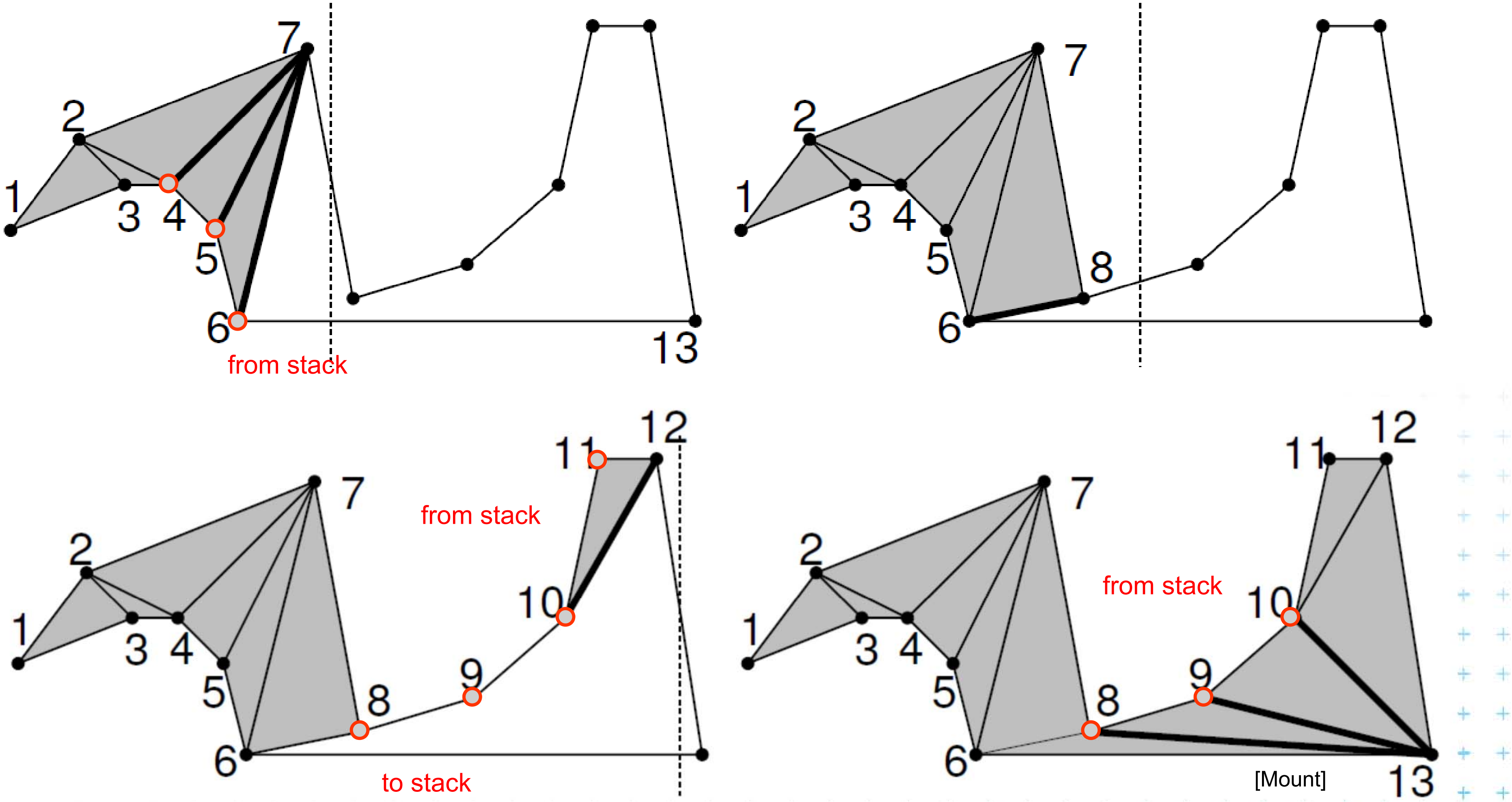
2. Triangulation of the monotone polygon

- Sweep left to right - in O(n) time
- Triangulate everything you can by adding diagonals between visible points
- Remove triangulated region from further consideration – mark as DONE



[Mount]

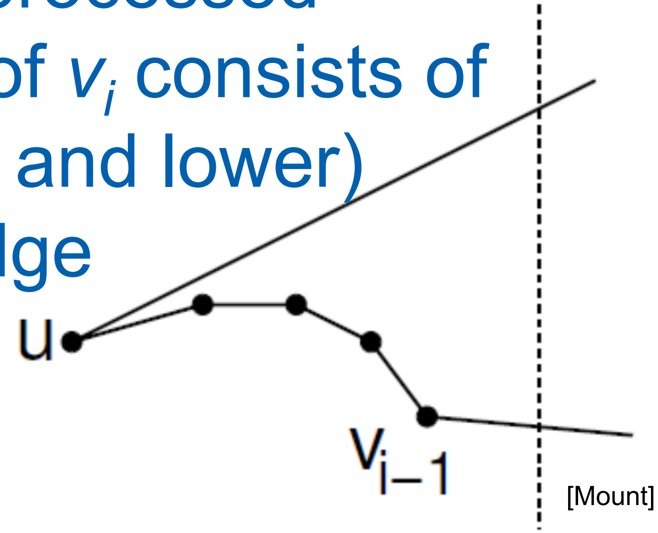
Triangulation of the monotone polygon



Main invariant of the untriangulated region

Main invariant

- Let v_i be the vertex being just processed
- The untriangulated region left of v_i consists of two x-monotone chains (upper and lower)
- Each chain has at least one edge
- If it has more than one edge
  – these edges form a reflex chain
  = sequence of vertices
  with interior angle ≥ 180°



Initial invariant

- Left vertex of the last added diagonal is *u*
- Vertices between *u* and v_i are waiting in the stack

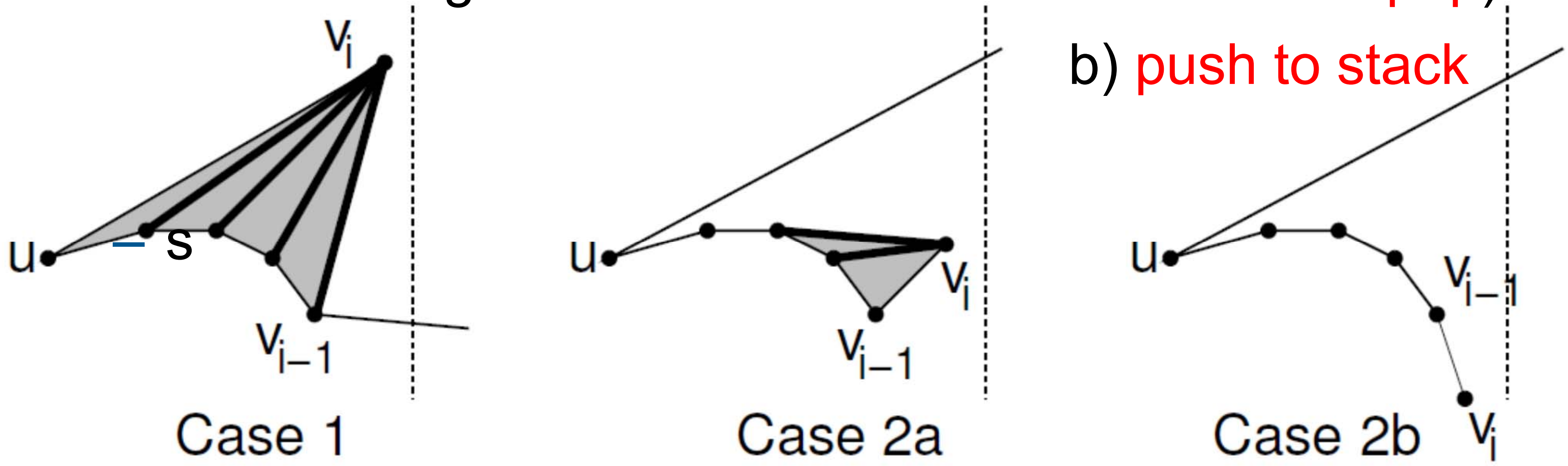
Triangulation cases

- Case 1: v_i lies on the opposite chain
  - – Add diagonals from next(u) to v_{i-1}
  - – Set u = v_{i-1}. Last diagonal (invariant) is $v_i v_{i-1}$
- Case 2: v is on the same chain as v_{i-1}

  a) walk back, adding diagonals joining v_i to prior vertices until the angle becomes > 180° or *u* is reached - pop)

  b) push to stack

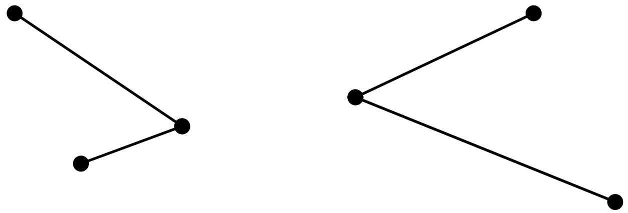
– s



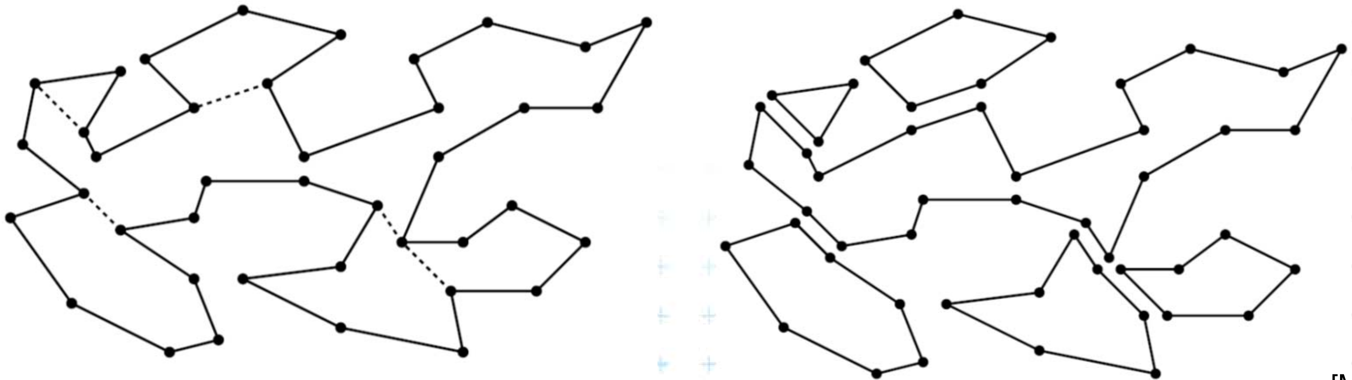
[Mount]

1. Polygon subdivision into monotone pieces

- X-monotonicity breaks the polygon in vertices with edges directed both left or both right



- The monotone polygons parts are separated by the splitting diagonals (joining vertex and helper)



Splitting diagonals Monotone decomposition

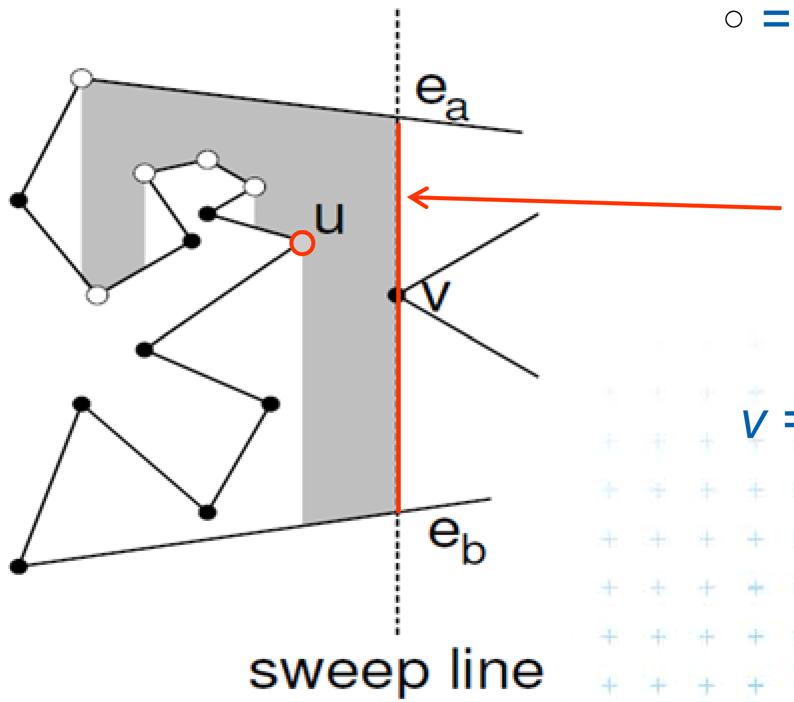
[Mount]

Data structures for subdivision

- Events
  - Endpoints of edges, known from the beginning
  - Can be stored in sorted list – no priority queue
- Sweep status
  - List of edges intersecting sweep line (top to bottom)
  - Stored in O(log n) time dictionary (like balanced tree)
- Event processing
  - Six event types based on local structure of edges around vertex *v*

Helper – definition

helper(e_a)
= the rightmost vertically visible processed vertex u
below edge e_a on polygonal chain between edges e_a & e_b
is visible to every point along the sweep line between e_a & e_b



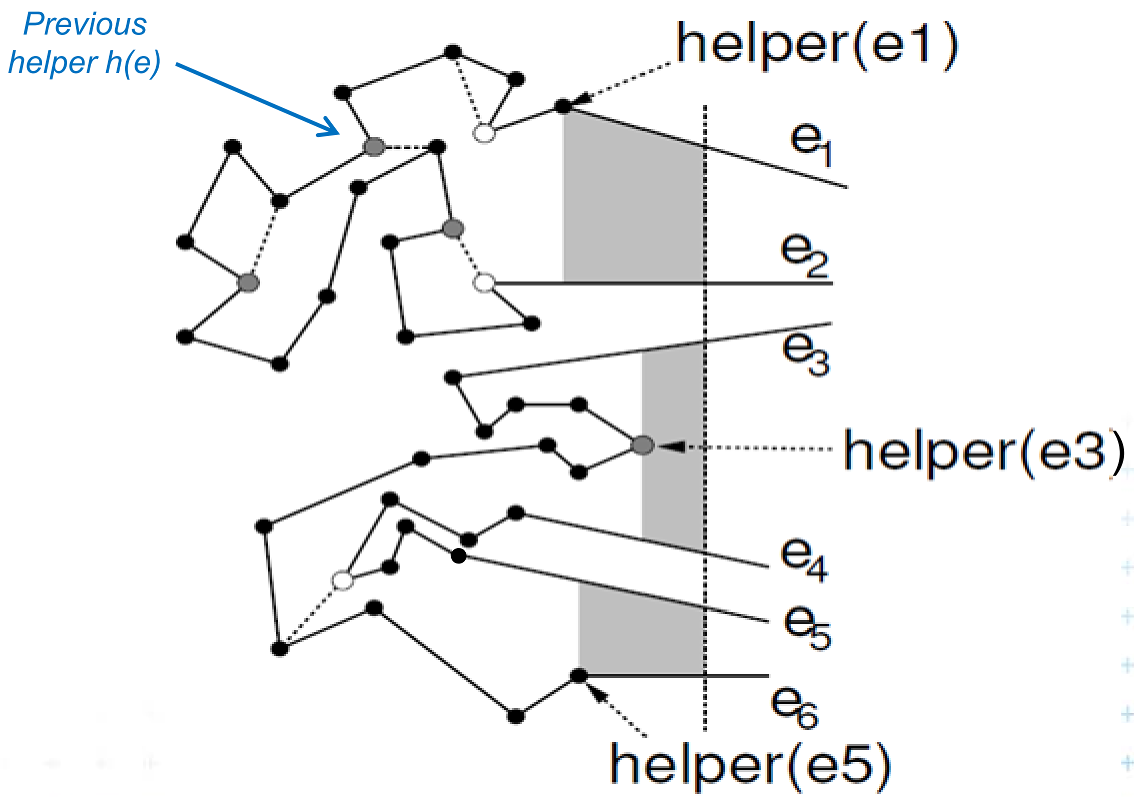
○ = vertically visible processed vertex

all these vertices see u = helper(e_a)

v = current vertex (sweep line stop)

