DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM

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Based on [Berg] and [Mount]

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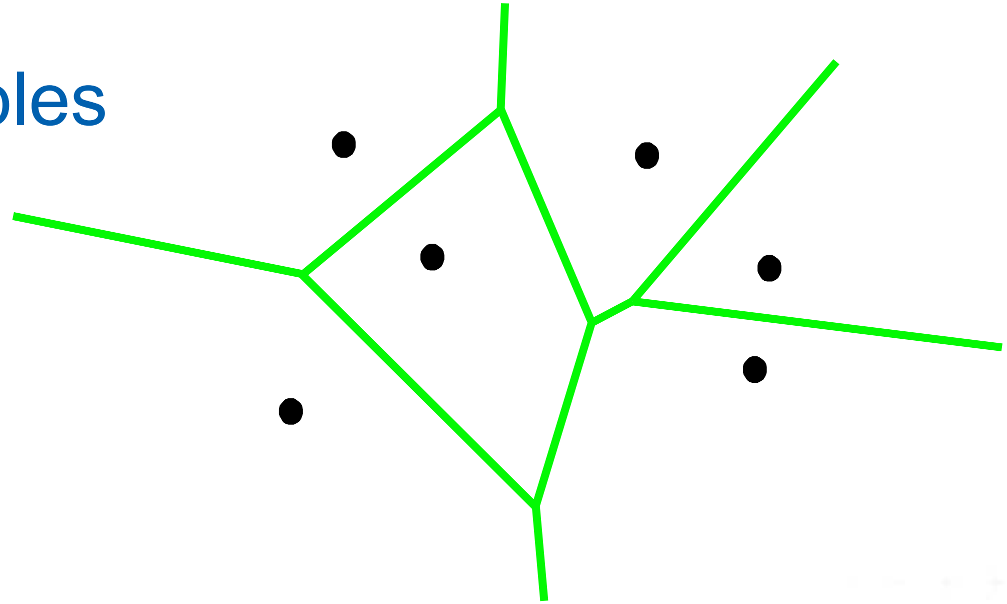
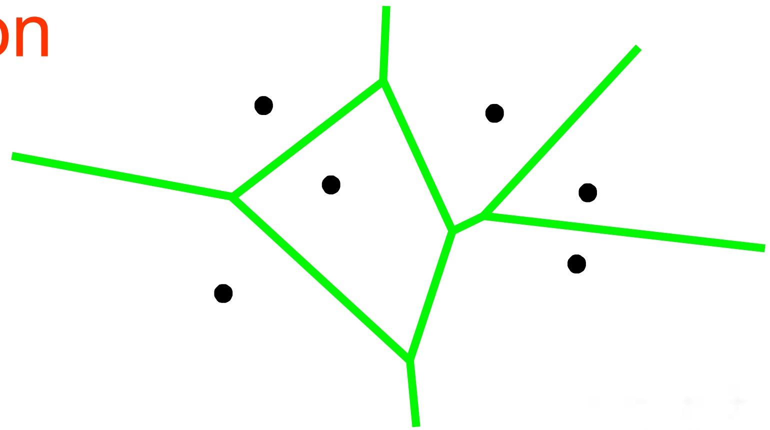
Talk overview

- Definition and examples
- Applications
- Algorithms in 2D
  - D&C O(n log n)
  - Sweep line O(n log n)

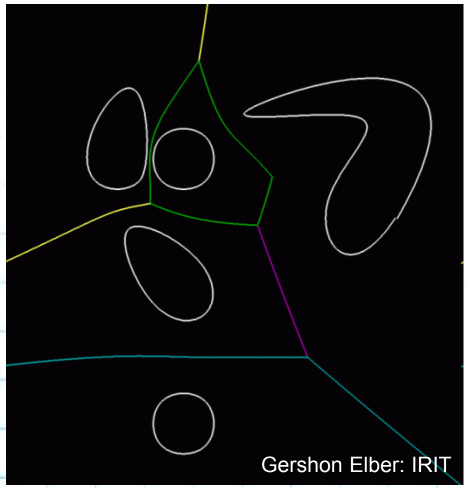


Voronoi diagram (VD)

- One of the most important structure in Comp. geom.
- Encodes proximity information
  What is close to what?

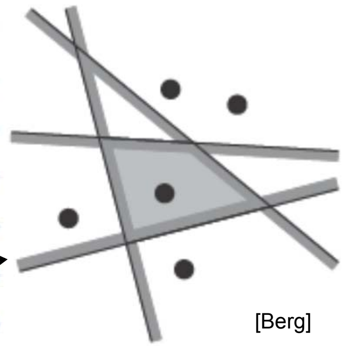


- Standard VD – this lecture
  - Set of points - nDim
  - Euclidean space & metric
- Generalizations
  - Set of line segments or curves
  - Different metrics
  - Higher order VD's (furthest point)



Voronoi cell (for points in plane)

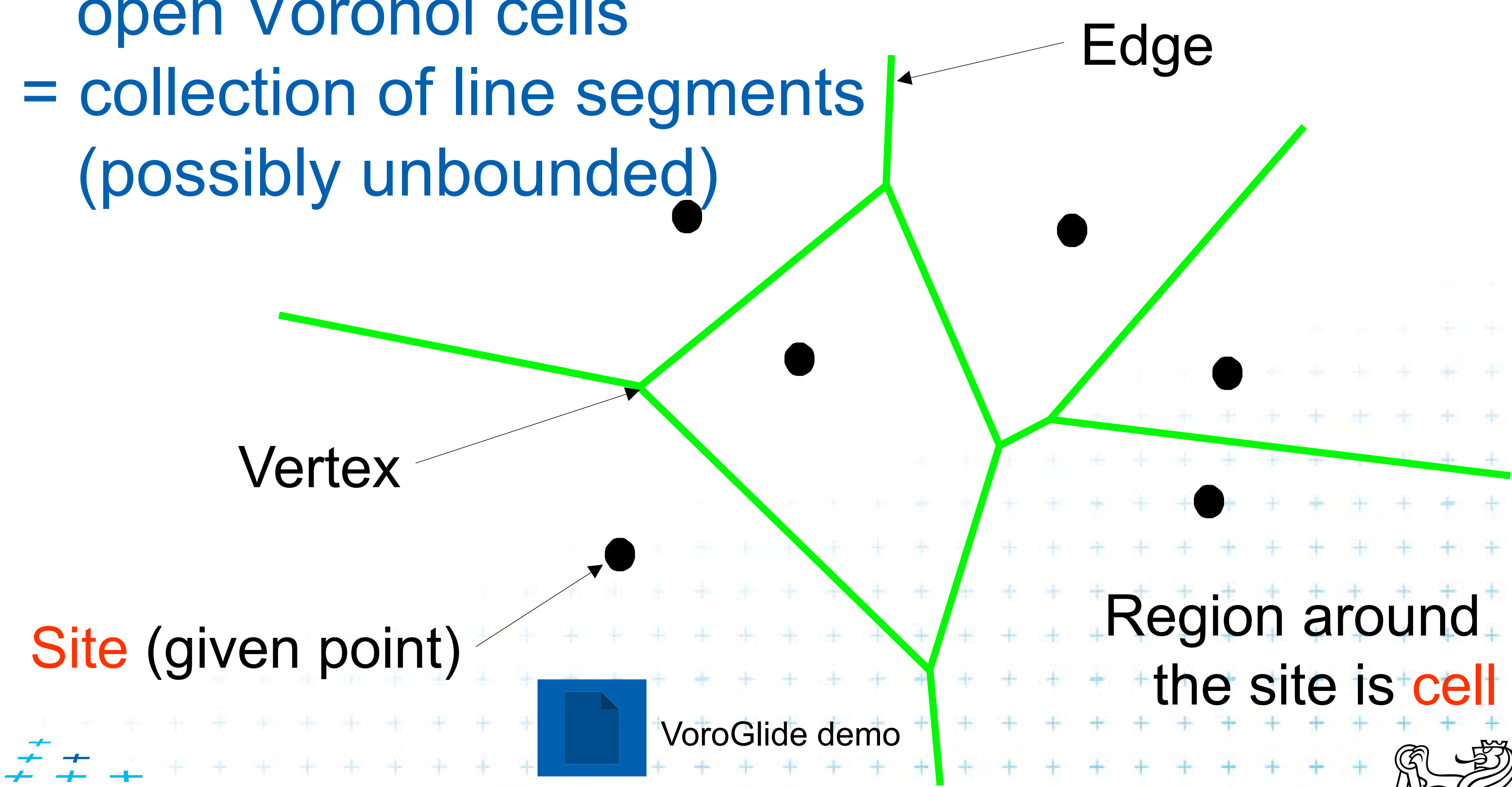
- Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of points (*sites*) in dDim space … 2D space (plane) here
- Voronoi cell $V(p_i)$ – is open!
  = set of points closer to p_i than to any other site:
  $$V(p_i) = \{q, \|p_i q\| < \|p_j q\|, \forall j \neq i\}$$, where
  $\|pq\|$ is the Euclidean distance between p and q
  = intersection of open halfplanes
  $$V(p_i) = \bigcap_{j \neq i} h(p_i, p_j)$$
  $h(p_i, p_j)$ = open halfplane
  = set of pts strictly closer to p_i than to p_j



[Berg]

Voronoi diagram (in plane)

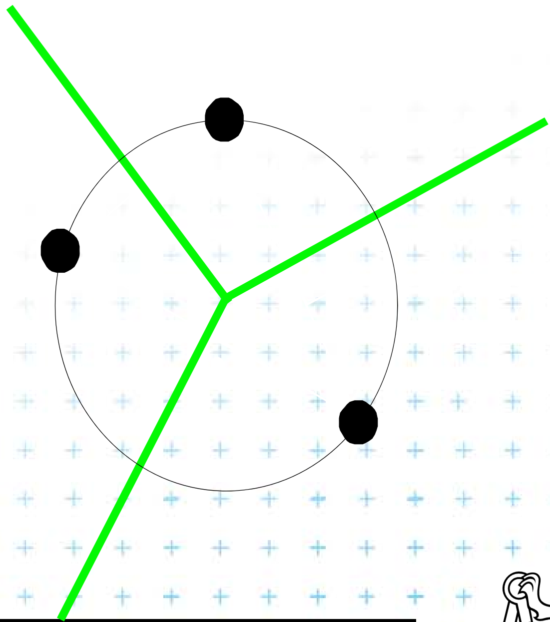
- Voronoi diagram Vor(*P*) of points *P*
  = what is left of the plane after removing all the open Voronoi cells
  = collection of line segments (possibly unbounded)



VoroGlide demo

Voronoi diagram examples

1 point 2 points 3 points



Cell

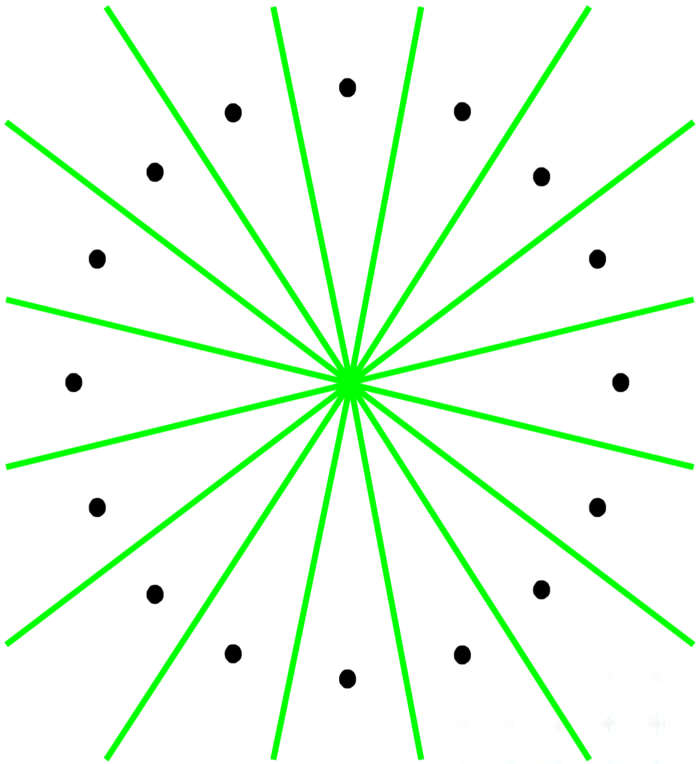
- The whole plain for 1 point
- Halfplane or strip for collinear points
- Convex (possibly unbounded) polygon

Edges of VD

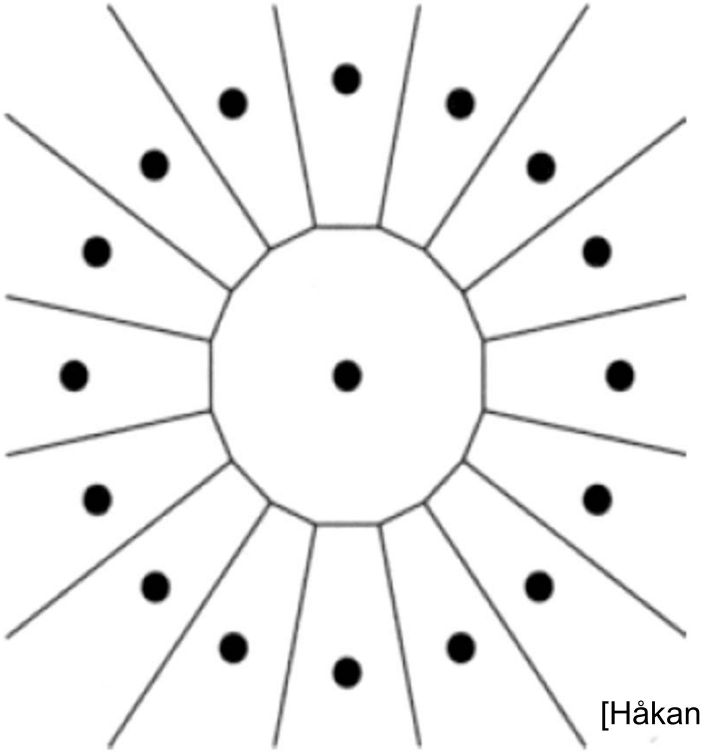
- || lines for collinear points
- Halflines (for CH points)
- Line segments (for bounded cells)

Voronoi diagram examples

16 points



17 points

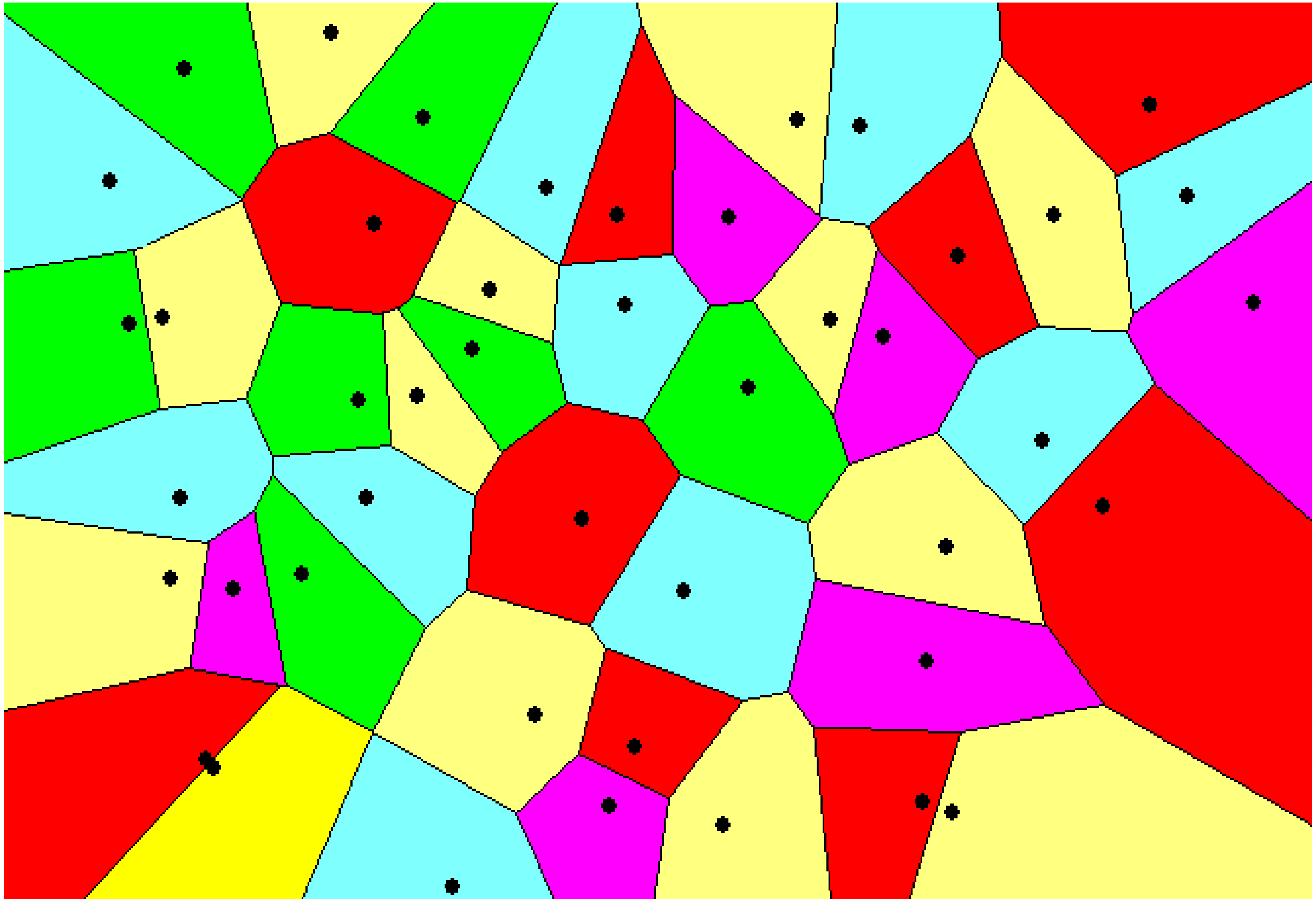


[Håkan Jonsson]

Vertex with O(n) incident edges
From total $|n_e| \leq 3n - 6$

Cell with O(n) vertices
From total $|n_v| \leq 2n\text{-}5$

Voronoi diagram examples



Voronoi diagram (in plane)

= planar graph

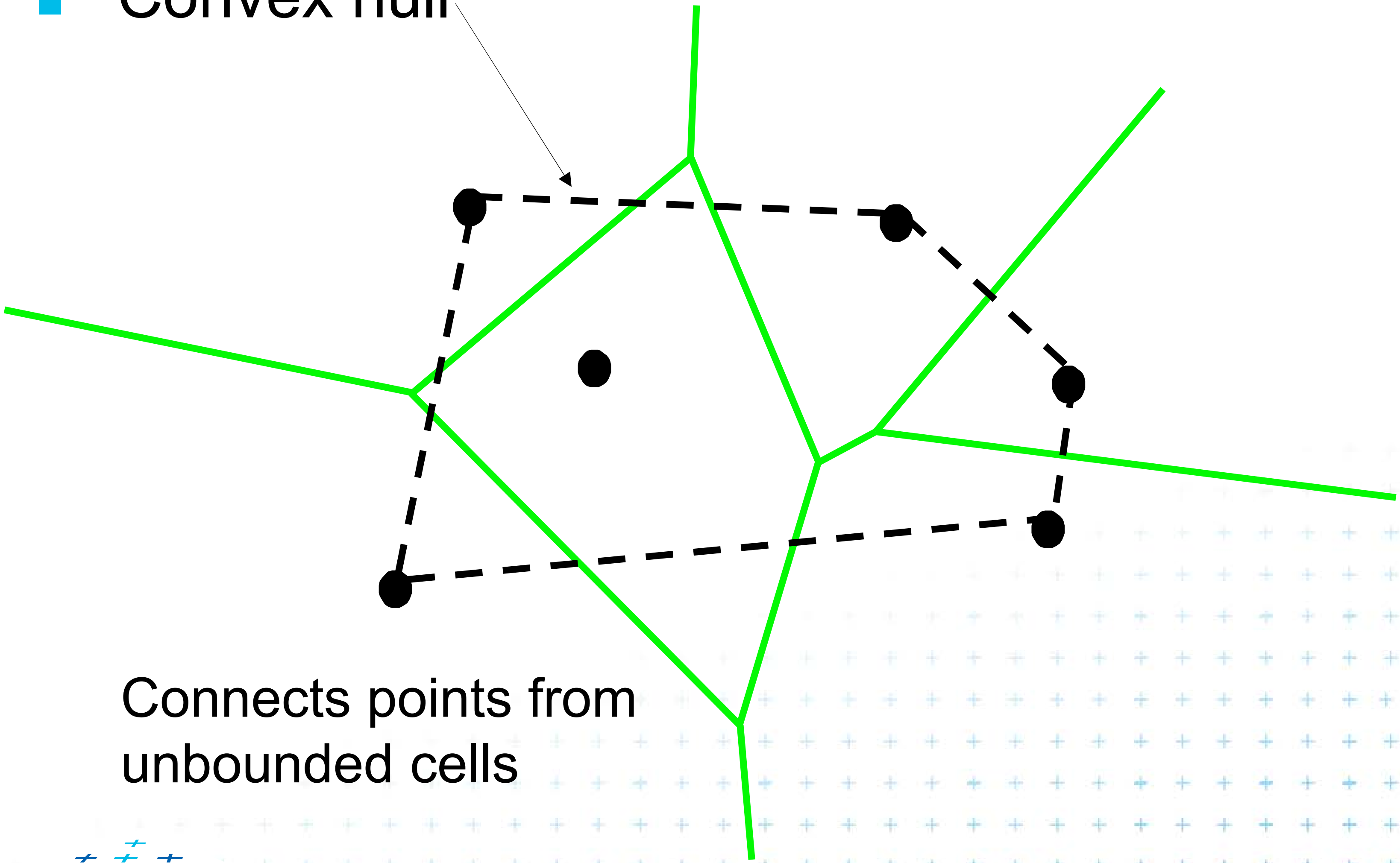
- Subdivides plane into n cells (n = num. of input sites |P|)
- Edge = locus of equidistant pairs of points (cells)
  = part of the bisector of these points
- Vertex = center of the circle defined by ≥ 3 points
  => vertices have degree ≥ 3
- Number of vertices $n_v \leq 2n - 5$ => $O(n)$
- Number of edges $n_e \leq 3n - 6$ => $O(n)$
  (only $O(n)$ from $O(n^2)$ intersections of bisectors)
- In higher dimensions complexity from $O(n)$ up to $O(n^{\lceil d/2 \rceil})$
- Unbounded cells belong to sites (points) on convex hull