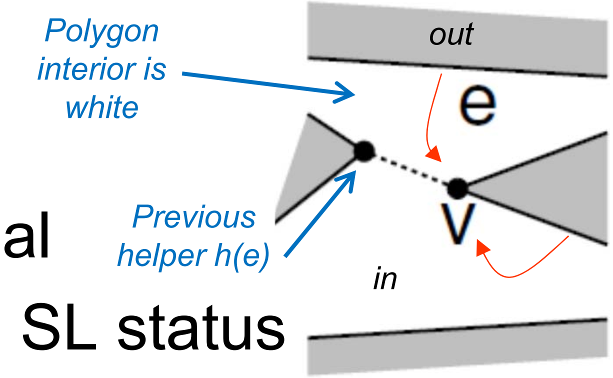
Helper

helper(e_a)
is defined only for edges intersected by the sweep line



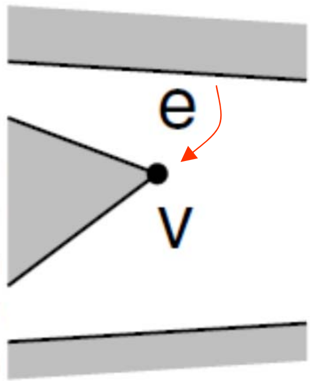
Six event types of vertex *v*

1. Split vertex



- Find edge *e* above *v*,
  connect *v* with helper(e) by diagonal
- Add 2 new edges incident to *v* into SL status
- Set new helper(e) = helper(lower edge of these two) = *v*

2. Merge vertex



- Find two edges incident with *v* in SL status
- Delete both from SL status
- Let *e* is edge immediately above *v*
- Make helper(e) = *v*

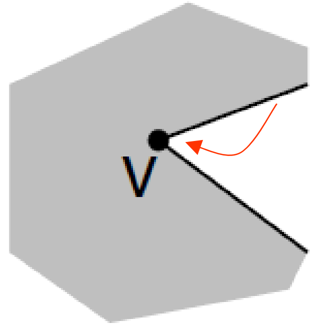
[Mount]

(Interior angle >180° for both – split & merge vertices)

Six event types of vertex *v*

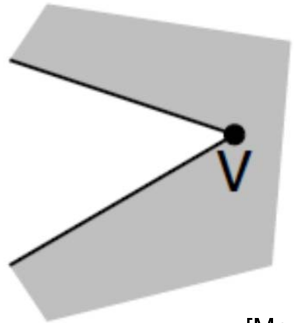
3. Start vertex

- Both incident edges lie right from *v*
- But interior angle <180°
- Insert both edges to SL status
- Set helper(upper edge) = *v*



4. End vertex

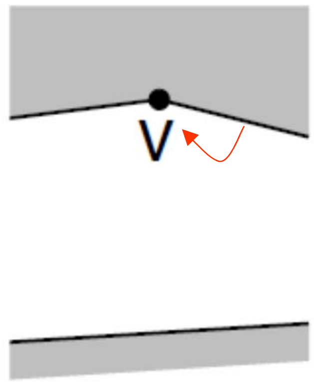
- Both incident edges lie left from *v*
- But interior angle <180°
- Delete both edges from SL status
- No helper set – we are out of the polygon



[Mount]

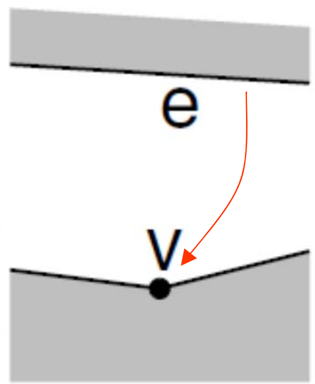
Six event types of vertex *v*

5. Upper chain-vertex



- one side is to the left, one side to the right, interior is below
- replace the left edge with the right edge in SL status
- Make *v* helper of the new (upper) edge

6. Lower chain-vertex



[Mount]

- one side is to the left, one side to the right, interior is above
- replace the left edge with the right edge in SL status
- Make *v* helper of the edge *e above*

Polygon subdivision complexity

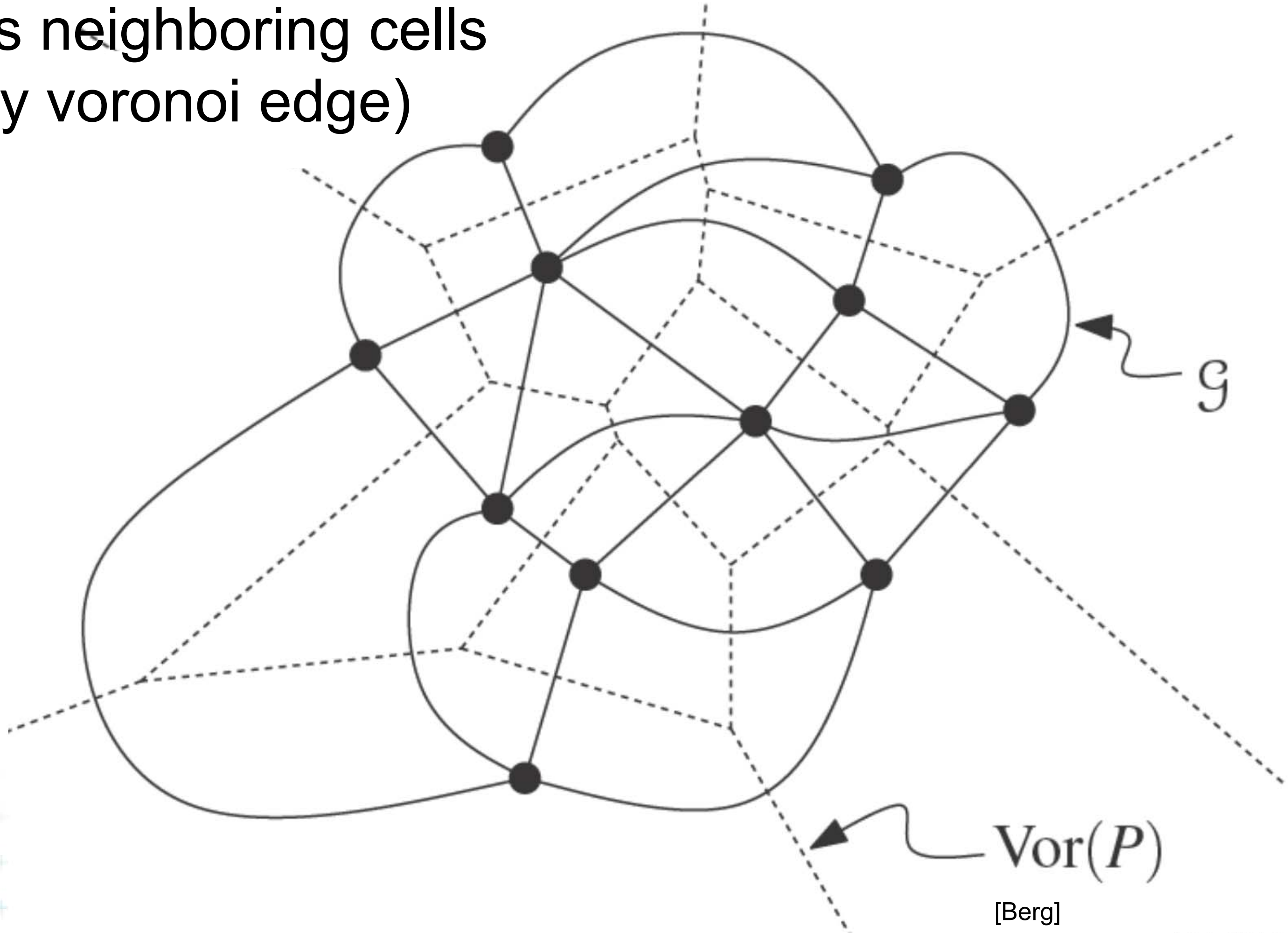
- Simple polygon with *n* vertices can be partitioned into x-monotone polygons in
  - O(*n* log *n*) time (n steps of SL, log n search each)
  - O(*n*) storage

- Complete simple polygon triangulation
  - O(*n* log *n*) time for partitioning into monotone polygons
  - O(*n*) time for triangulation
  - O(*n*) storage

Dual graph G for a Voronoi diagram

Graph G: Node for each Voronoi-diagram cell $V(p)$ ~ VD site p

Arc connects neighboring cells
(arc for every voronoi edge)

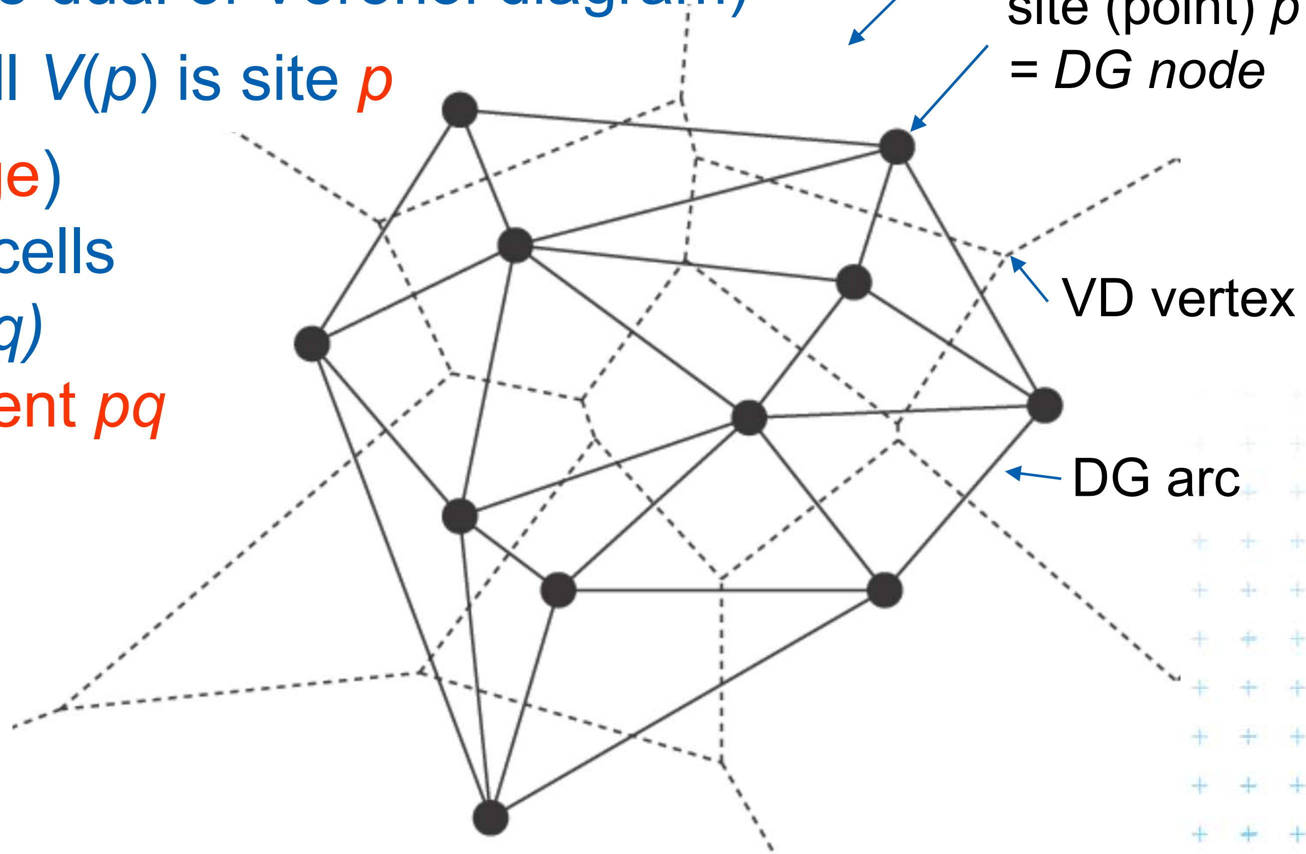


[Berg]

Delaunay graph *DG*(*P*) [*Борис Николаевич Делоне*]

= straight line embedding of *G*
(straight-line dual of Voronoi diagram)

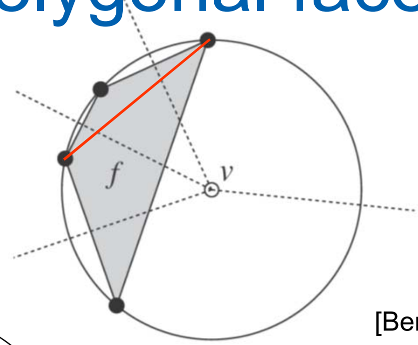
- Node for cell *V*(*p*) is site *p*
- Arc (DT edge) connecting cells *V(p) and V(q)* is the segment *pq*



[Berg]

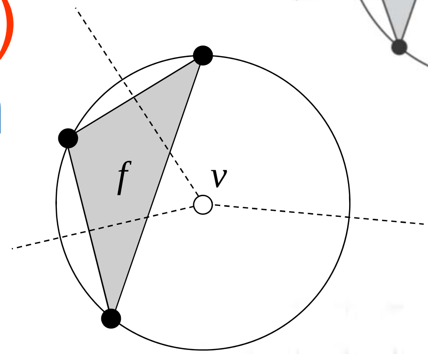
Delaunay graph and Delaunay triangulation

- Delaunay *graph DG(P)* has convex polygonal faces (with number of vertices ≥3, equal to the degree of Voronoi vertex)



[Berg]

- Delaunay *triangulation DT(P)* = Delaunay graph for sites in general position



  – No four sites on a circle
  – Faces are triangles (Voronoi vertices have degree = 3)
  – DT is unique (DG not! Can be triangulated differently)

DG(P) sites not in general position

  – Triangulate larger faces – such triangulation is not unique

Delaunay triangulation properties 1/2

Circumcircle property

- The circumcircle of any triangle in DT is empty (no sites)
  Proof: It's center is the Voronoi vertex
- Three points *a,b,c* are vertices of the same face of *DG*(*P*)
  iff circle through *a,b,c* contains no point of *P* in its interior

Empty circle property and legal edge

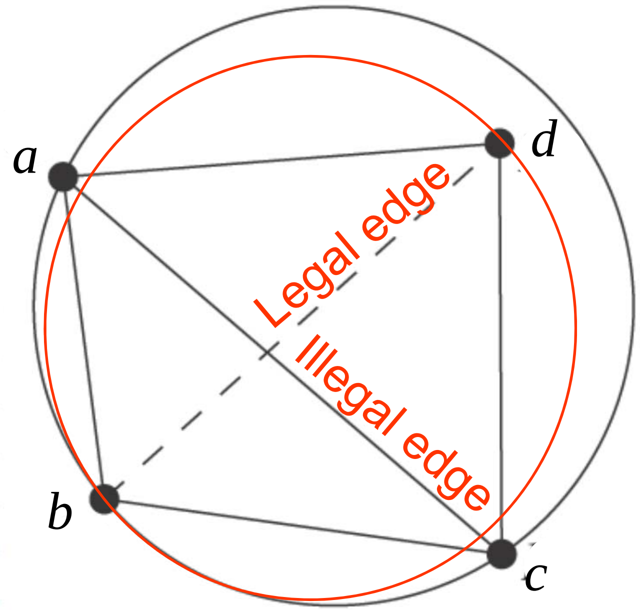
- Two points *a,b* form an edge of *DG*(*P*) – it is a legal edge
  iff ′ closed disc with *a,b* on its boundary that contains
  no other point of *P* in its interior … disc minimal diameter = dist(a,b)

Closest pair property

- The closest pair of points in *P* are neighbors in *DT*(*P*)

Delaunay triangulation properties 2/2

- DT edges do not intersect
- Triangulation *T* is legal, **iff** *T* is a Delaunay triangulation (i.e., if it does not contain illegal edges)
- Edge that was legal before may become illegal if one of the triangles incident to it changes
- In convex quadrilateral *abcd* (*abcd* do not lie on common circle) exactly one of *ac, bd* is an illegal edge = principle of edge flip operation



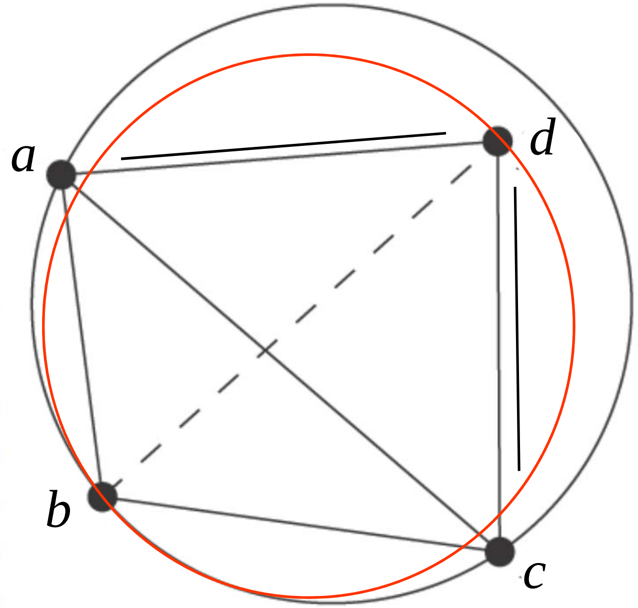
[Berg]

Edge flip operation

Edge flip

= a local operation, that increases the angle vector

- Given two adjacent triangles $\triangle abc$ and $\triangle cda$ such that their union forms a convex quadrilateral, the edge flip operation replaces the diagonal ac with bd.