Voronoi diagram O(n) complexity derivation

- For n collinear sites $n_v = 0$, $n_e = (n-1)$ – both hold
- For non-collinear sites
  - Add extra VD vertex v in infinity $m_v = n_v + 1$
  - Apply Euler's formula: $m_v - m_e + m_f = 2$
  - Obtain $(n_v + 1) - n_e + n = 2$
  - Every VD edge has 2 vertices
  - Each VD vertex has degree ≥ 3
  - Sum of vertex degrees = 2x number of edges n_e (each counted twice) $2n_e \geq 3\,(n_v + 1)$ => $n_v \leq 2n - 5$

Voronoi diagram and convex hull

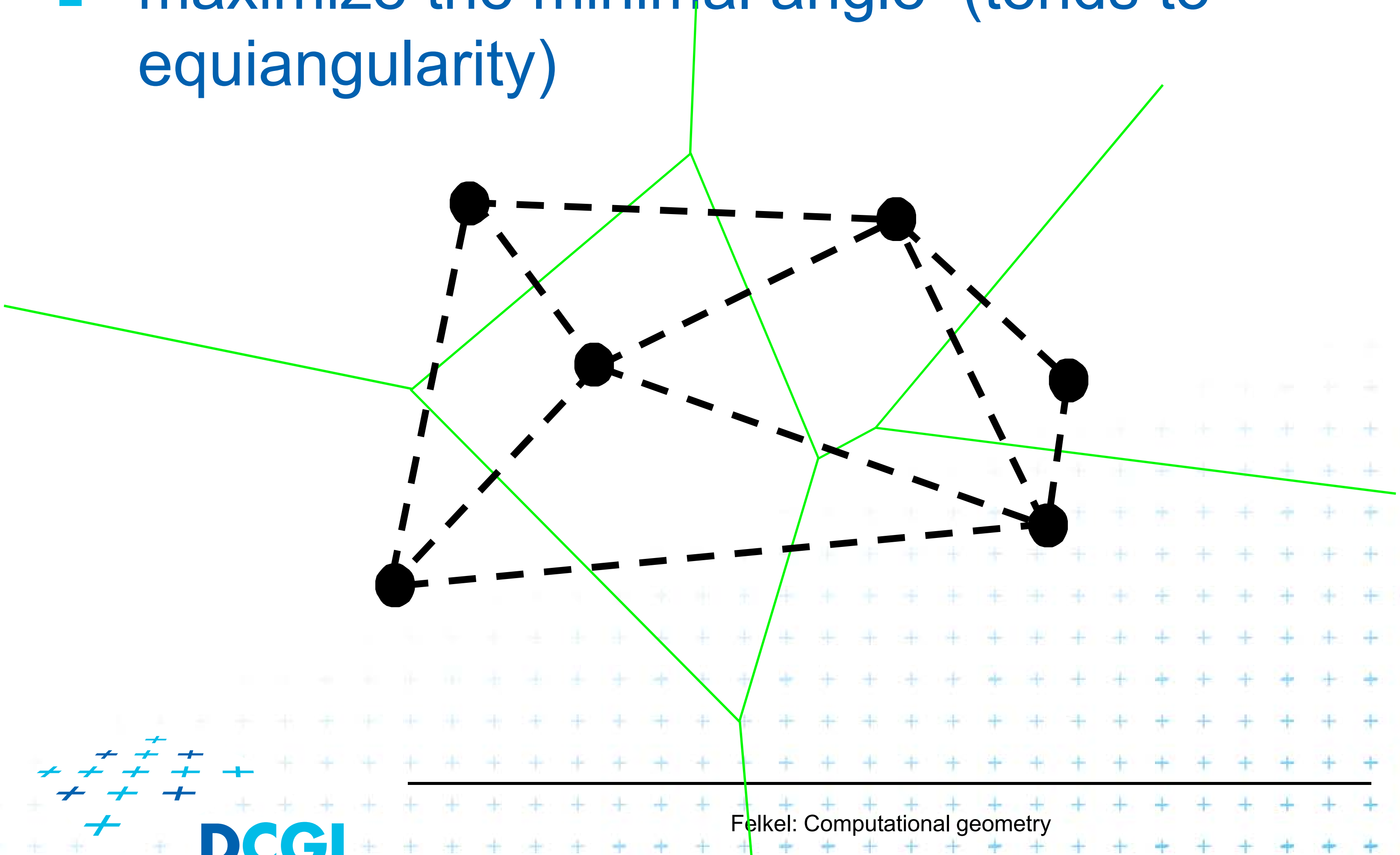
- Convex hull



Connects points from unbounded cells

Delaunay triangulation

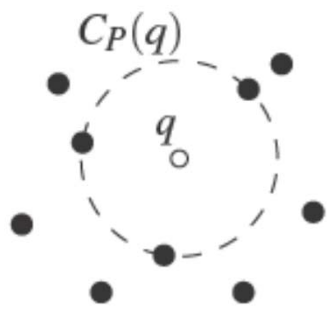
- point set triangulation (straight line dual to VD)
- maximize the minimal angle (tends to equiangularity)



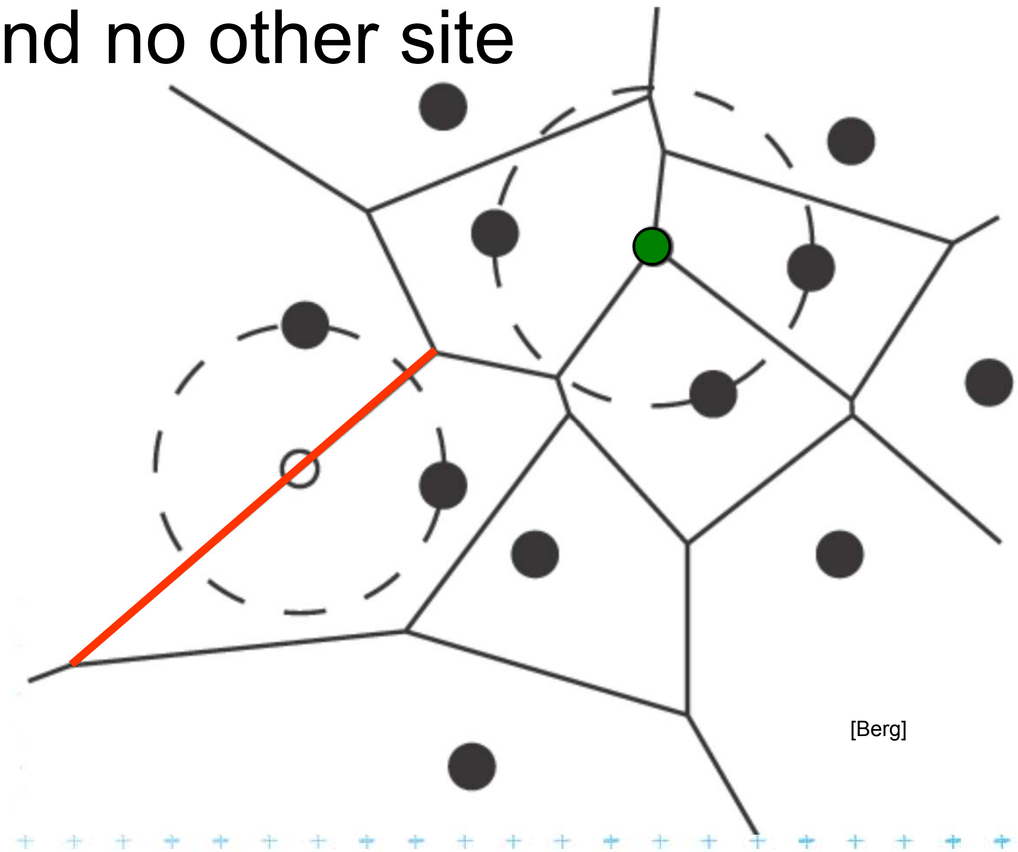
Edges, vertices and largest empty circles

Largest empty circle $C_P(q)$ with center in

1. In VD vertex q: has 3 or more sites on its boundary
2. On VD edge: contains exactly 2 sites on its boundary and no other site



[Berg]

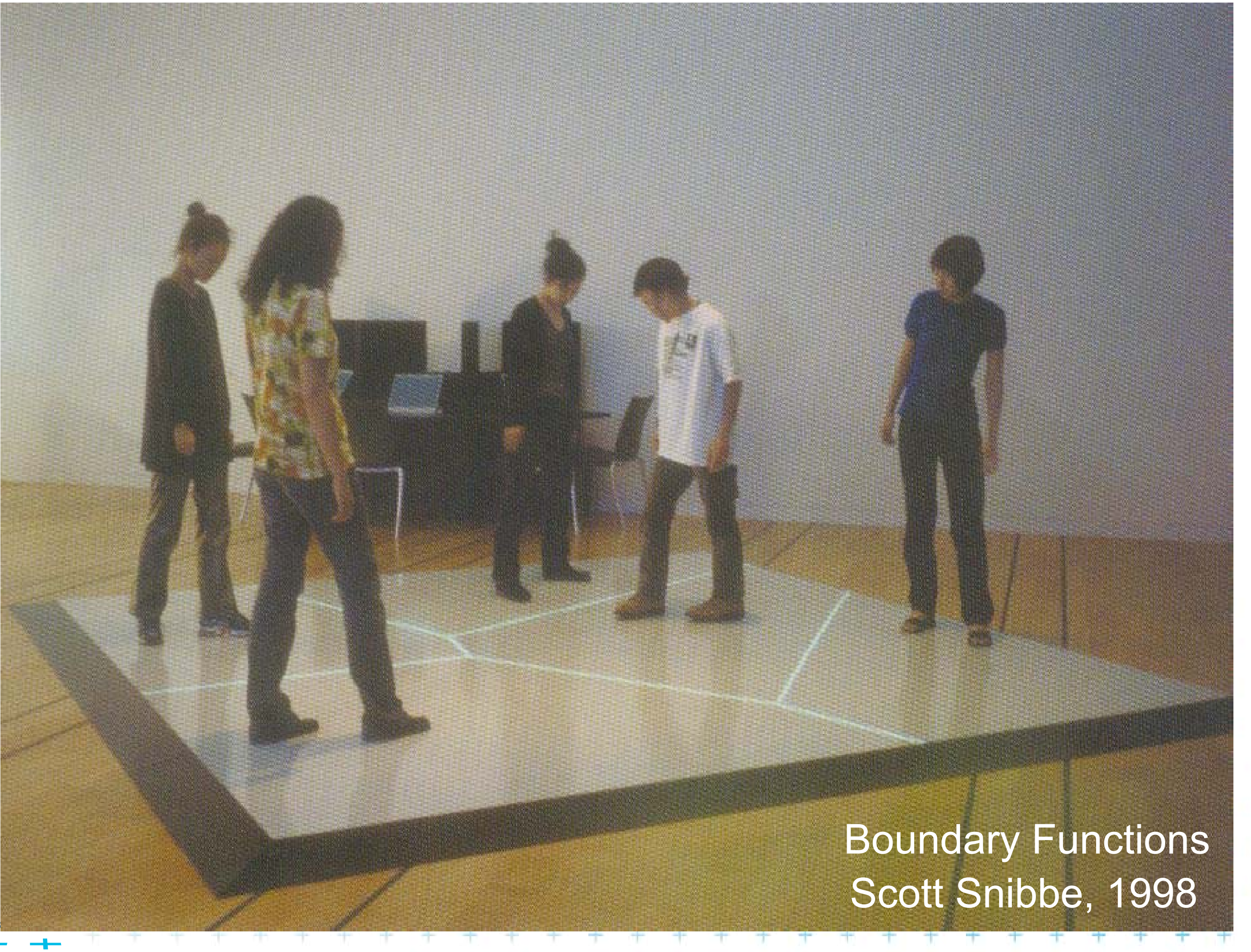


[Berg]

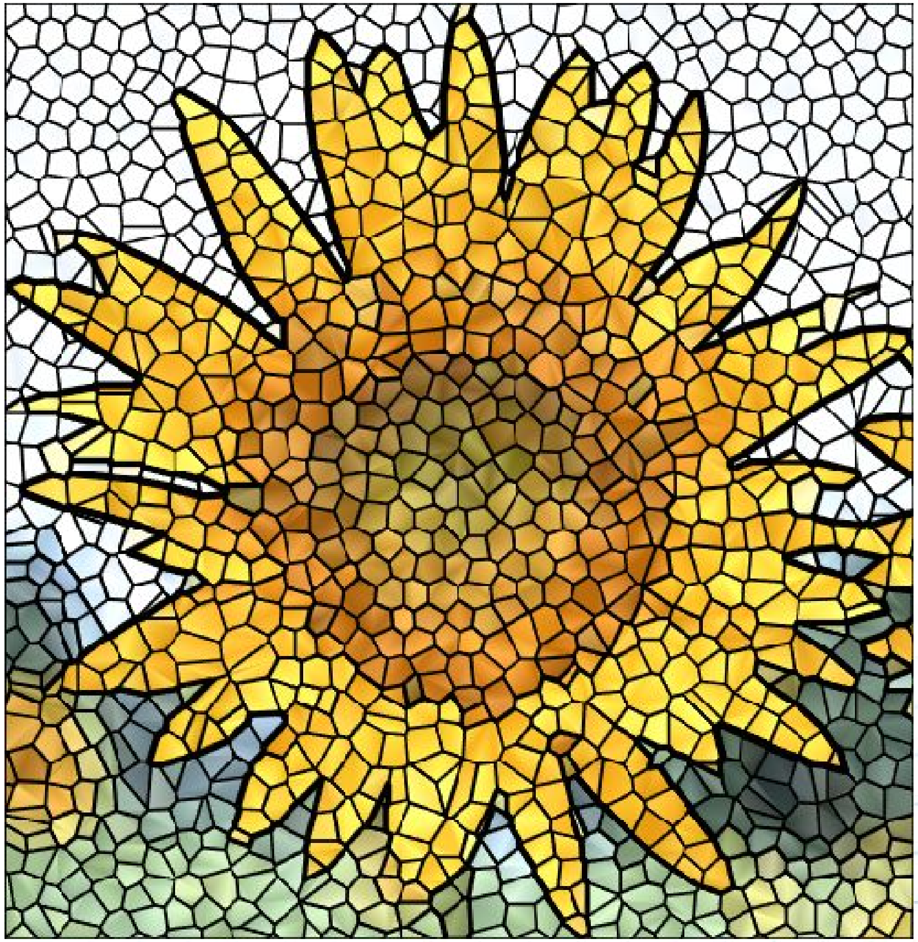
Some applications

- Nearest neighbor queries in Vor(P) of points P
  - Point q µ P ... search sites across the edges around the cell q
  - Point q Ù P ... point location queries – see Lecture 2 (the cell where point *q* falls)
- Facility location (shop or power plant)
  - Largest empty circle (better in Manhattan metric VD)
- Neighbors and Interpolation
  - Interpolate with the nearest neighbor,
    in 3D: surface reconstruction from points
- Art

Voronoi Art



Voronoi Art



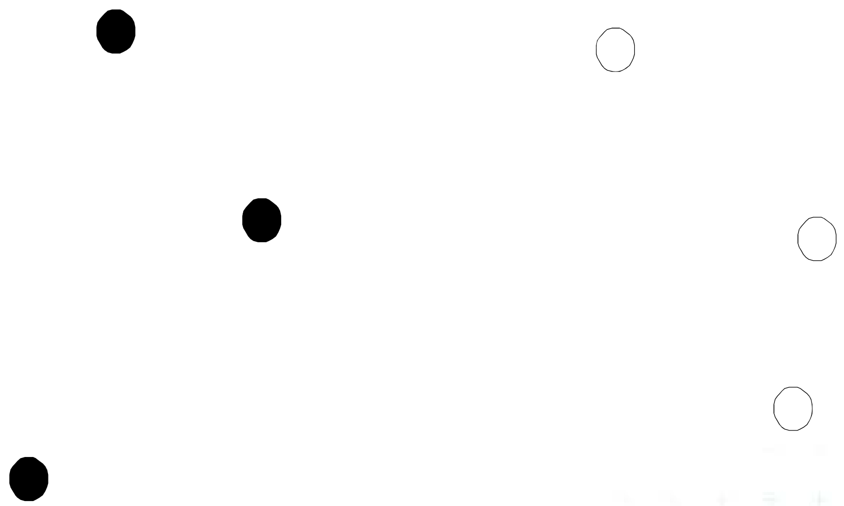
Courtesy [Gold]

Algorithms in 2D

- D&C — O(n log n)
- Fortune's Sweep line — O(n log n)

Voronoi diagram (VD)

Divide and Conquer method

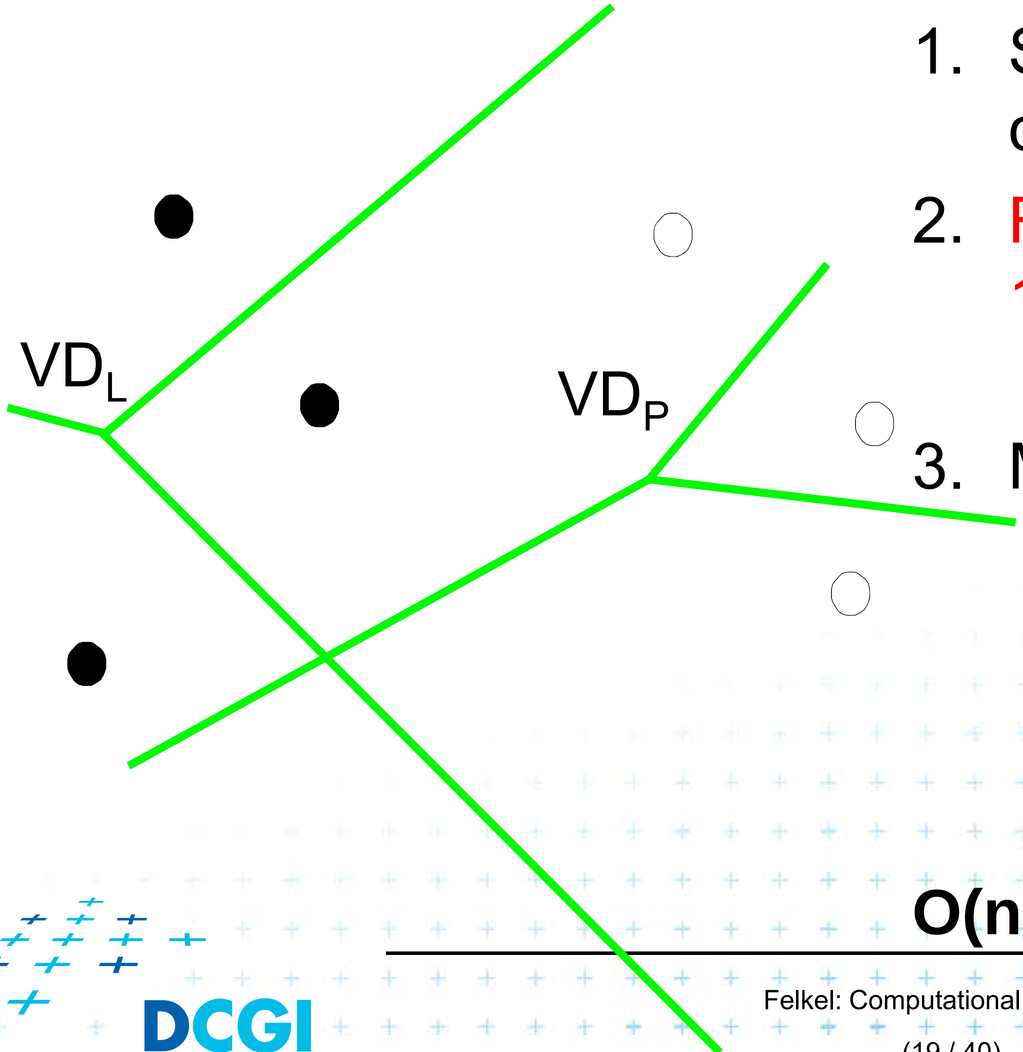


1. Split points based on x-coord into L and R
2. Recursion on L and R
   1-3 points => return
   >3 points => recursion
3. Merge VD_L and VD_R
   - monotone chain
   - trim intersected edges
   - Add new edges from the chain

O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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O(n log n)

Voronoi diagram (VD)