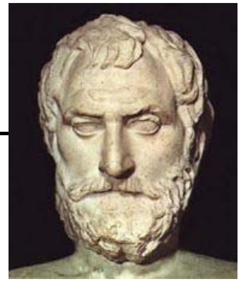
[Berg]

Delaunay triangulation

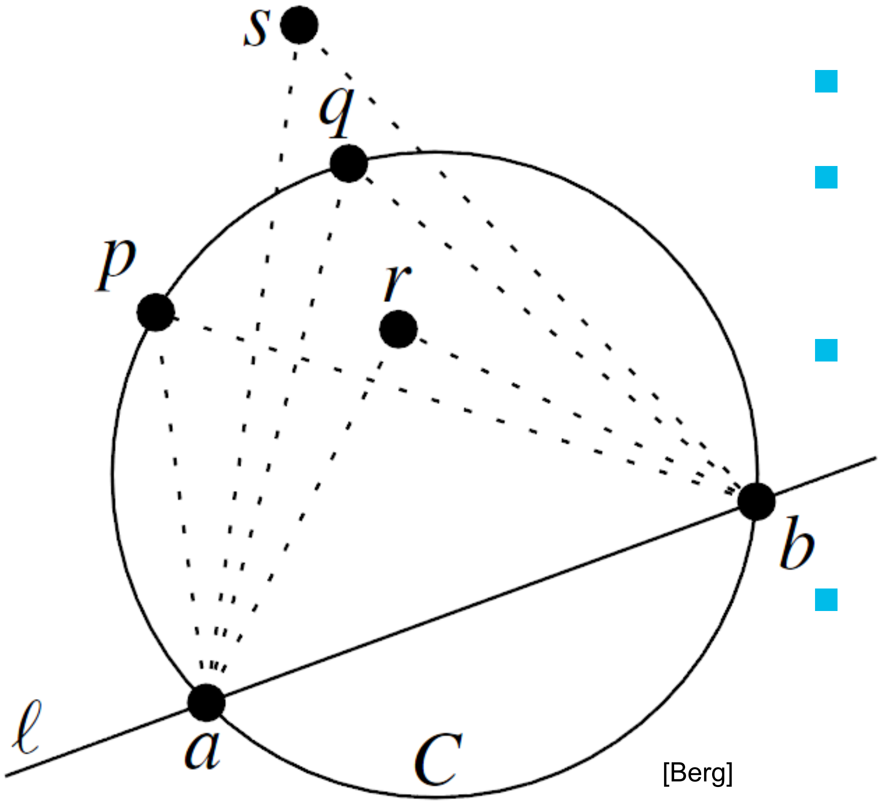
- Let *T* be a triangulation with *m* triangles (and 3*m angles*)
- Angle-vector

  = non-decreasing ordered sequence $(\alpha_1, \alpha_2, \ldots, \alpha_{3m})$
  inner angles of triangles, $\alpha_i \le \alpha_j$, for $i < j$
- Delaunay triangulation has the lexicographically largest angle sequence
  - It maximizes the minimal angle (the first angle in angle-vector)
  - It maximizes the second minimal angle, …
  - It maximizes all angles
  - It is an angle optimal triangulation

Thales's theorem (624-546 BC)



Respective Central Angle Theorem



[Berg]

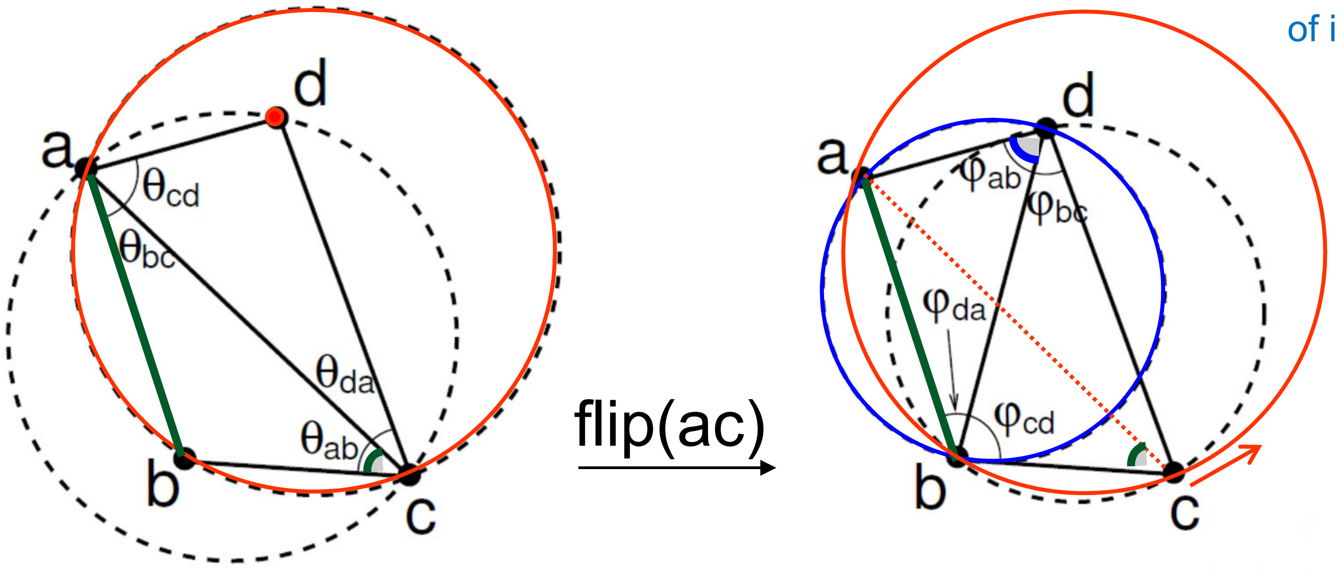
- Let C = circle,
- l =line intersecting C in points a, b
- p, q, r, s = points on the same side of l
  p,q on C , r is *in*, s is out
- Then for the angles holds:
  $\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb$

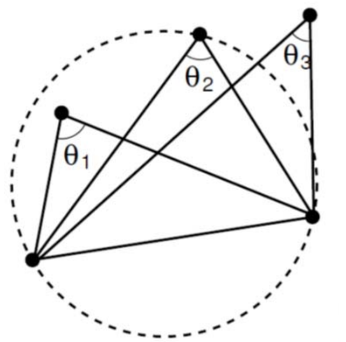
http://www.mathopenref.com/arccentralangletheorem.html

Edge flip of illegal edge and angle vector

- The minimum angle increases after the edge flip

of illegal edge ac > bd





$\theta_1 > \theta_2 > \theta_3$

[Mount]

$bd < ac$ $\quad \varphi_{ab} > \theta_{ab} \quad \varphi_{bc} > \theta_{bc} \quad \varphi_{cd} > \theta_{cd} \quad \varphi_{da} > \theta_{da}$

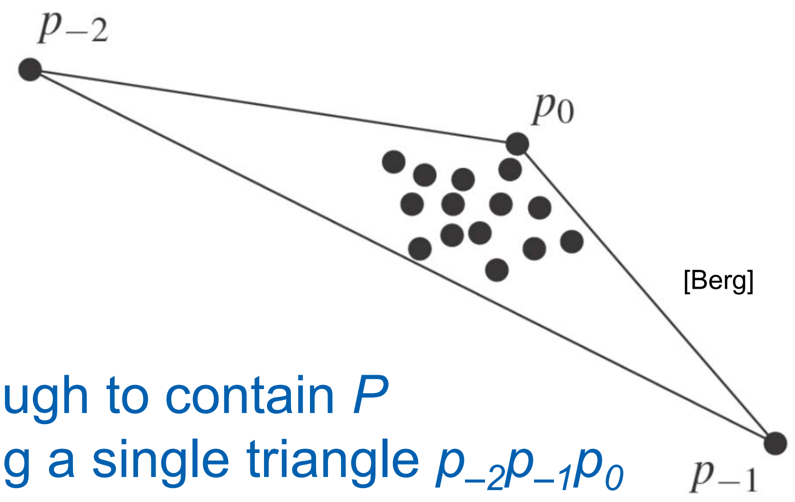
=> After limited number of edge flips

- Terminate with lexicographically maximum triangulation
- It satisfies the empty circle condition => Delauney T.

Incremental algorithm principle

1. Create a large triangle containing all points (to avoid problems with unbounded cells)
   – must be larger than the largest circle through 3 points
   – will be discarded at the end
2. Insert the points in random order
   – Find triangle with inserted point *p*
   – Add edges to its vertices (these new edges are correct)
   – Check correctness of the old edges (triangles) "around *p*" and legalize (flip) potentially illegal edges
3. Discard the large triangle and incident edges

Incremental algorithm in detail



DelaunayTriangulation(*P*)
Input: Set P of n points in the plane
Output: A Delaunay triangulation T of P

1. Let p_{-2}, p_{-1}, p_0 form a triangle large enough to contain P
2. Initialize T as the triangulation consisting a single triangle $p_{-2}p_{-1}p_0$
3. Compute random permutation $p_1, p_2, \dots, p_n$ of $P \setminus \{p_0\}$
4. **for** $r = 1$ **to** n **do**
5. $\quad T = \text{Insert}(p_r, T)$
6. Discard p_{-1}, p_{-2}, p_{-3} with all incident edges from T
7. **return** T

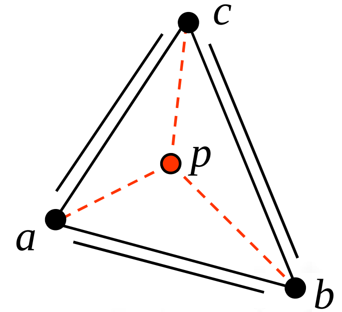
Incremental algorithm – insertion of a point

Insert(***p*, *T*** **)**

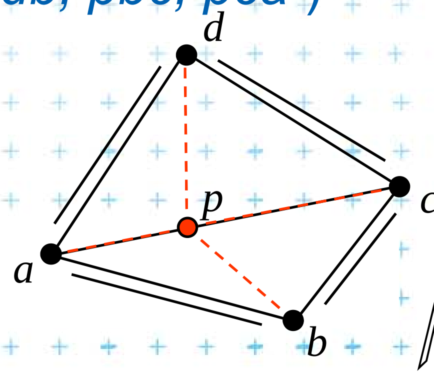
Input: Point *p* being inserted into triangulation *T*

Output: Correct Delaunay triangulation after insertion of *p*

1. Find a triangle *abc* µ *T* containing *p*
2. **if** *p* lies in the interior of *abc* **then**
3. Insert edges *pa, pb, pc* into triangulation *T* (splitting *abc* into 3 triangles *pab, pbc, pca*)
4. LegalizeEdge(*p, ab, T*)
5. LegalizeEdge(*p, bc, T*)
6. LegalizeEdge(*p, ca, T*)
7. **else** // *p* lies on the edge of *abc*, say *ab,* point *d* is right from edge *ab*
8. Remove *ab* and insert edges *pa, pb, pc, pd* into triangulation *T* (splitting *abc* and *abd* into 4 triangles *pad, pdb, pbc, pca*)
9. LegalizeEdge(*p, ab, T*)
10. LegalizeEdge(*p, bc, T*)
11. LegalizeEdge(*p, cd, T*)
12. LegalizeEdge(*p, da, T*)
13. **return** *T*



[Berg]



[Berg]

Incremental algorithm – edge legalization

LegalizeEdge(*p, ab, T*)

Input: Edge *ab* being checked after insertion of point *p* to triangulation *T*

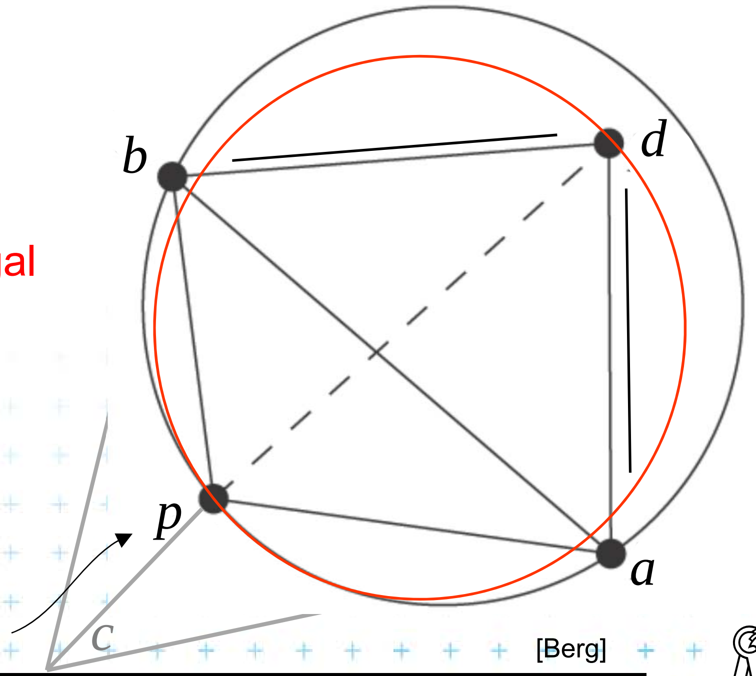
Output: Delaunay triangulation of *p* + *T*

1. **if**(*ab* is edge on the exterior face) **return**
2. let *d* be the vertex to the right of edge *ab*
3. if(inCircle(*p, a, b, d*)) // *d* is in the circle around *pab* => *d* is illegal
4. Flip edge *ab* for *pd*
5. LegalizeEdge(*p, ad, T*)
6. LegalizeEdge(*p, db, T*)

Insertion of *p* may make edges *ab*, *bc* & *ca* illegal (circle around *pab* will contain point *d*)

After edge flip, the edge *pd* will be legal (the circumcircles of the resulting triangles *pdb,* and *pad* will bee empty)

We must check and possibly flip edges *ad, db*

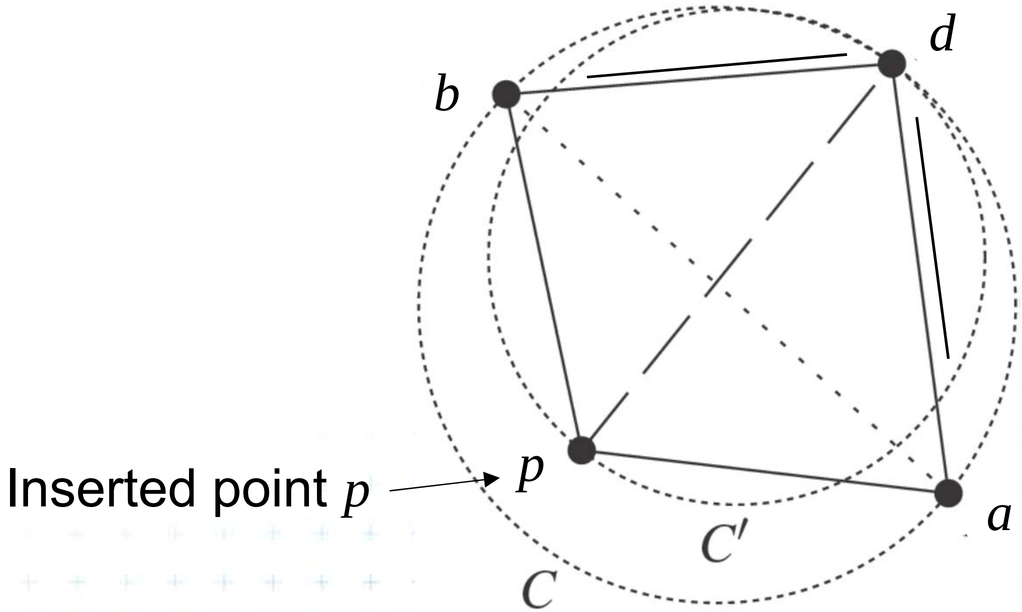


Inserted point *p*

[Berg]