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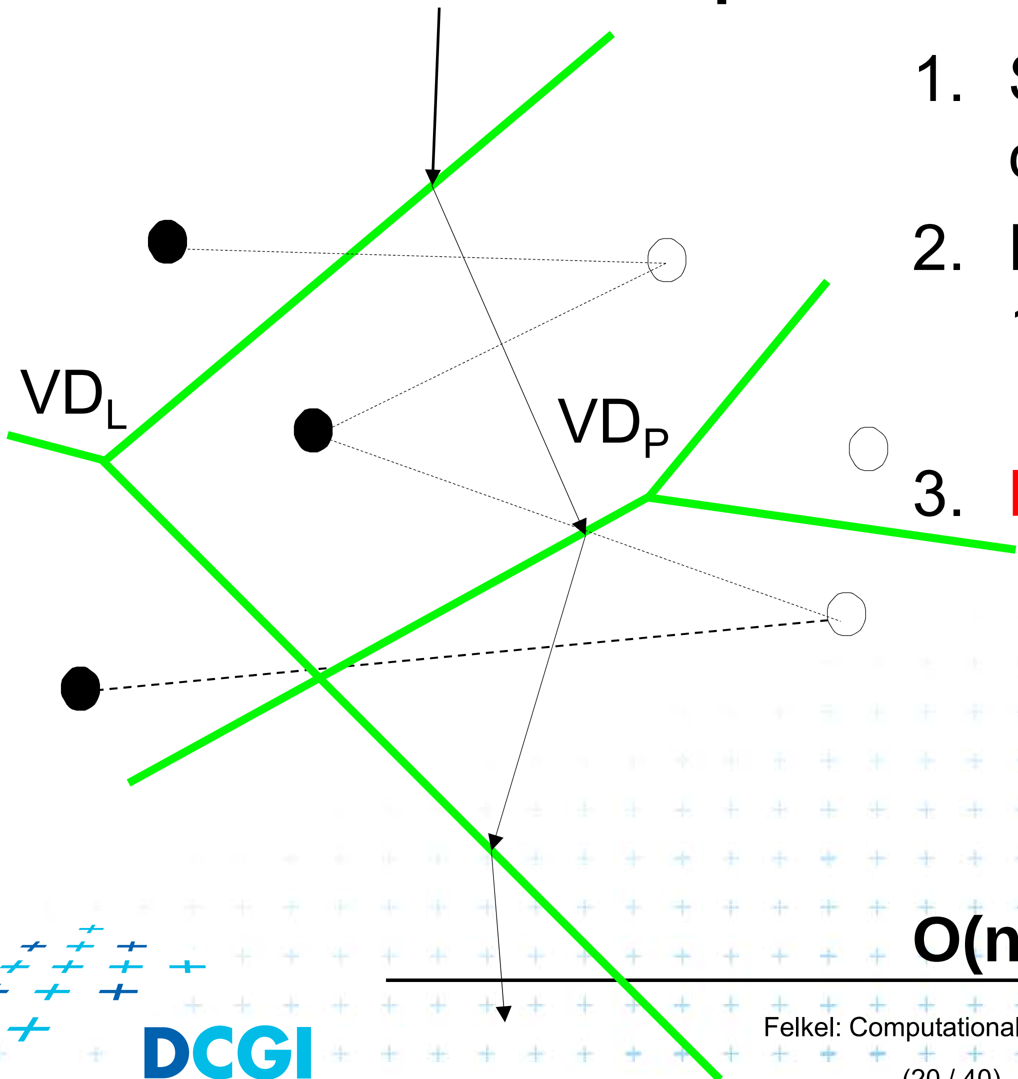

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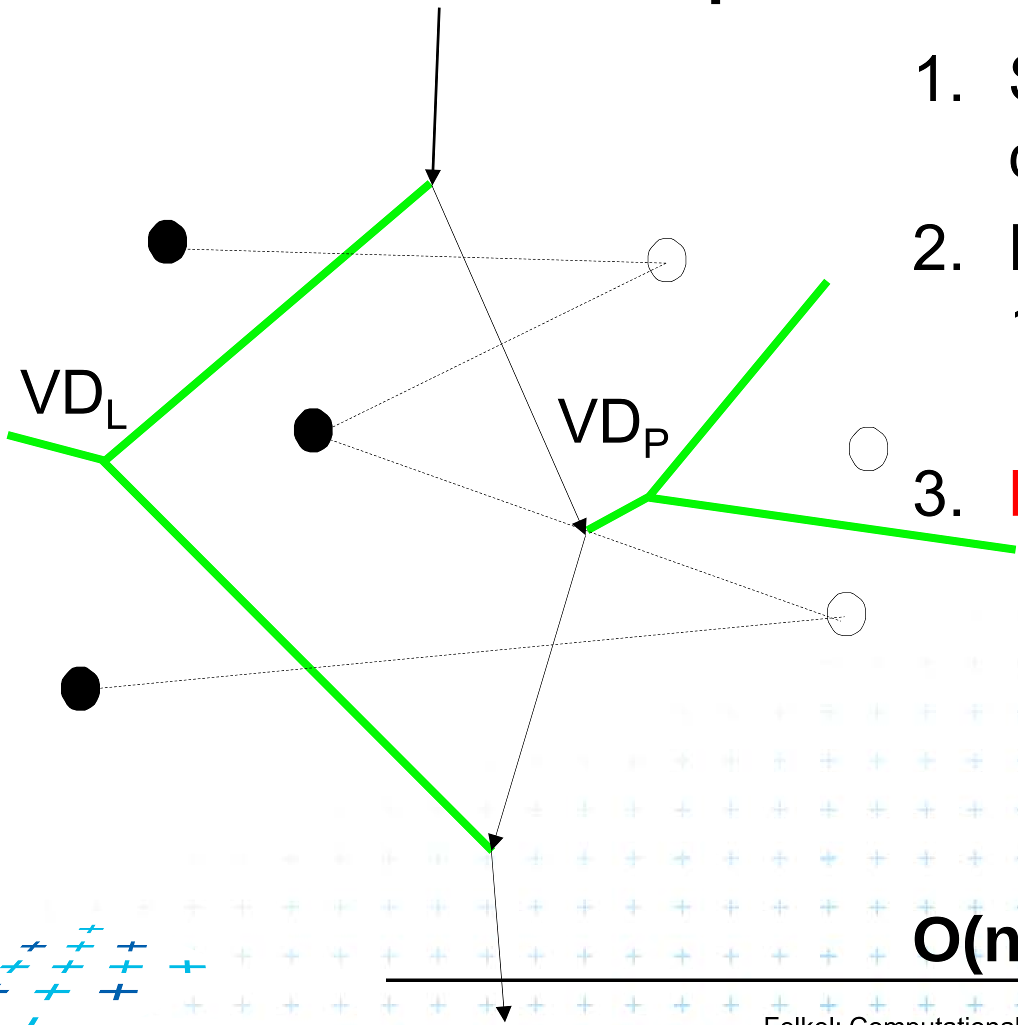
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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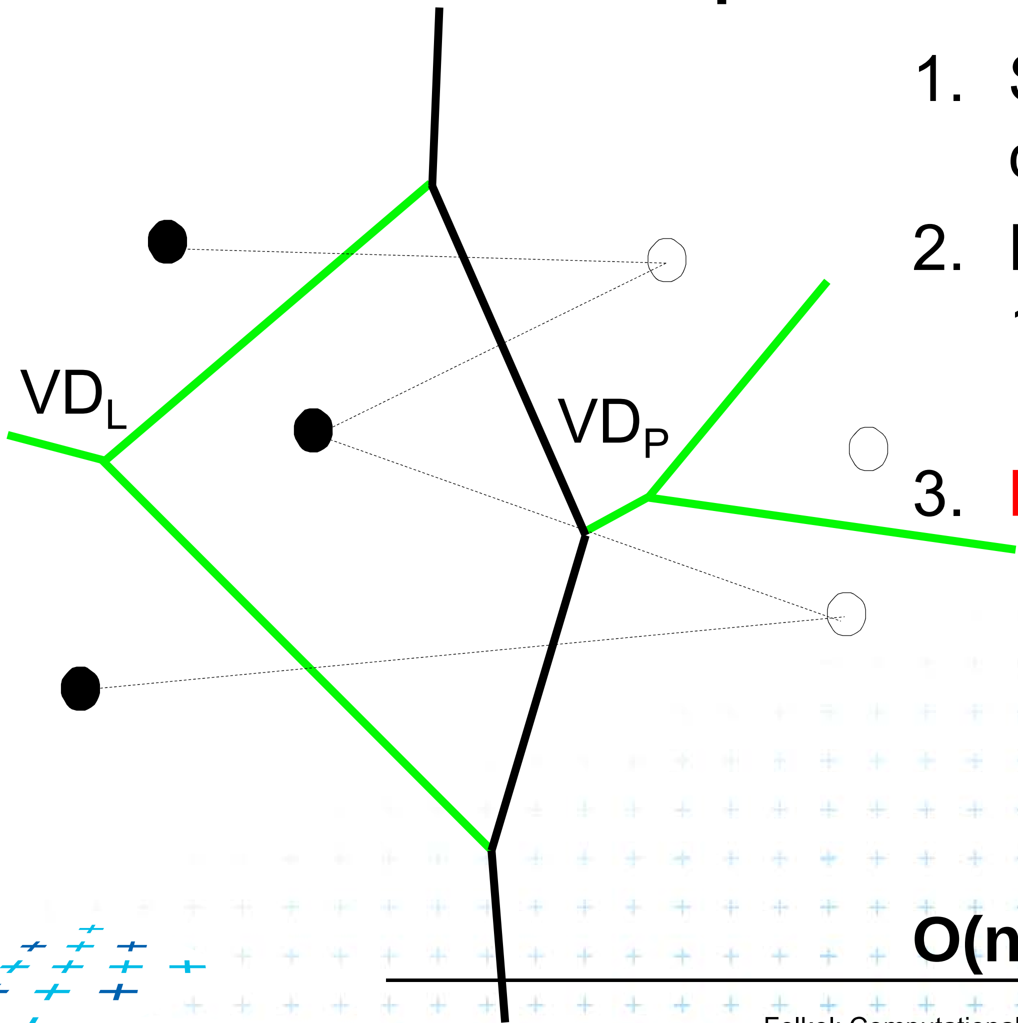
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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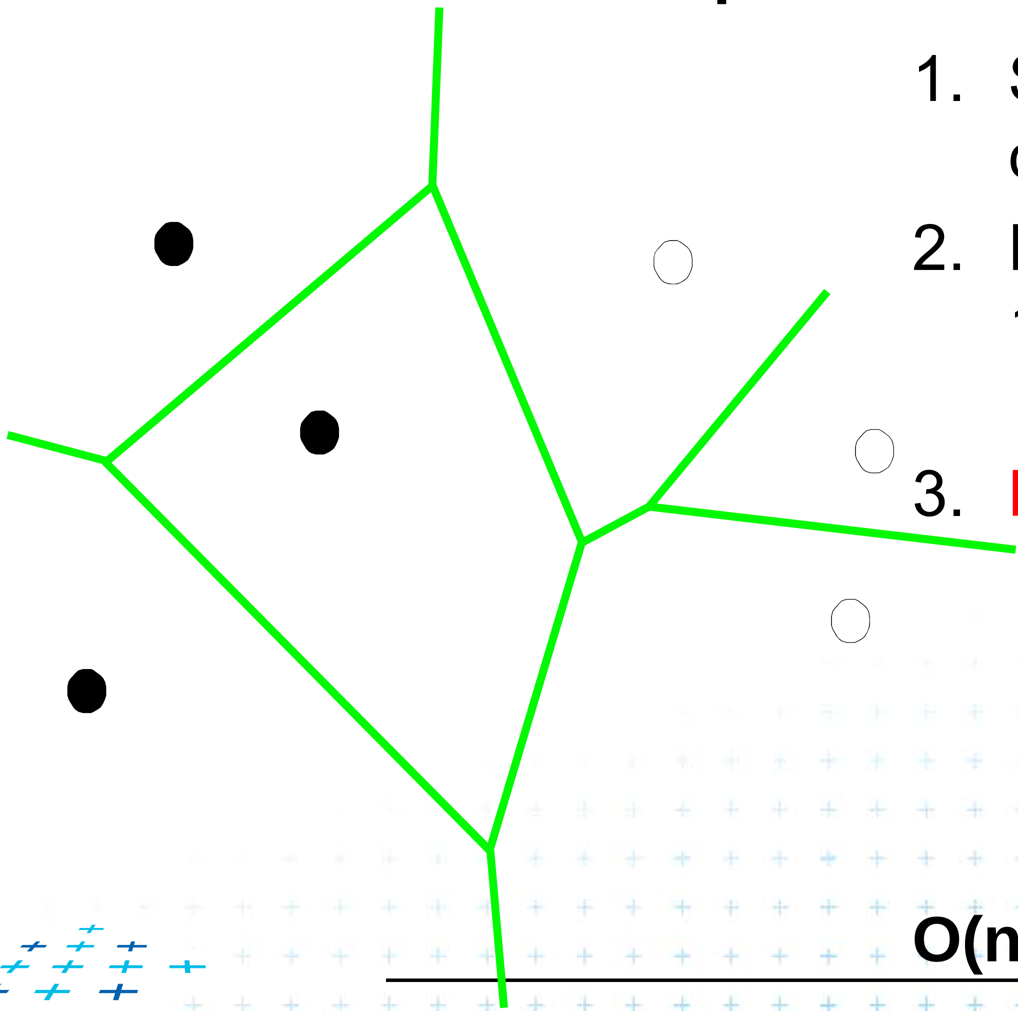
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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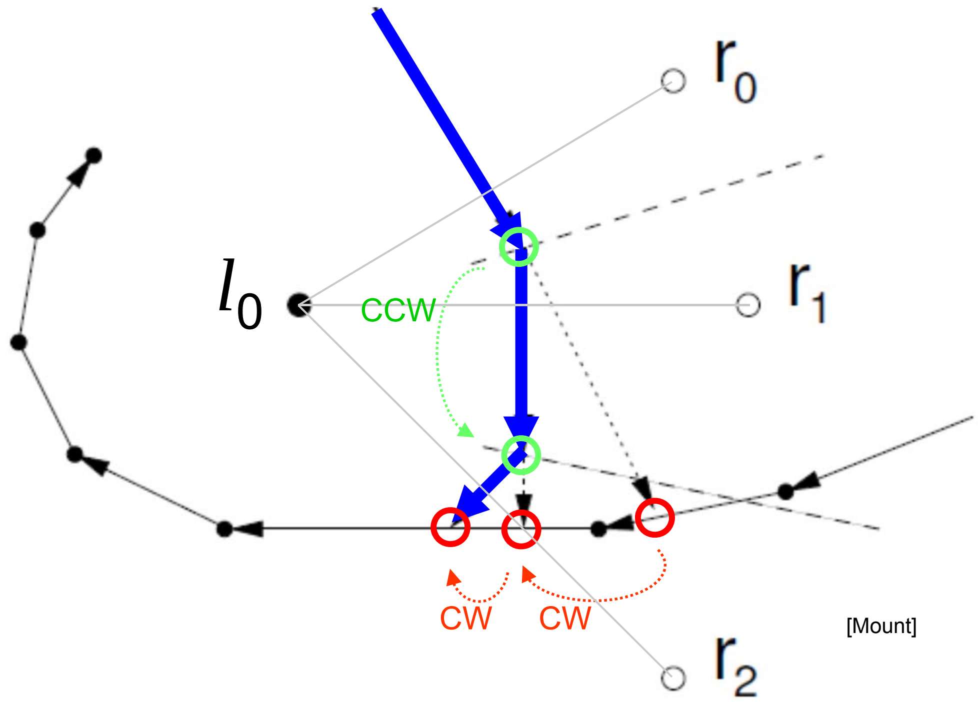
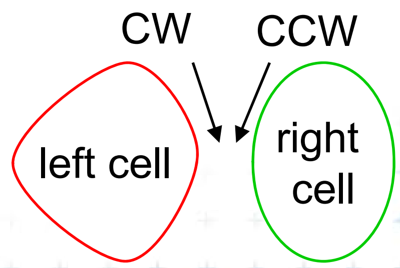
O(n log n)

Monotone chain search in O(n)

- Avoid repeated rescanning of cell edges
- Start in the last tested edge of the cell (each edge tested ~once)
- Continue CW in the l_i left, CCW in the r_i right cell
- Image shows CW search on cell l_0:

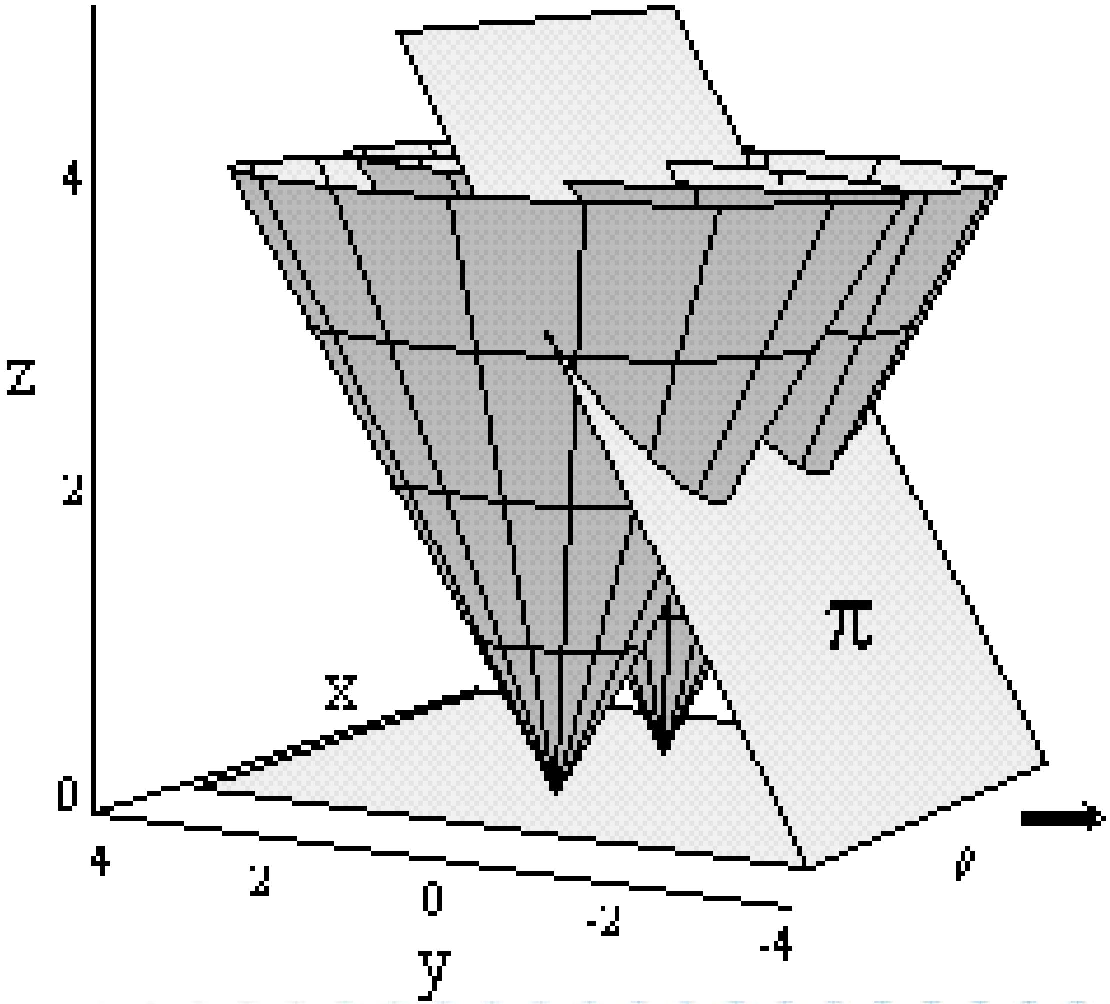




Divide and Conquer method complexity

- Initial sort $O(n \log n)$
- $O(\log n)$ recursion levels
  - O(n) each merge (chain search, trim, add edges to VD)
- Altogether $O(n \log n)$

Fortune's sweep line algorithm – idea in 3D



Cones in sites

Scanning plane π

Both slanted 45°

Intersection
projection to xy:

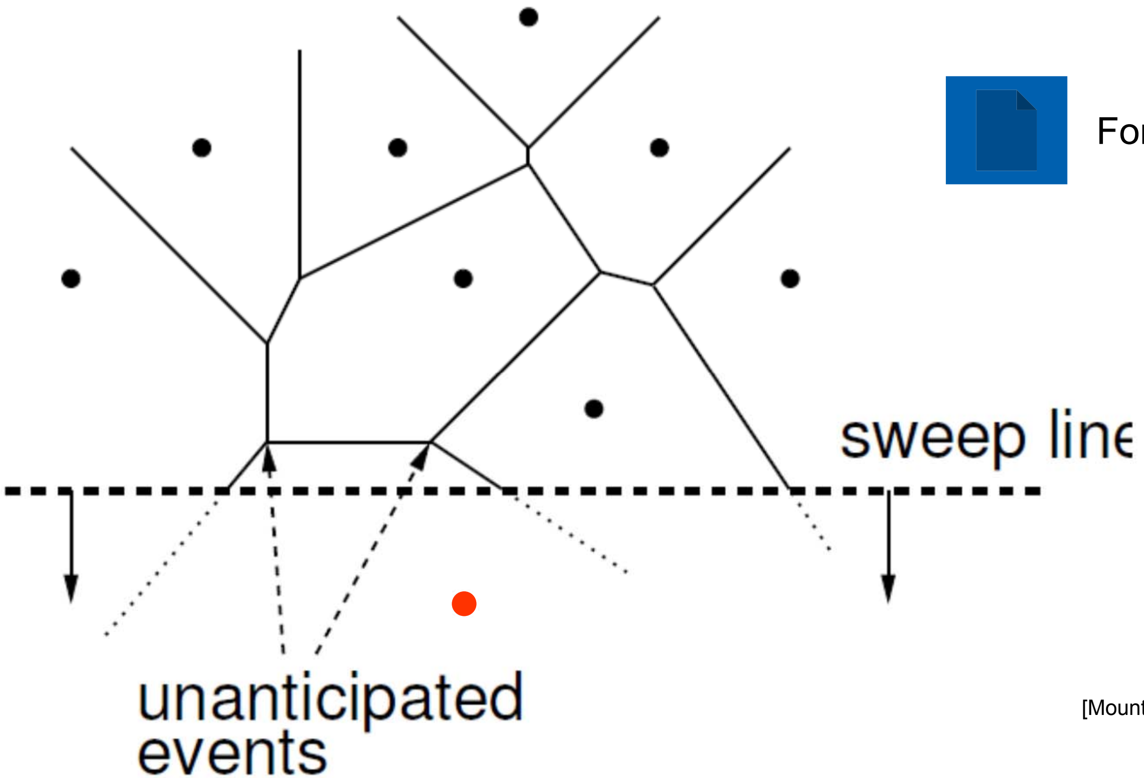
- Cone x plane = parabolic arcs
- Cone x cone = edges of VD