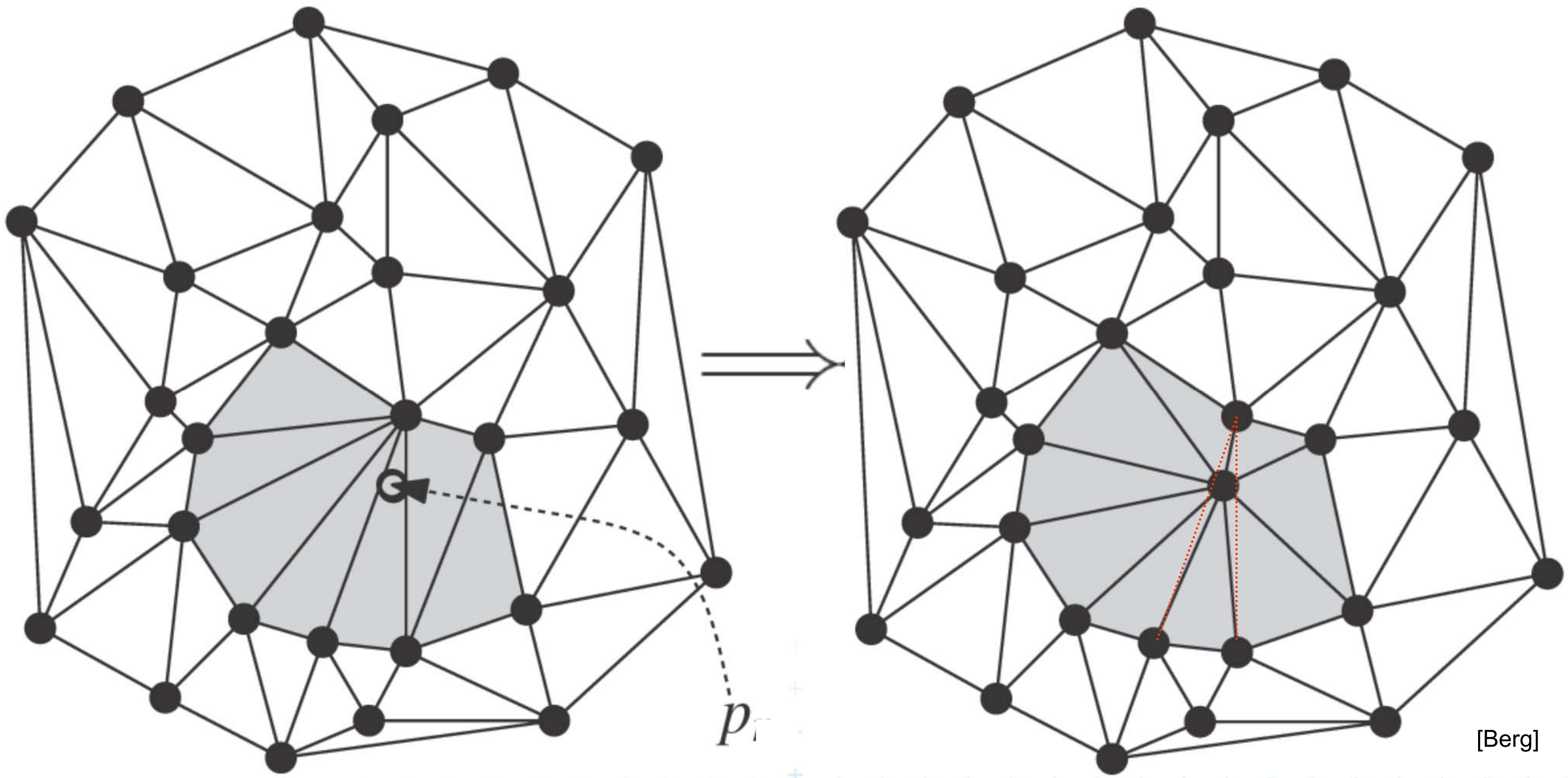
Correctness of edge flip of illegal edge

- Assume point *p* is in *C* (it violates DT criteria for *adb*)
- *adb* was a triangle of DT => *C* was an empty circle
- Create circle C’ trough point *p*, C’ is inscribed to C, C’ð C
  => *C’* is also an empty circle
  => new edge *pd* is a Delaunay edge



[Berg]

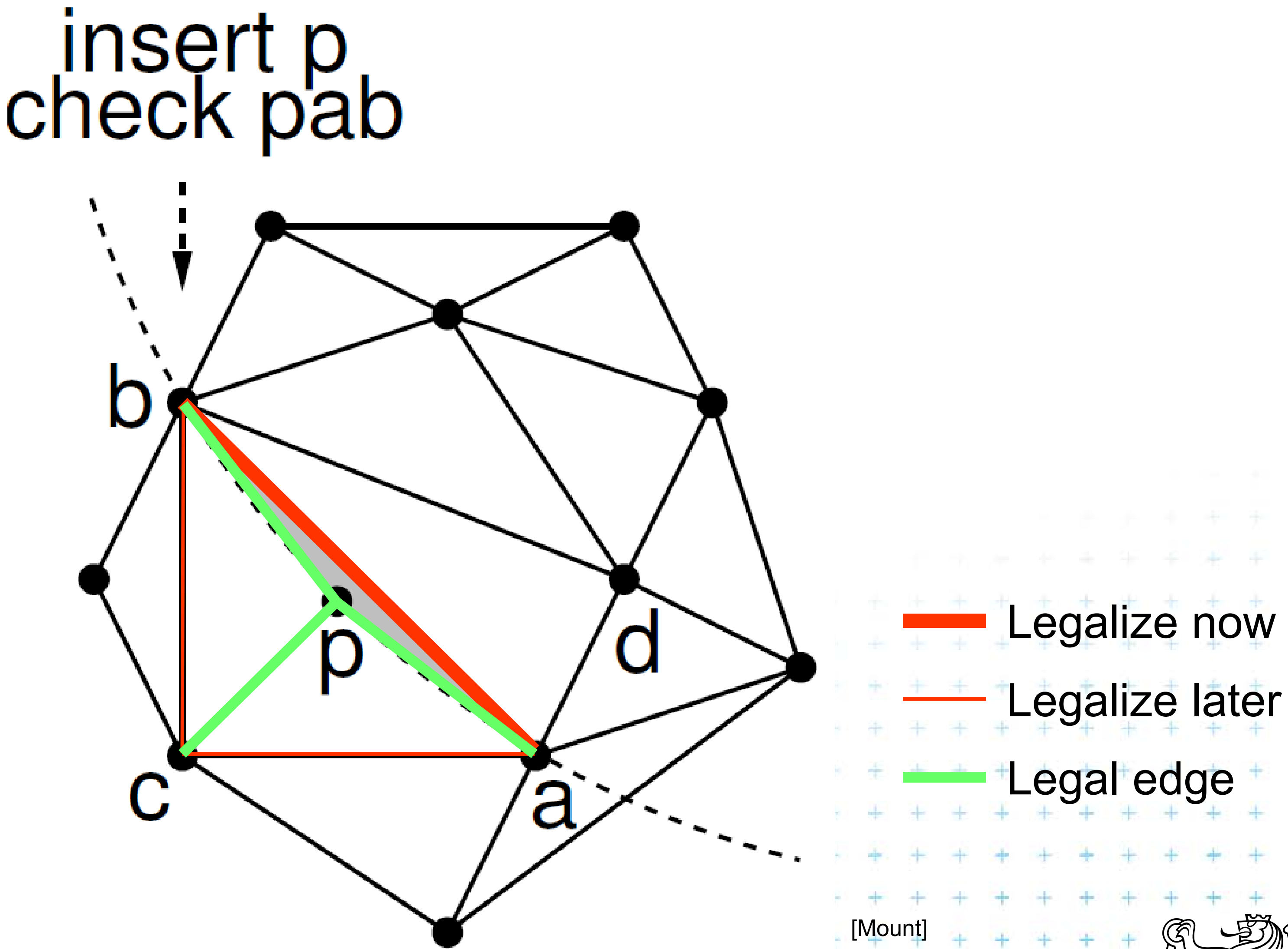
DT- point insert and mesh legalization



[Berg]

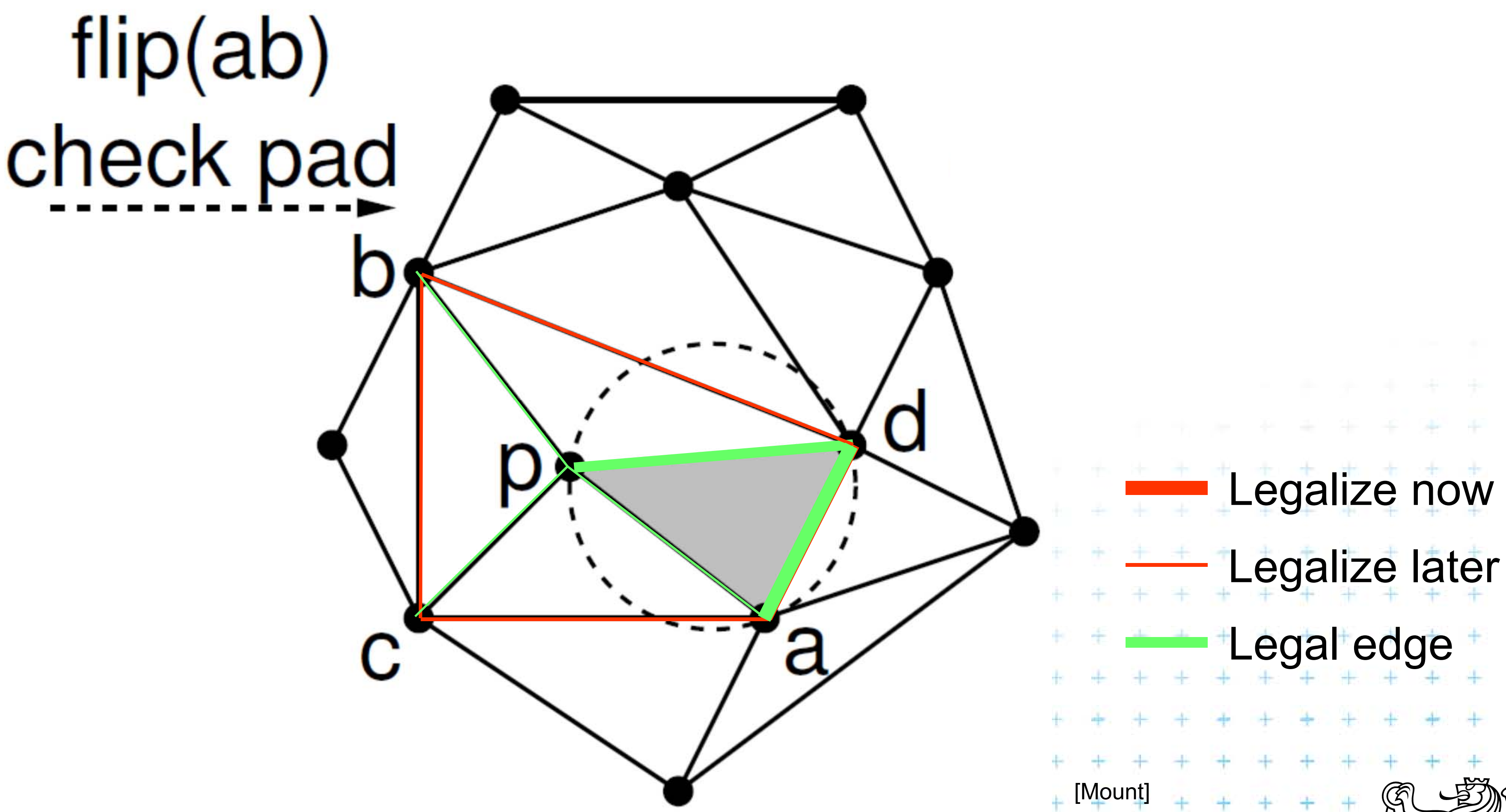
Every new edge created due to insertion of *p* will be incident to *p*

Delaunay triangulation – other point insert

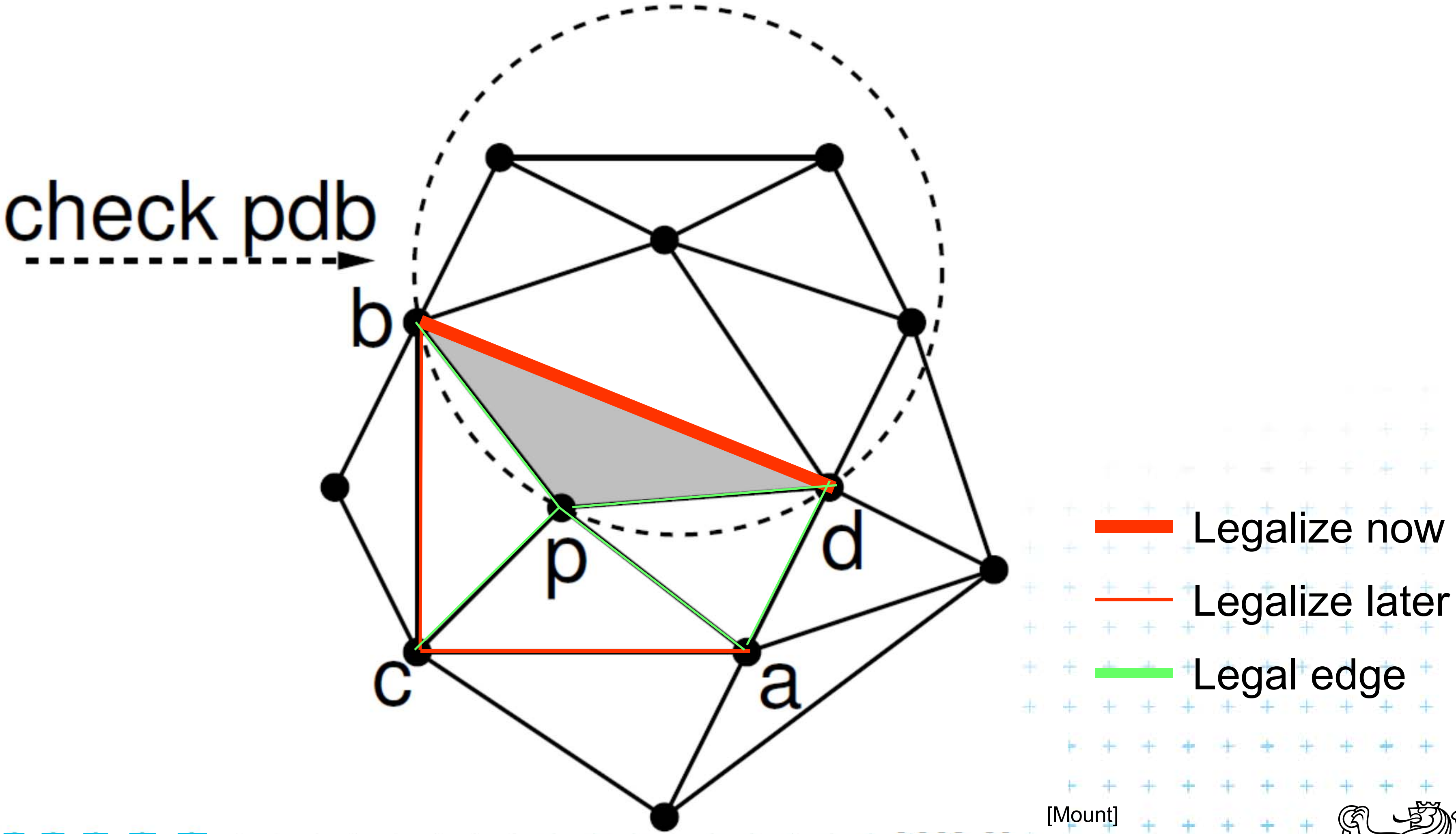


[Mount]

Delaunay triangulation – other point insert

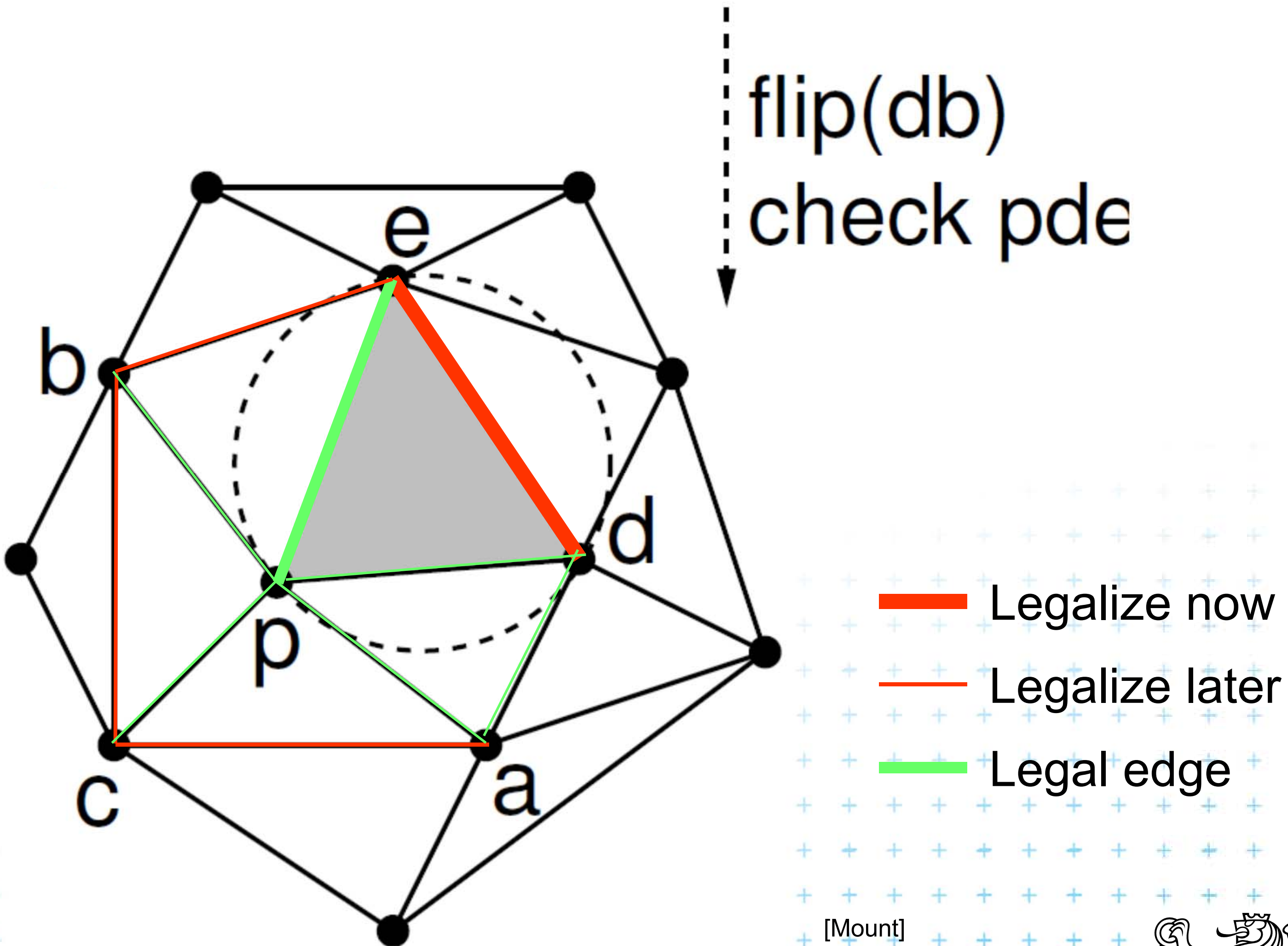


Delaunay triangulation – other point insert



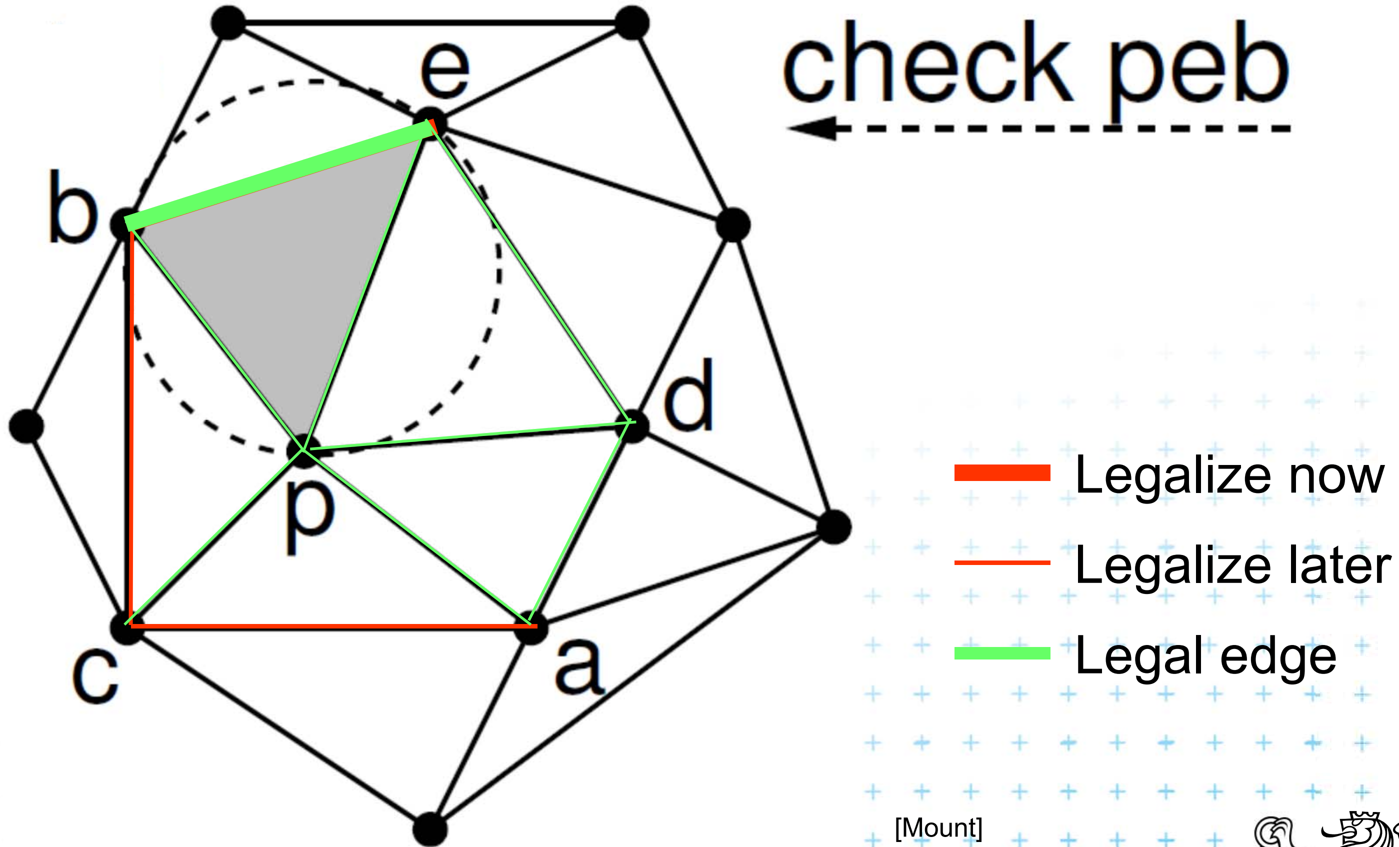
[Mount]

Delaunay triangulation – other point insert



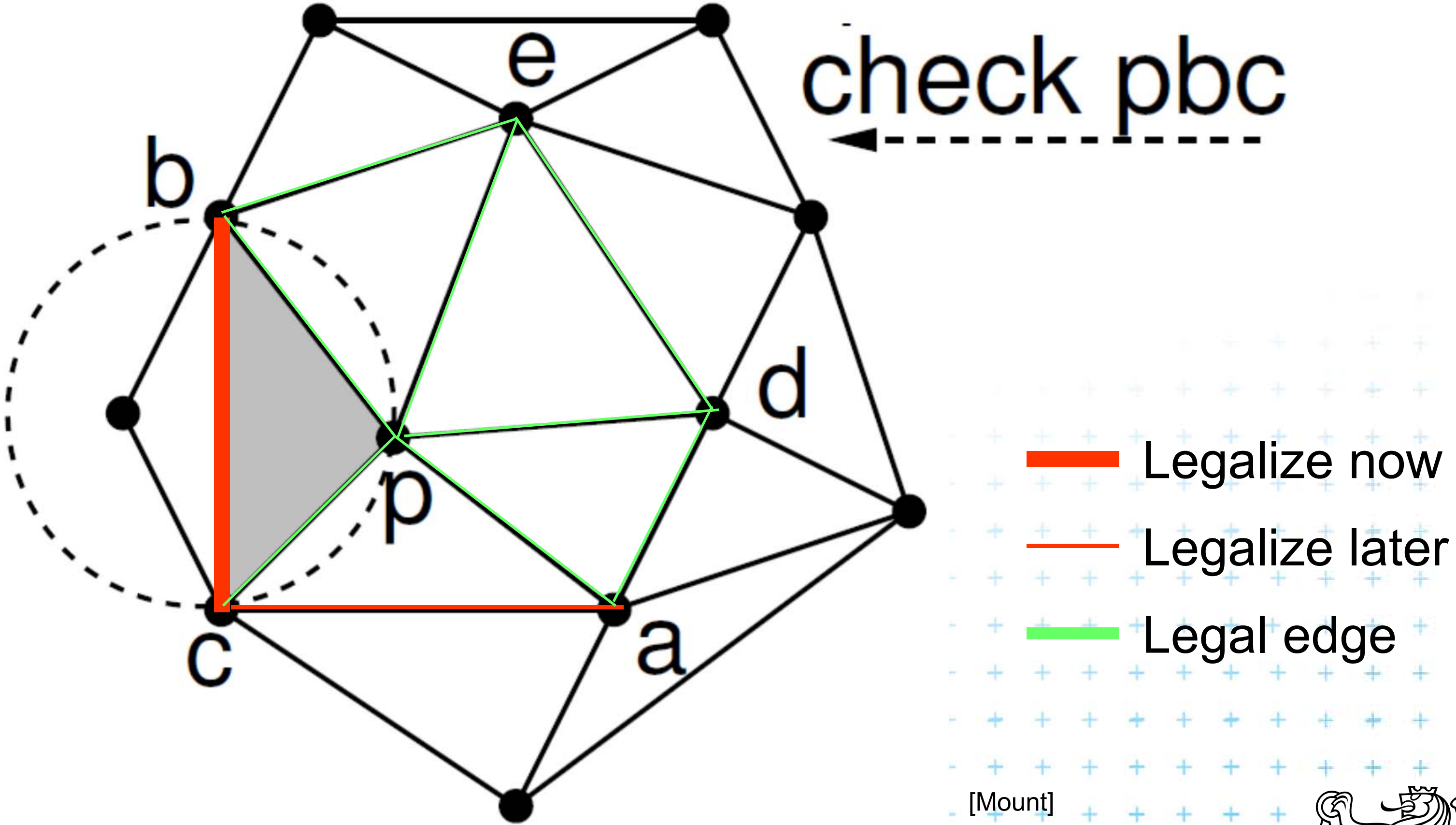
[Mount]

Delaunay triangulation – other point insert



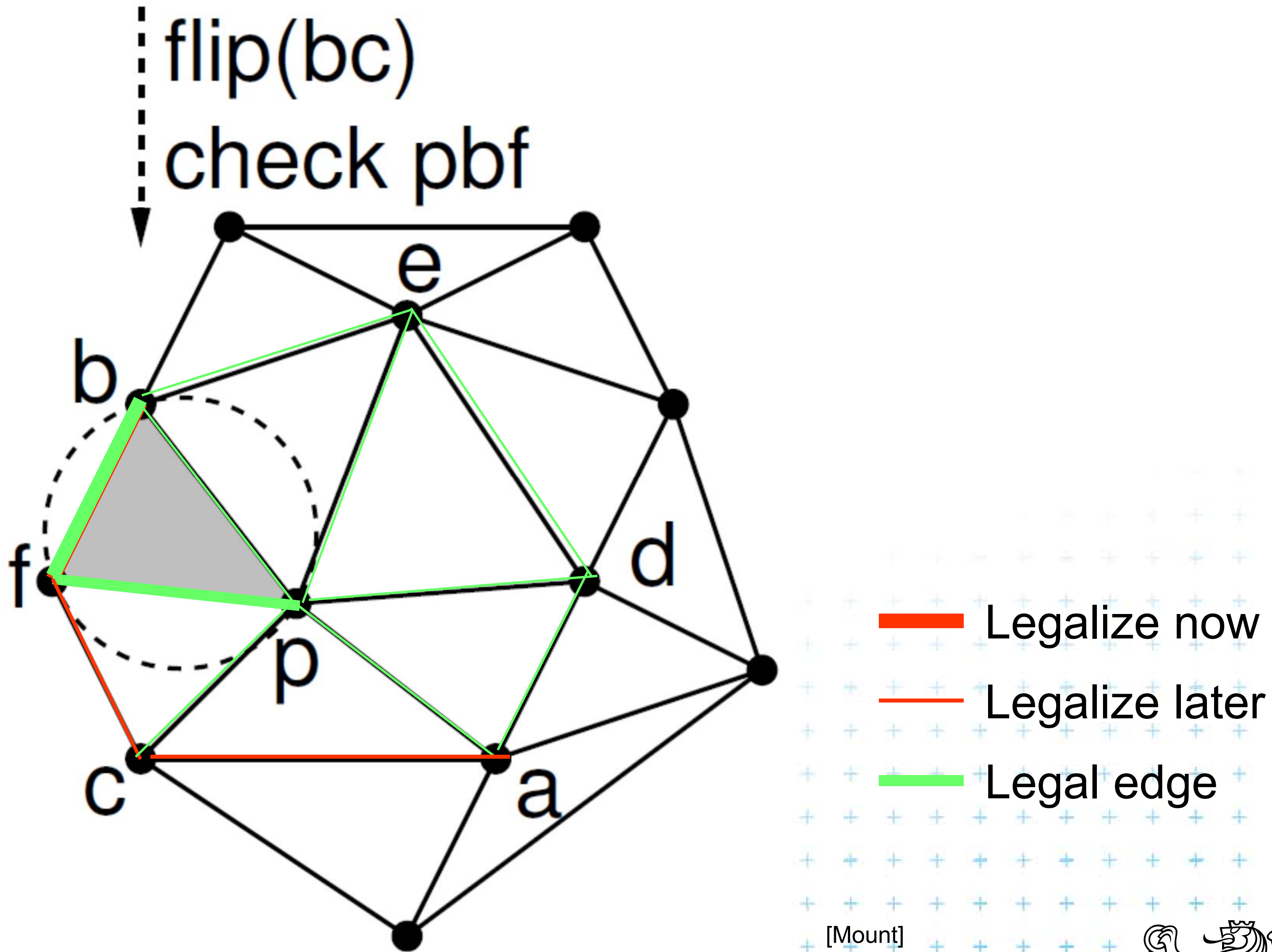
[Mount]