Fortune's sweep line algorithm

DONE

TODO

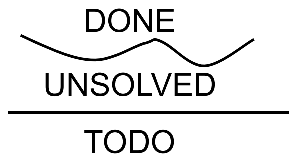
- Differs from "typical" sweep line algorithm
- Unprocessed sites ahead from sweep line may generate Voronoi vertex behind the sweep line



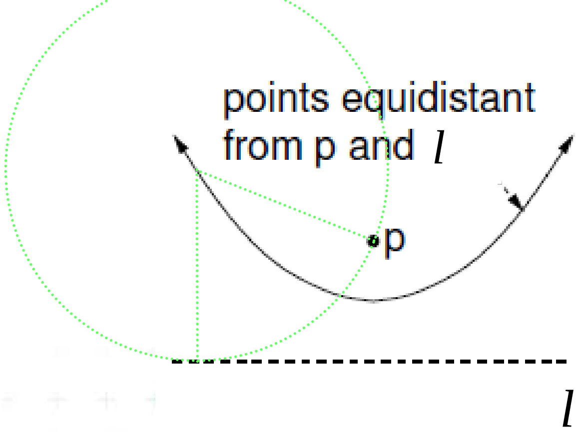
Fortune's applet

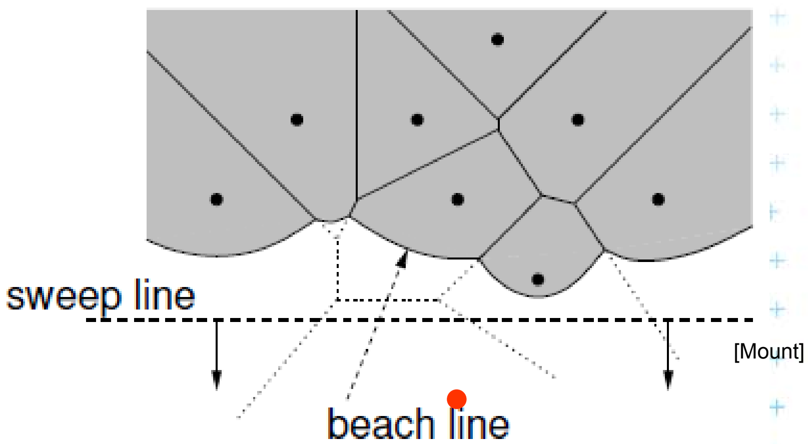
[Mount]

Fortune's sweep line algorithm idea



- Subdivide the halfplane above the sweep line *l* into 2 regions
  1. Points closer to some site above than to sweep line *l* (solved part)
  2. Points closer to sweep line *l* than any point above (unsolved part – can be changed by sites below *l*)
- Border between these 2 regions is a beach line



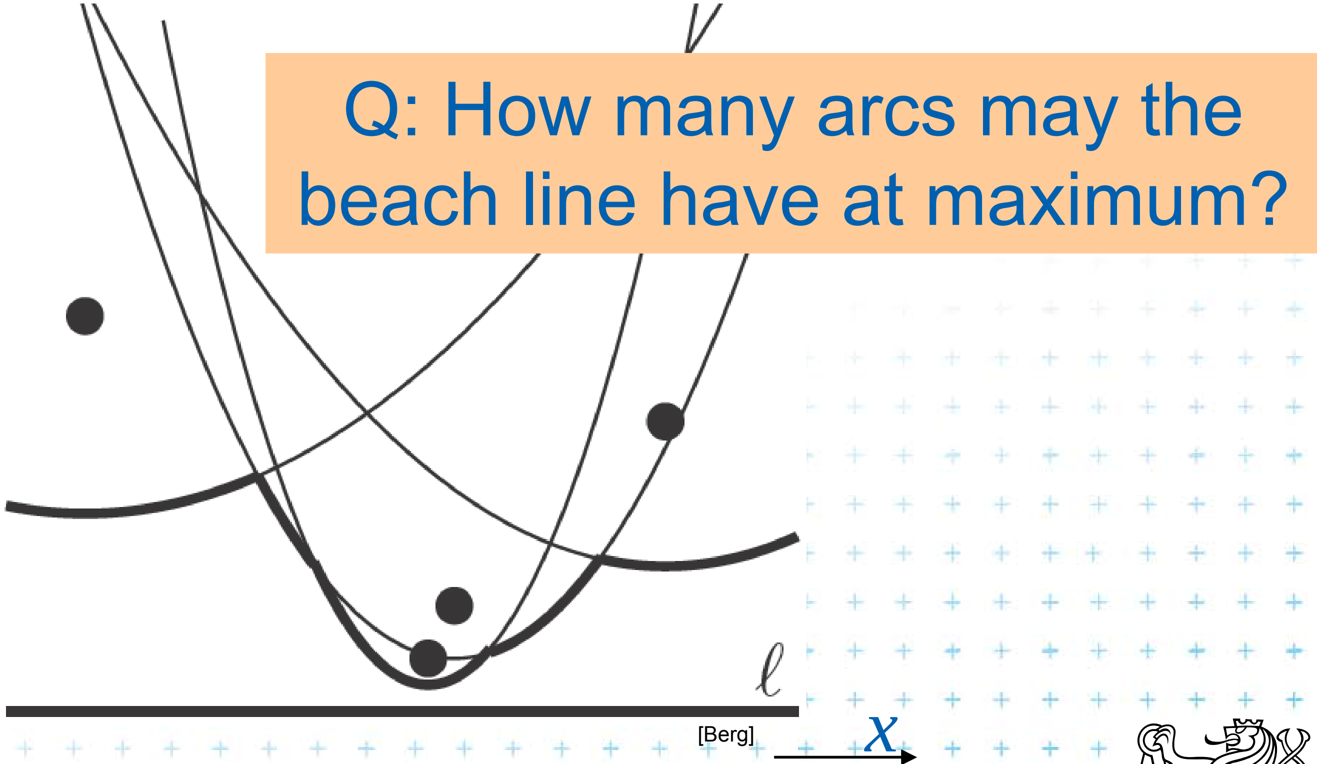


Sweep line and beach line

- Straight sweep line l
  - Separates processed and unprocessed sites (points)
- Beach line (Looks like waves rolling up on a beach)
  - Separates *solved* and *unsolved* regions above sweep line (separates sites above l that can be changed from sites that cannot be changed by sites below l)
  - *x*-monotonic curve made of parabolic arcs (max 2n-1)
  - Follows the sweep line
  - Prevents us from missing unanticipated events until the sweep line encounters the corresponding site

Beach line

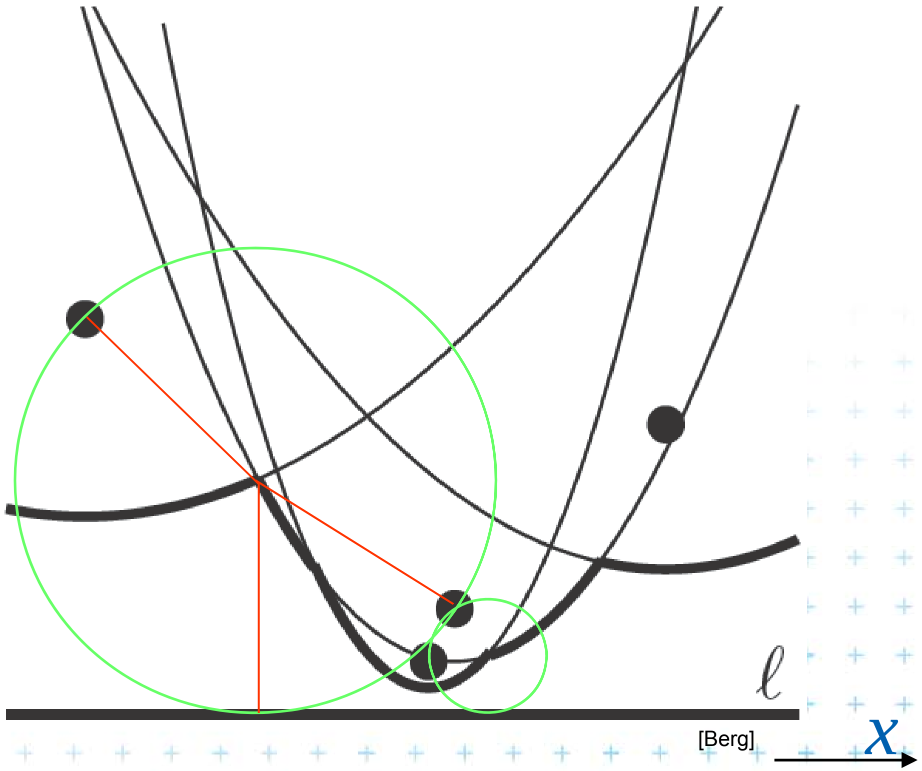
- Every site p_i above l defines a complete parabola
- Beach line is the function, that passes through the lowest points of all the parabolas (lower envelope)



Break point (*bod zlomu*)

= Intersection of two arcs on the beach line

- Equidistant to 2 sites and sweep line l
- Lies on Voronoi edge of the final diagram