Delaunay triangulation – other point insert



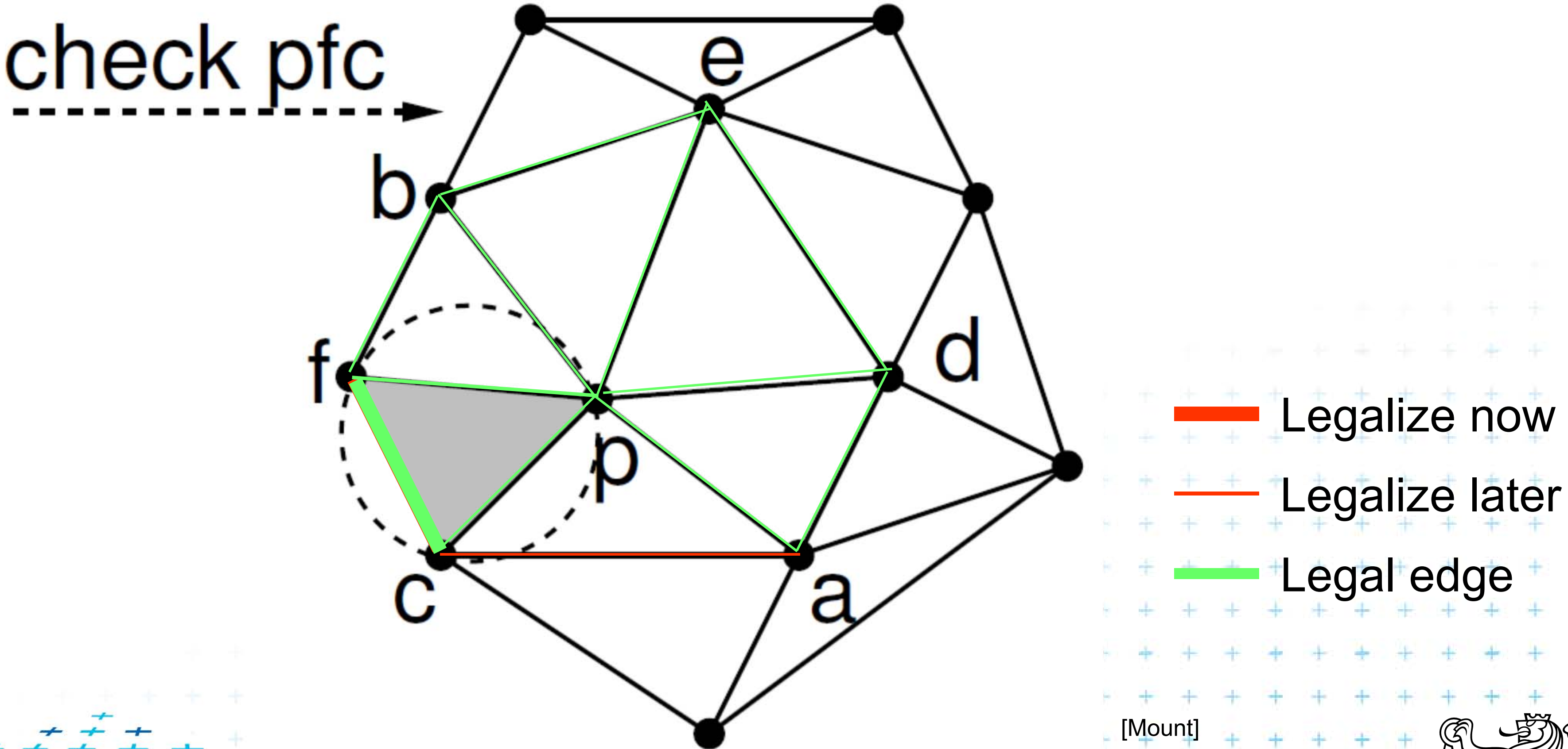
[Mount]

Delaunay triangulation – other point insert



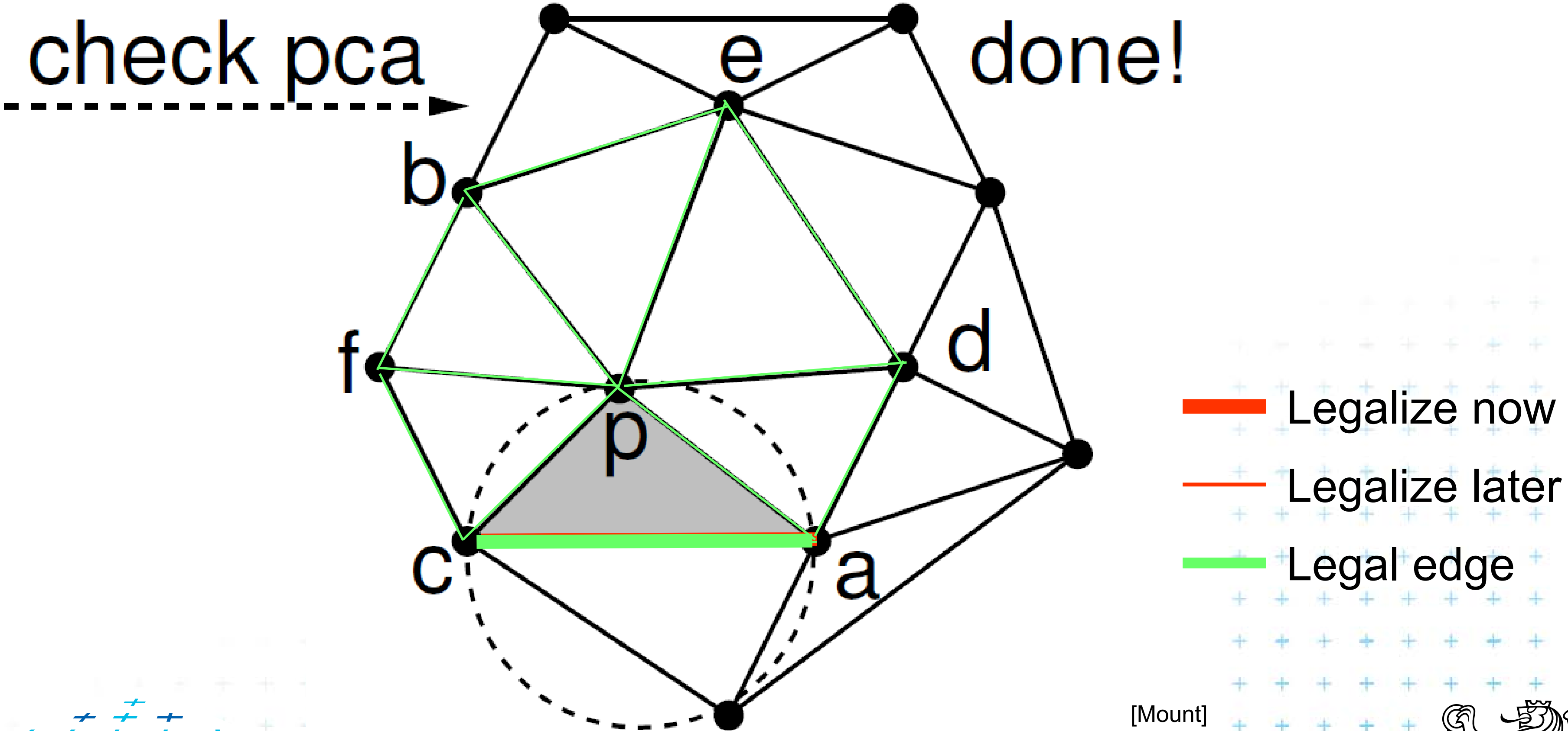
[Mount]

Delaunay triangulation – other point insert



[Mount]

Delaunay triangulation – other point insert



[Mount]

Correctness of the algorithm

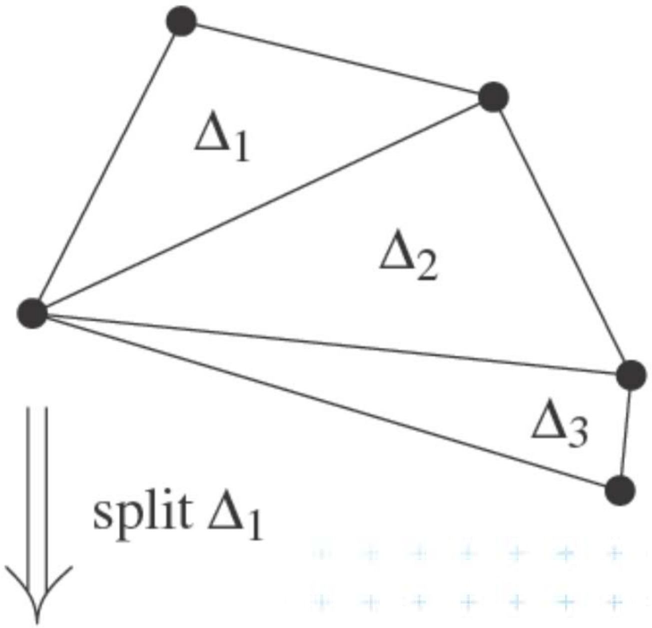
- Every new edge (created due to insertion of *p)*
  - is incident to *p*
  - must be legal
    => no need to test them
- Edge can only become illegal if one of its incident triangle changes
  - Algorithm tests any edge that may become illegal
    => the algorithm is correct
- Every edge flip makes the angle-vector larger
  => algorithm can never get into infinite loop

Point location data structure

- For finding a triangle *abc* µ *T* containing *p*
  - Leaves for active (current) triangles
  - Internal nodes for destroyed triangles
  - Links to new triangles
- Search *p*: start in root (initial triangle)
  - In each inner node of *T*:
    - Check all children (max three)
    - Descend to child containing *p*

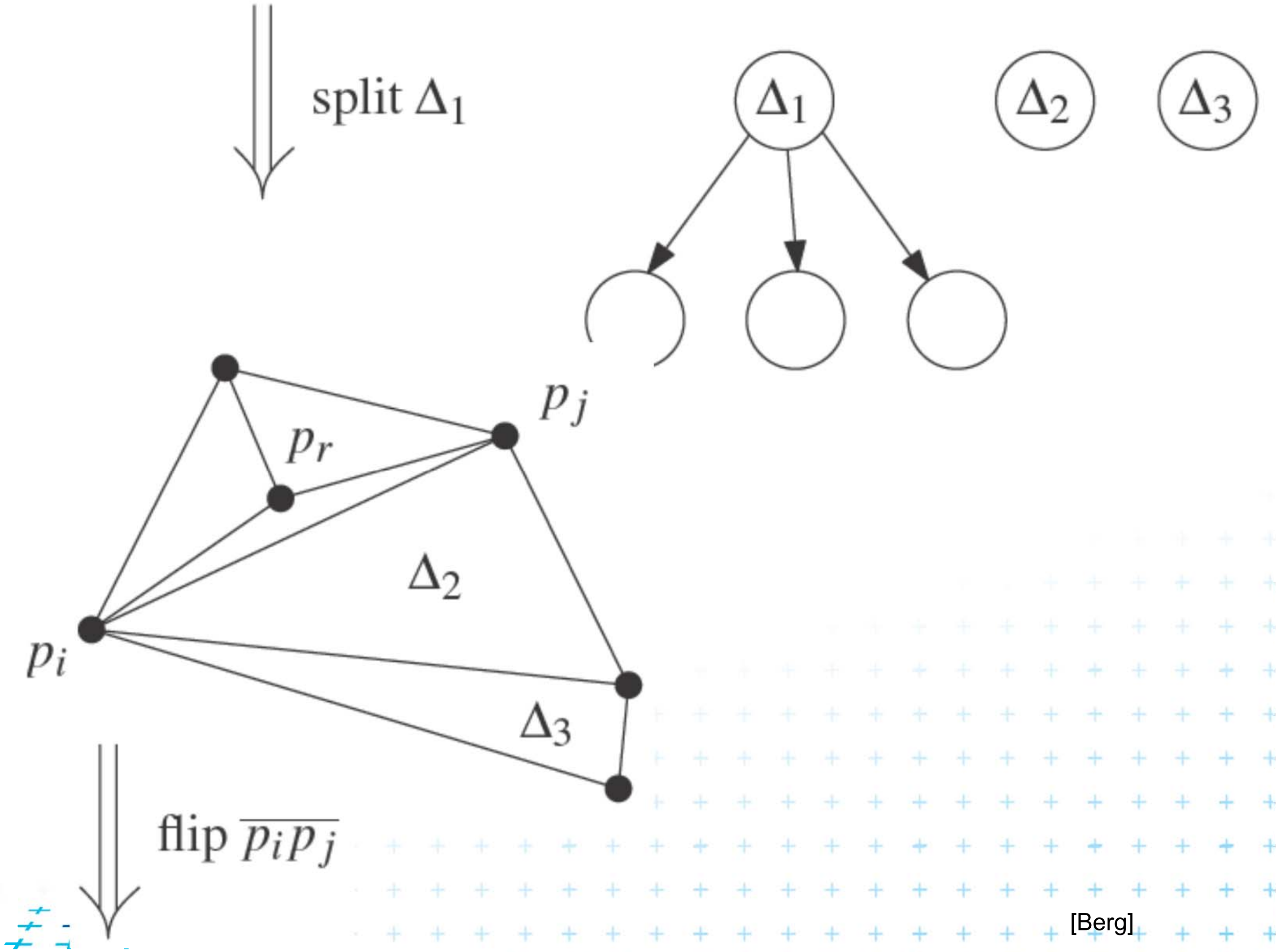
Point location data structure

Simplified
- it should also contain the root node



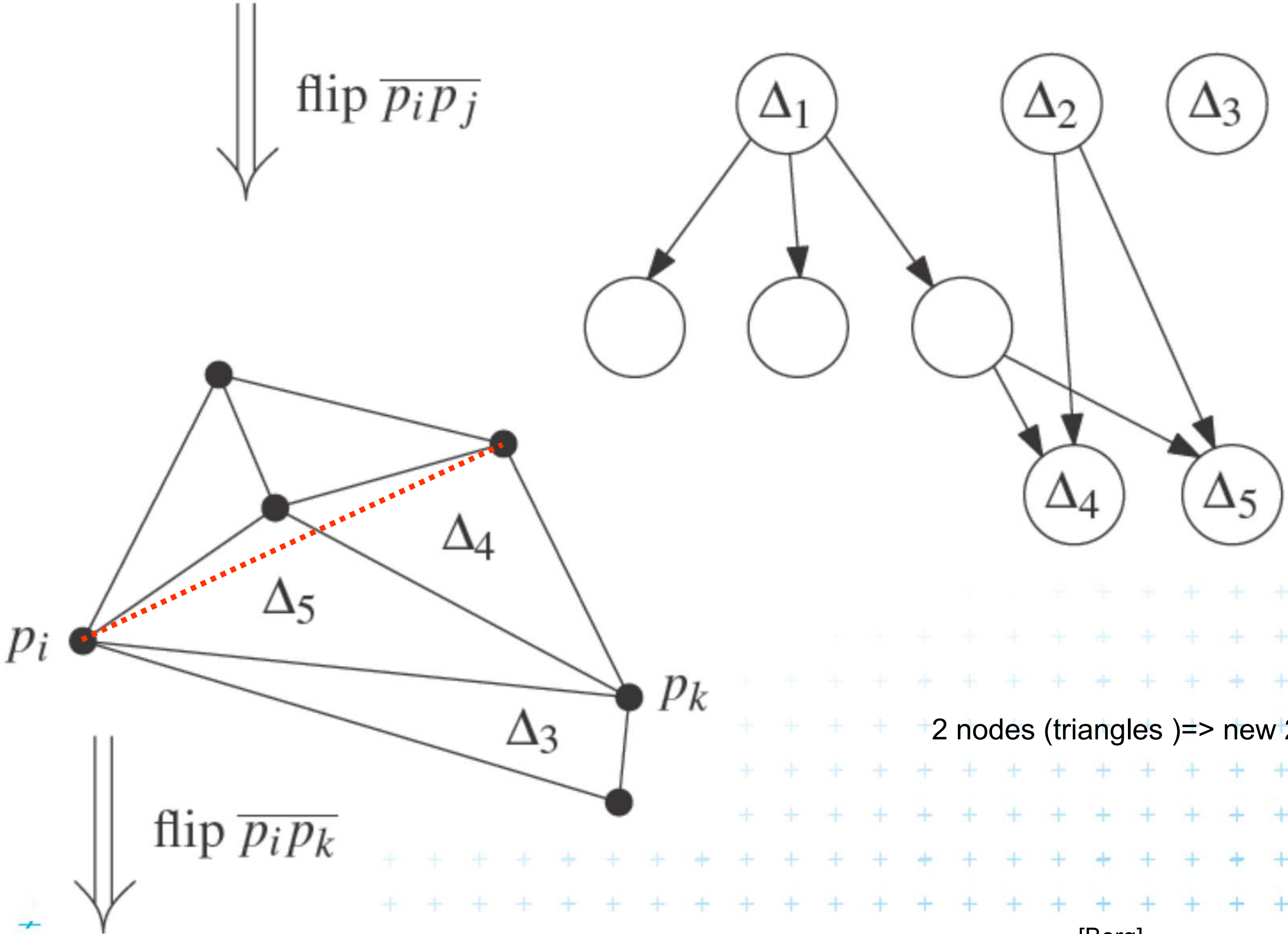
[Berg]

Point location data structure



[Berg]

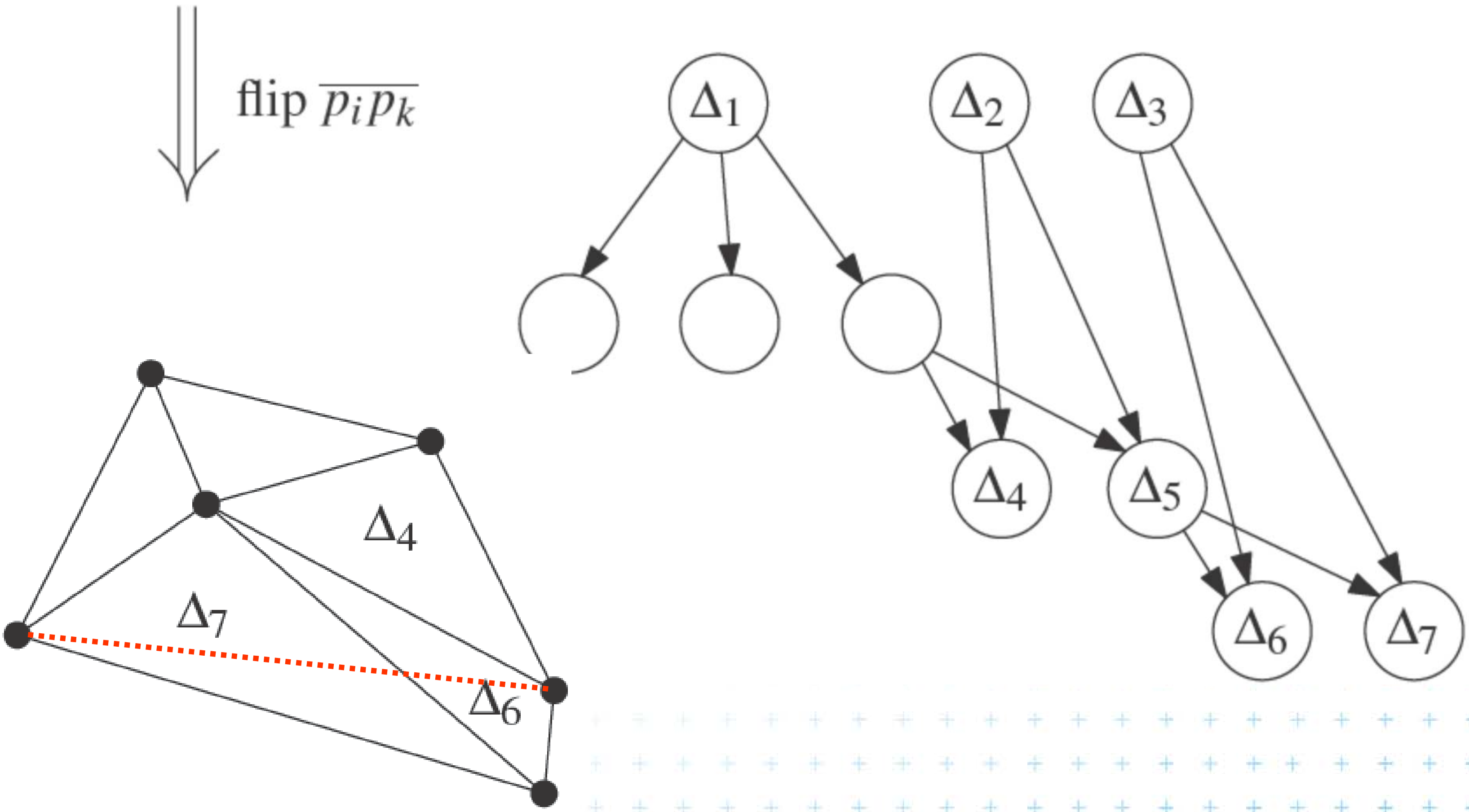
Point location data structure



2 nodes (triangles)=> new 2 nodes

[Berg]

Point location data structure

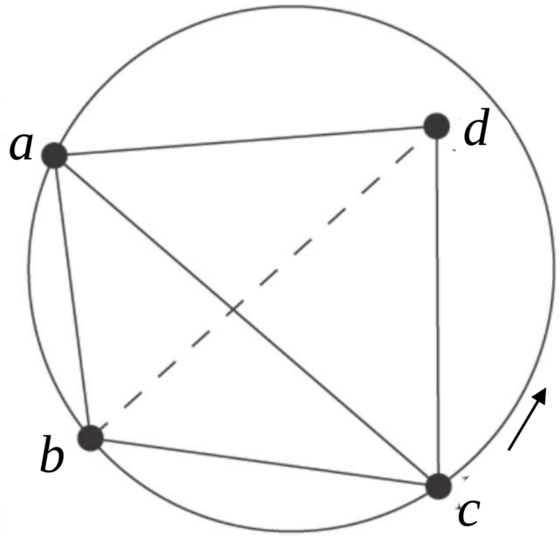


[Berg]

InCircle test

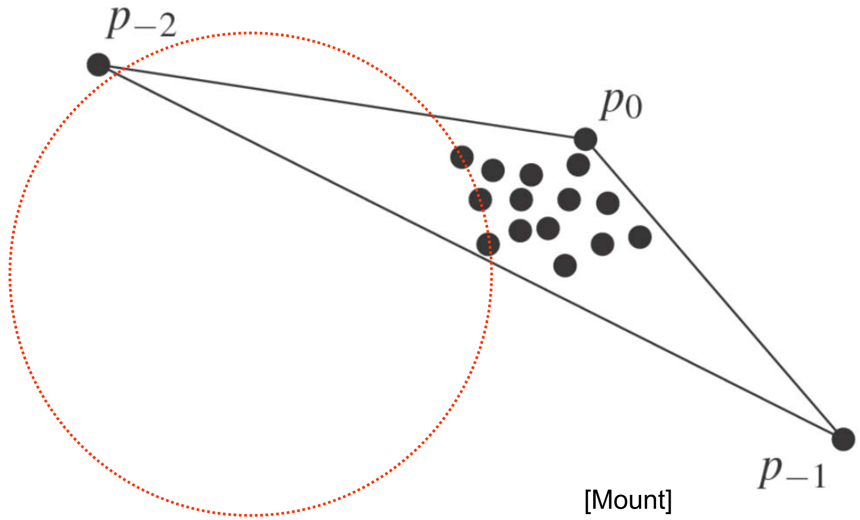
- *a,b,c* are counterclockwise in the plane
- Test, if *d* lies to the left of the oriented circle through *a,b,c*

$$\text{inCircle}(a,b,c,d) = \det \begin{pmatrix} a_x & a_y & a_x^2 + a_y^2 & 1 \\ b_x & b_y & b_x^2 + b_y^2 & 1 \\ c_x & c_y & c_x^2 + c_y^2 & 1 \\ d_x & d_y & d_x^2 + d_y^2 & 1 \end{pmatrix} > 0$$



[Mount]

Creation of the initial triangle



- For given points set P
- Initial triangle $p_{-2}p_{-1}p_0$
  - Must contain all points of P
  - Must not be (none of its points) in any circle defined by non-collinear points of P
- l_{-2} = horizontal line above P
- l_{-1} = horizontal line below P
- p_{-2} = lies on l_{-2} as far left that p_{-2} lies outside every circle
- p_{-1} = lies on l_{-1} as far right that p_{-1} lies outside every circle defined by 3 non-collinear points of P

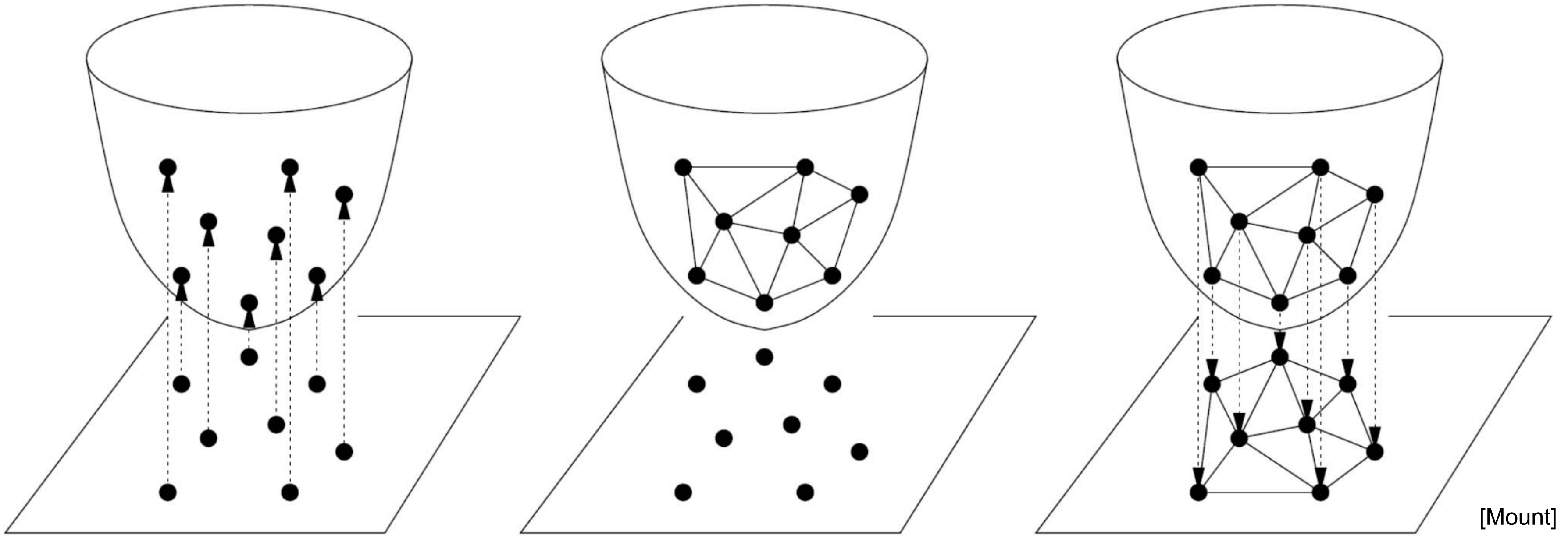
- Symbolical tests with this triangle => p_{-1} and p_{-2} always out

Complexity of incremental DT algorithm

- Delaunay triangulation of a set *P* in the plane can be computed in
  - O(n log n) expected time
  - using O(n) storage
- For details see [Berg, Section 9.4]

Delaunay triangulations and Convex hulls

- Delaunay triangulation in R^d can be computed as part of the convex hull in R^{d+1} (lower CH)
- 2D: Connection is the paraboloid: $z = x^2 + y^2$



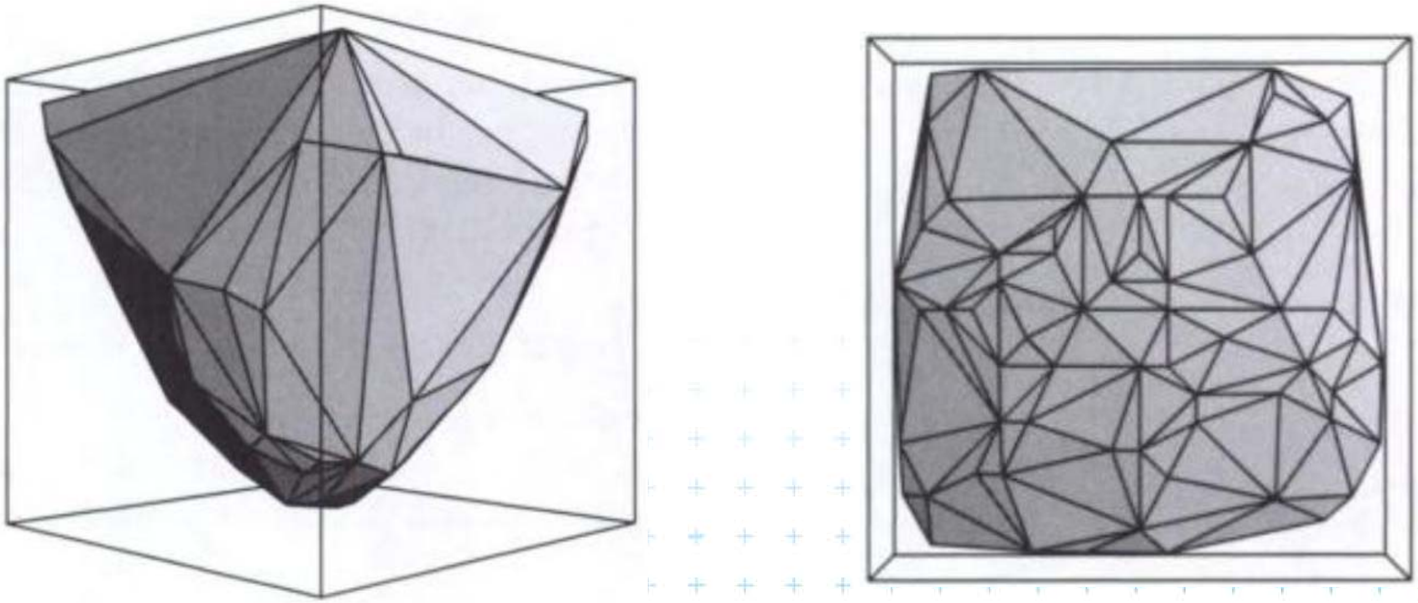
Project onto paraboloid. Compute convex hull. Project hull faces back to plane.

Vertical projection of points to paraboloid

- Vertical projection of 2D point to paraboloid in 3D

$$(x, y) \to (x, y, x^2 + y^2)$$

- Lower convex hull
  = portion of CH visible from $z = -\infty$



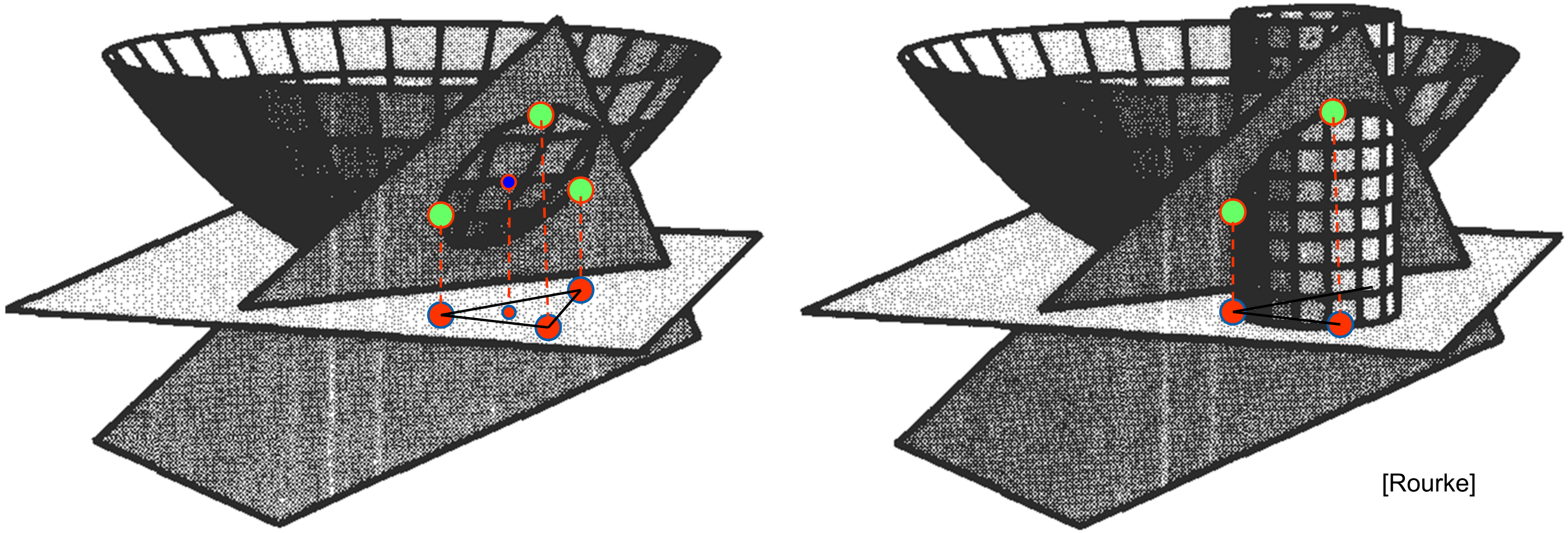
[Rourke]

Relation between CH and DT

- Delaunay condition (2D)
  Points *p,q,r* µ *S* form a Delaunay triangle **iff** the circumcircle of *p,q,r* is empty (contains no point)
- Convex hull condition (3D)
  Points *p',q',r'* µ *S'* form a face of *CH*(*S'*) **iff** the plane passing through *p',q',r'* is supporting *S'*
  – all other points lie to one side of the plane
  – plane passing through *p',q',r'* is supporting hyperplane of the convex hull CH(S')

DCGI

Relation between CH and DT



[Rourke]

- 4 distinct points *p,q,r,s* in the plane, and let *p', q', r', s'* be their respective projections onto the paraboloid, $z = x^2 + y^2$.
- The point *s* lies within the circumcircle of *pqr* **iff** s' lies on the lower side of the plane passing through *p', q', r'*.

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

- Evaluates to $2a$ and $2b$
- Plane: $z = 2ax + 2by + \gamma$

$$\gamma = -(a^2 + b^2)$$

$$a^2 + b^2 = 2a.a + 2b.b + \gamma$$

- Tangent plane through point $(a, b, a^2 + b^2)$

$$z = 2ax + 2by - (a^2 + b^2)$$

[Mount]

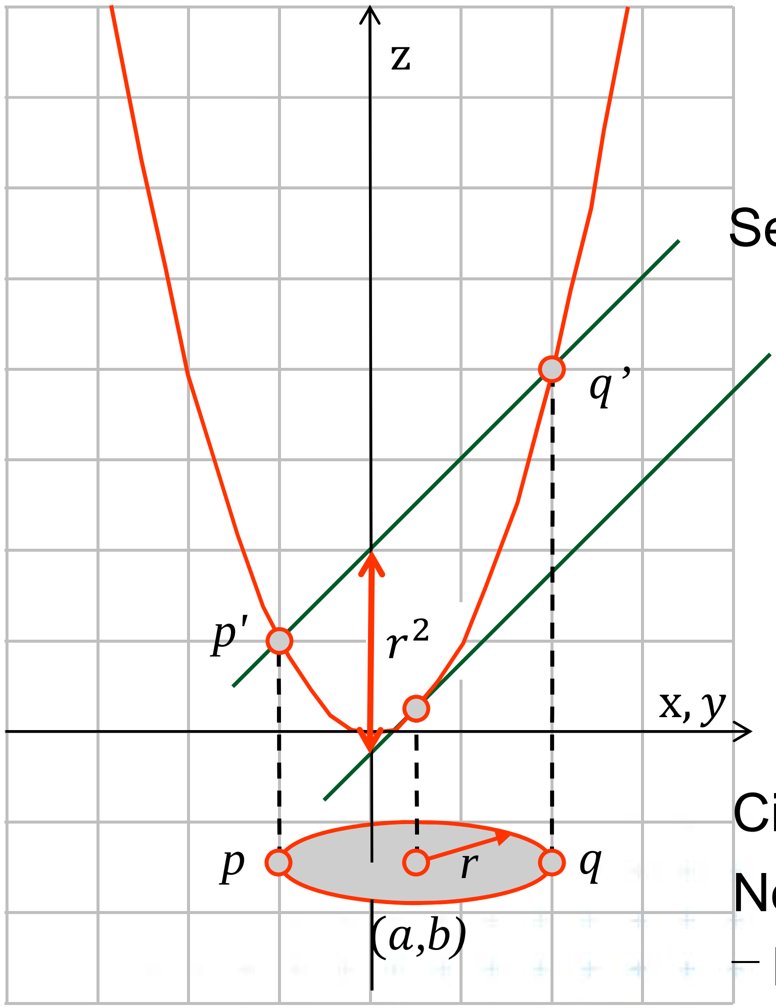
Plane intersecting the paraboloid (secant plane)

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
  $z = 2ax + 2by - (a^2 + b^2)$
- Shift this plane r^2 upwards –> secant plane intersects the paraboloid in an ellipse in 3D
  $z = 2ax + 2by - (a^2 + b^2) + r^2$
- Eliminate *z* (project to 2D) $z = x^2 + y^2$
  $x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + r^2$
- This is a circle projected to 2D with center (a, b):

$$(x - a^2) + (y - b^2) = r^2$$

[Mount]

Tangent and secant planes



Cross section of the paraboloid

Secant plane

Tangent plane

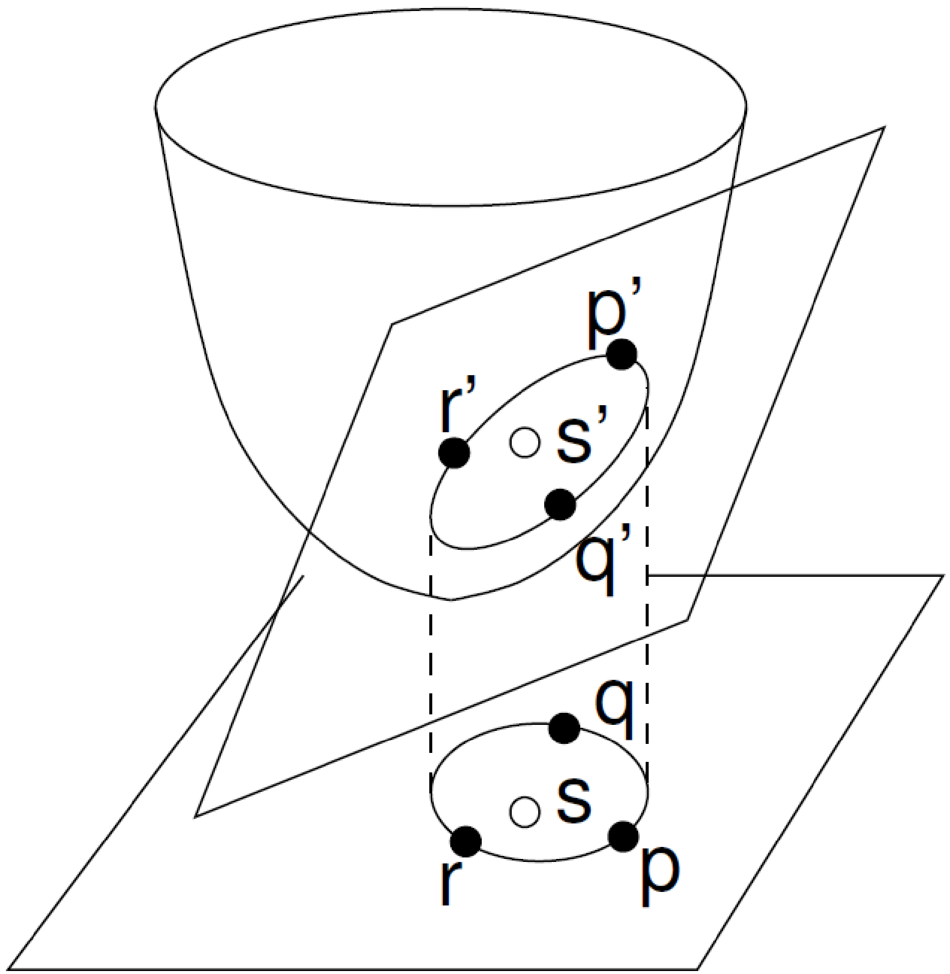
Circle in *xy* plane

Note: the circle is moved a little down

– points *p* and *q should lie in the* xy *plane*

– the circle too

Secant plane defined by three points



[Mount]

Test inCircle – meaning in 3D

- Points *p,q,r* are counterclockwise in the plane
- Test, if *s* lies in the circumcircle of Δ*pqr* is equal to
  - = test, weather *s'* lies within a lower half space of the plane passing through *p',q',r'* (3D)
  - = test, if quadruple *p',q',r',s'* is positively oriented (3D)
  - = test, if *s lies* to the left of the oriented circle through *abc* (2D)

$$\mathrm{in}(p,q,r,s) = \det \begin{pmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{pmatrix} > 0.$$

[Mount]

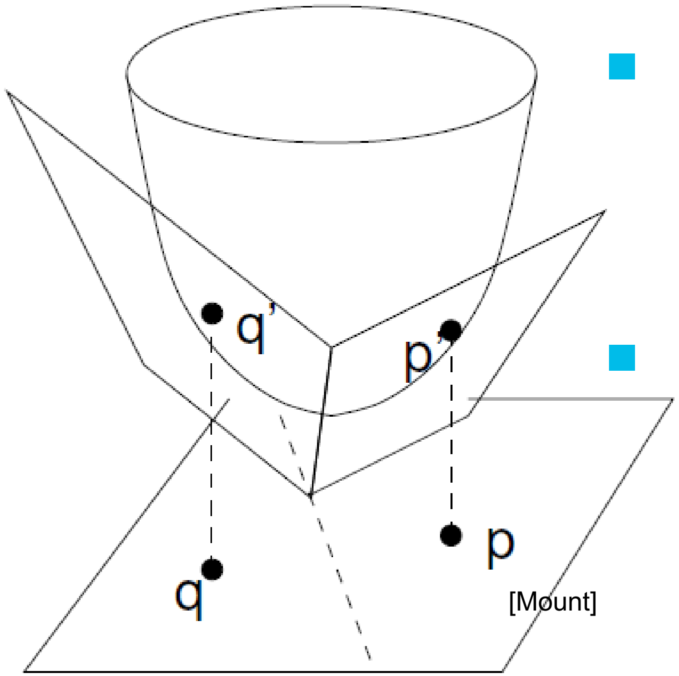
An the Voronoi diagram?

- VD and DT are dual structures
- Points and lines in the plane
  are dual to
  points and planes in 3D space
- VD of points in the plane
  can be transformed to
  intersection of halfspaces in 3D space

Voronoi diagram as upper envelope in R^{d+1}

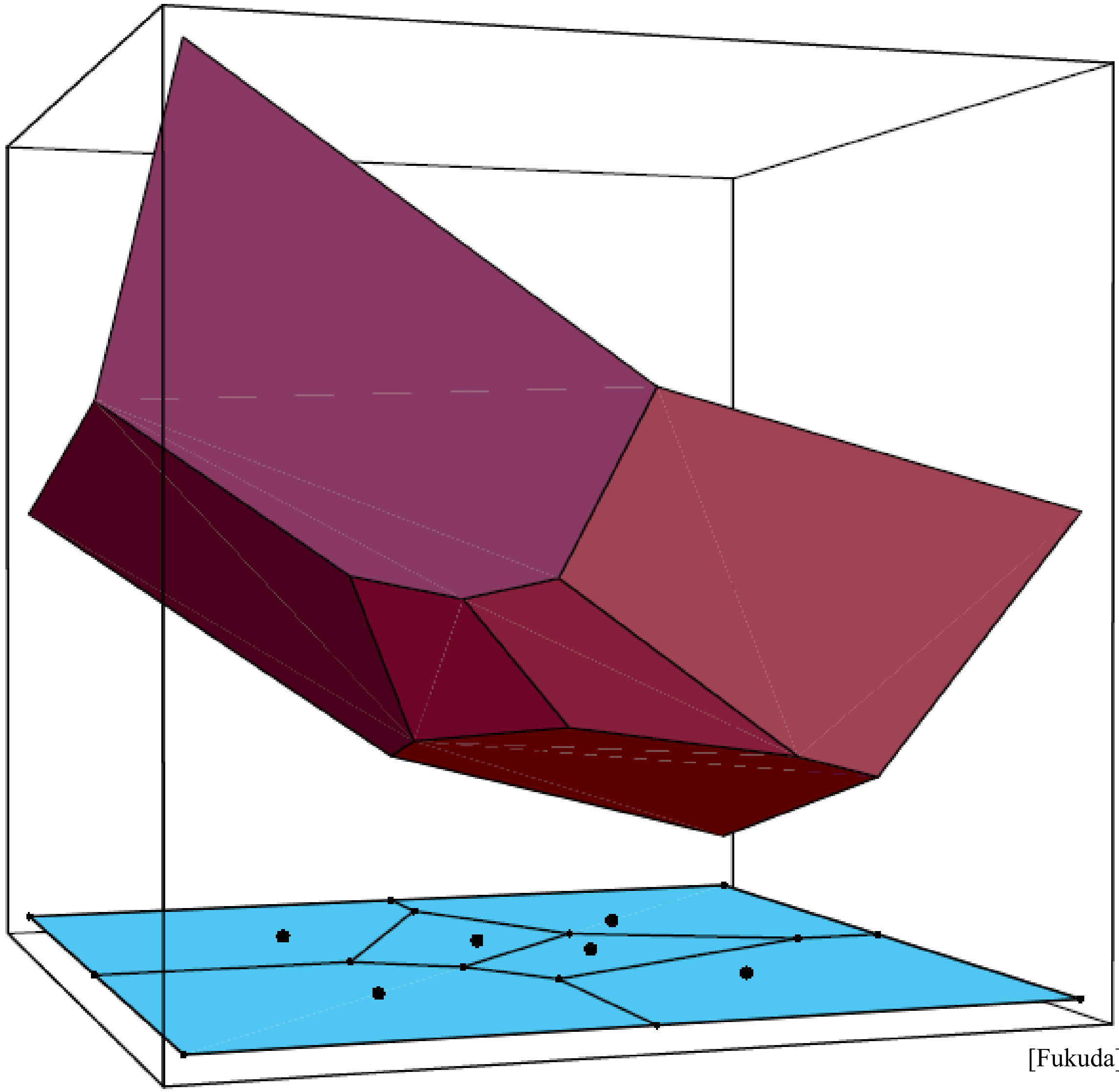
- For each point $p = (a, b)$ a tangent plane to the paraboloid is $z = 2ax + 2by - (a^2 + b^2) + r^2$
- $H^+(p)$ is the set of points above this plane

$$H^+(p) = \{(x, y, z) \mid z \geq 2ax + 2by - (a^2 + b^2) + r^2$$



- *VD of points in the plane can be computed as intersection of halfspaces $H^+(p_i)$*
- *This intersection of halfspaces*
  = unbounded convex polyhedron
  = upper envelope of halfspaces $H^+(p_i)$

Voronoi diagram as upper envelope in 3D



[Fukuda]

Derivation of projected Voronoi edge

- 2 points: $p = (a, b)$ and $q = (c, d)$ in the plane

  $z = 2ax + 2by - (a^2 + b^2)$
  $z = 2cx + 2dy - (c^2 + d^2)$

  Tangent planes to paraboloid

- Intersect the planes, project onto xy (eliminate *z*)

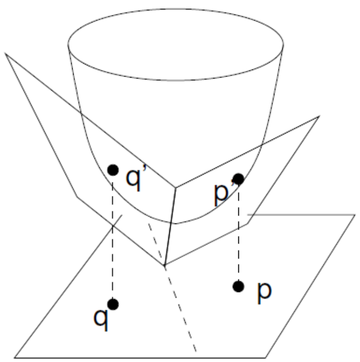
  $$x(2a - 2c) + y(2b - 2d) = (a^2 - c^2) + (b^2 - d^2)$$

- This line passes through midpoint between p and q

  $$\frac{a+c}{2}(2a - 2c) + \frac{b+d}{2}(2b - 2d) = (a^2 - c^2) + (b^2 - d^2)$$

- It is perpendicular bisector with slope

  $$-(a - c)/(b - d)$$



[Mount]

References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: Algorithms and Applications, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapters 3 and 9, http://www.cs.uu.nl/geobook/

[Mount] David Mount, - CMSC 754: Computational Geometry, Lecture Notes for Spring 2007, University of Maryland, Lectures 7,22, 13,14, and 30. http://www.cs.umd.edu/class/spring2007/cmsc754/lectures.shtml

[Rourke] Joseph O´Rourke: .: Computational Geometry in C, Cambridge University Press, 1993, ISBN 0-521- 44592-2 http://maven.smith.edu/~orourke/books/compgeom.html

[Fukuda] Komei Fukuda: Frequently Asked Questions in Polyhedral Computation. Version June 18, 2004 http://www.ifor.math.ethz.ch/~fukuda/polyfaq/polyfaq.html