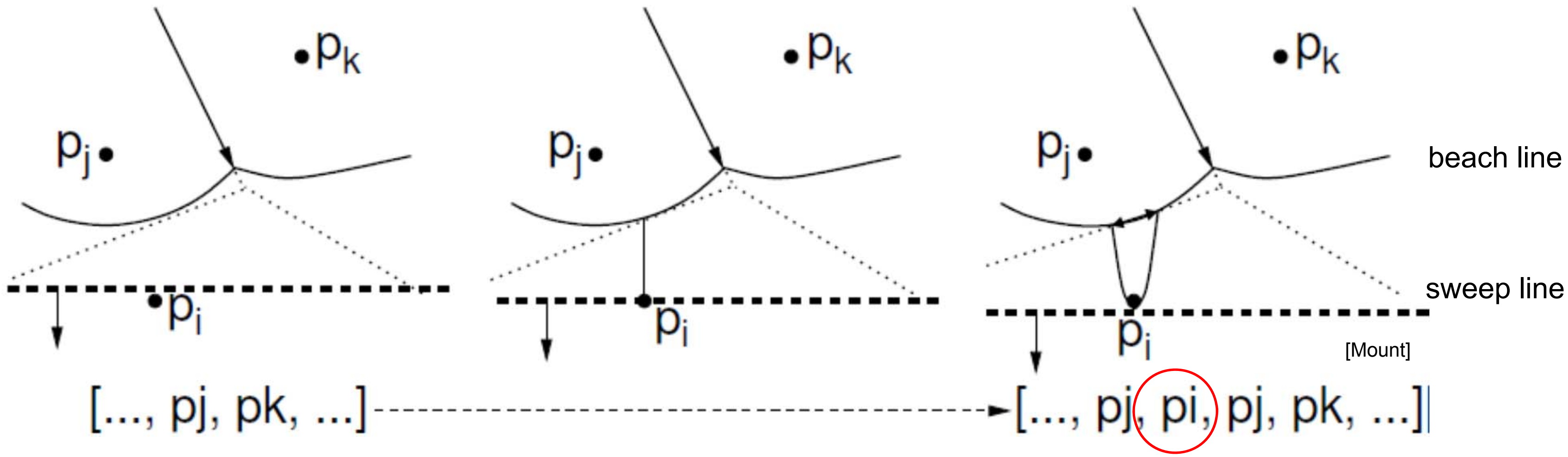


Events

There are two types of events:

- Site events (SE)
  - When the sweep line passes over a new site p_i,
    - *new arc* is added to the beach line
    - *new edge fragment* added to the VD.
  - All SEs known from the beginning (sites sorted by *y*)
- Voronoi vertex event ([Berg] calls a circle event)
  - When the parabolic arc shrinks to zero and disappears, *new Voronoi vertex* is created.
  - Created dynamically by the algorithm for triples or more neighbors on the beach line (triples changed by both types of events)

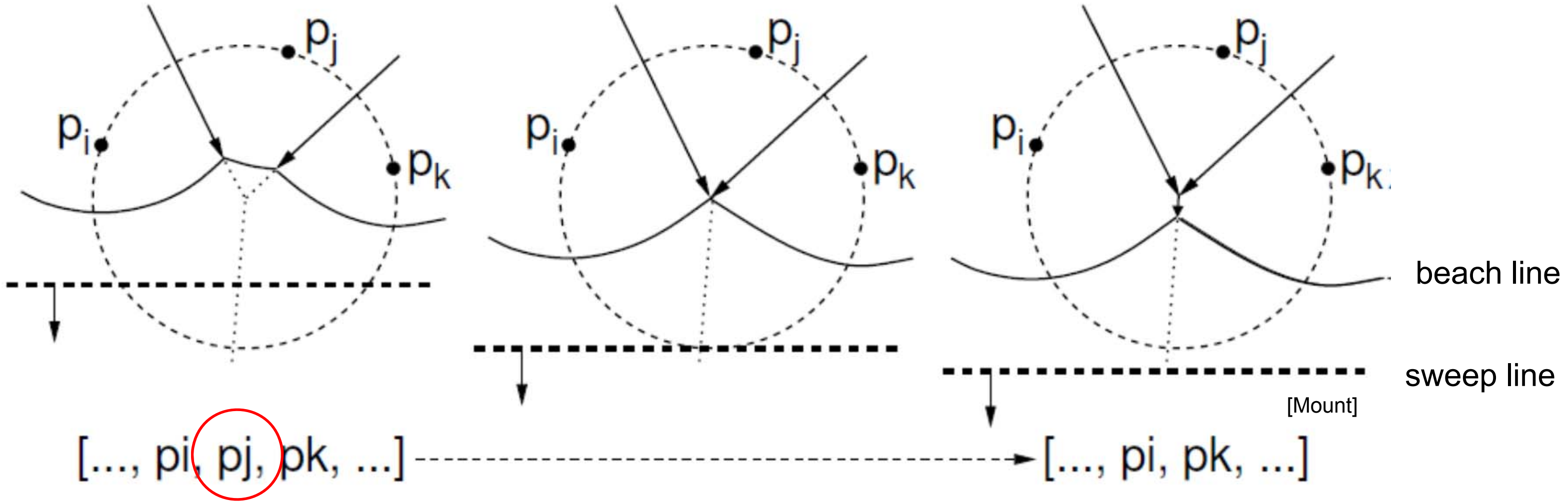
Site event



Generated when the sweep line passes over a site p_i

- New parabolic arc created,
  it starts as a vertical ray from p_i to the beach line
- As the sweep line sweeps on, the arc grows wider
- The entry $\langle .., p_j, ... \rangle$ on the sweep line status is replaced by the triple $\langle .., p_j, p_i, p_j, ... \rangle$
- Dangling future VD edge created on the bisector (p_i, p_j)

Voronoi vertex event (circle event)



Generated when *l* passes the lowest point of circle

- Sites p_i , p_j , p_k appear consecutively on the beach line
- Circumcircle lies partially below the sweep line (Voronoi vertex has not yet been generated)
- This circumcircle contains no point below the sweep line (no future point will block the creation of the vertex)
- Vertex & bisector (p_i, p_k) created, (p_i, p_j) & (p_j, p_k)deleted
- One parabolic arc removed from the beach line

Data structures

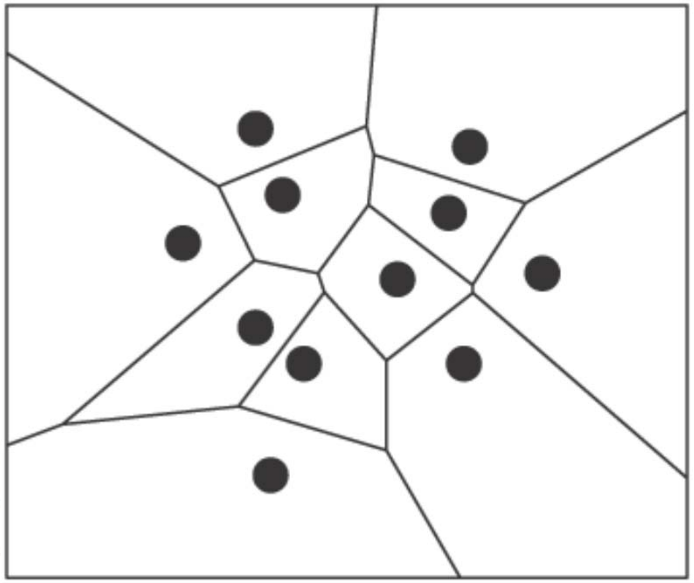
1. (Partial) Voronoi diagram
2. Beach line data structure
3. Event queue Q

1. hrany VD vznikají site event circle event?
2. vrcholy VD vznikají site event circle event?
3. Site events jsou známy předem ano ne?
4. Circle events jsou známy předem ano ne?

1. (Partial) Voronoi diagram data structure

Any PSLG data structure, e.g. DCEL (planar stright line graph)

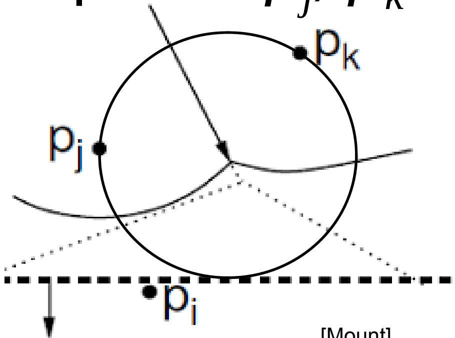
- Stores the VD during the construction
- Contain unbounded edges
  - dangling edges during the construction (managed by the beach line DS) and
  - edges of unbounded cells at the end
    => create a bounding box



[Berg]

2. Beach line tree data structure

- Used to locate the arc directly above a new site
- E.g. Binary tree T
  – Leaves - ordered arcs along the beach line (x-monotone)
    - T stores only the sites p_i in leaves, T does not store the parabolas
  – Inner tree nodes - breakpoints as ordered pairs $\langle p_j, p_k \rangle$
    - p_j, p_k are neighboring sites
    - Breakpoint position computed on the fly from p_j, p_k and y-coord of the sweep line



[Mount]

  – Pointers to other two DS
    - In leaves – pointer to event queue, point to node when arc disappears via Voronoi vertex event – if it exists
    - In inner nodes - pointer to half-edge in DCEL of VD, that is being traced out by the break point

3. Event queue Q

- Priority queue, ordered by y-coordinate
- For site event
  - stores the site itself
  - known from the beginning
- For Voronoi vertex event (circle event)
  - stores the lowest point of the circle
  - stores also pointer to the leaf in tree T (represents the parabolic arc that will disappear)
  - created by both events, when triples of points become neighbors (possible max three triples for a site)
  - p_i, p_j, p_k, p_l, p_m insert of p_k can create up to 3 triples and delete up to 2 triples (p_i, p_j, p_l) and (p_j, p_l, p_m)

Fortune's algorithm

FortuneVoronoi(*P*)

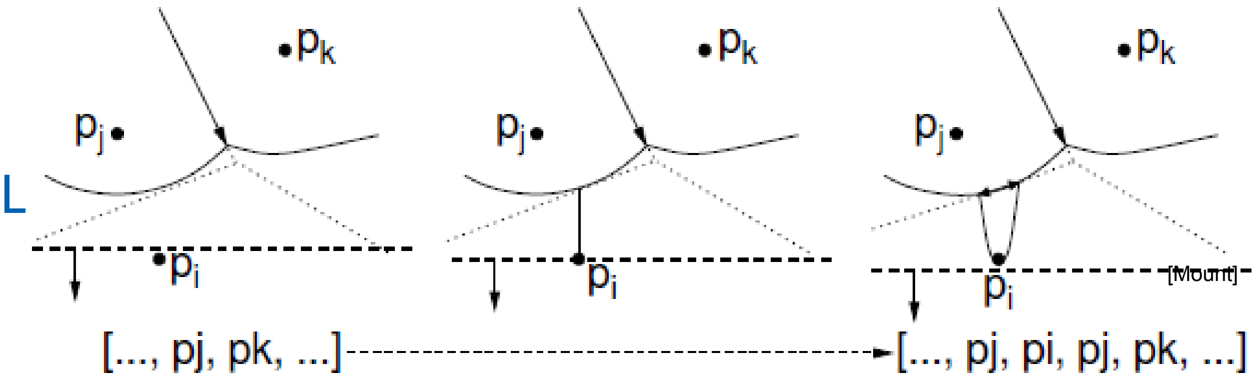
Input: A set of point sites $P = \{p_1, p_2, \dots, p_n\}$ in the plane

Output: Voronoi diagram Vor(*P*) inside a bounding box in a DCEL struct.

1. Init event queue Q with all *site events*
2. **while**(Q not empty) **do**
3. consider the event with largest *y*-coordinate in Q (next in the queue)
4. **if**(event is a *site event* at site p_i)
5. **then** HandleSiteEvent(p_i)
6. **else** HandleVoroVertexEvent(p_i), where p_i is the lowest point of the circle causing the event
7. remove the event from Q
8. Create a bbox and attach half-infinite edges in *T* to it in DCEL.
9. Traverse the halfedges in DCEL and add cell records and pointers to and from them

Handle site event

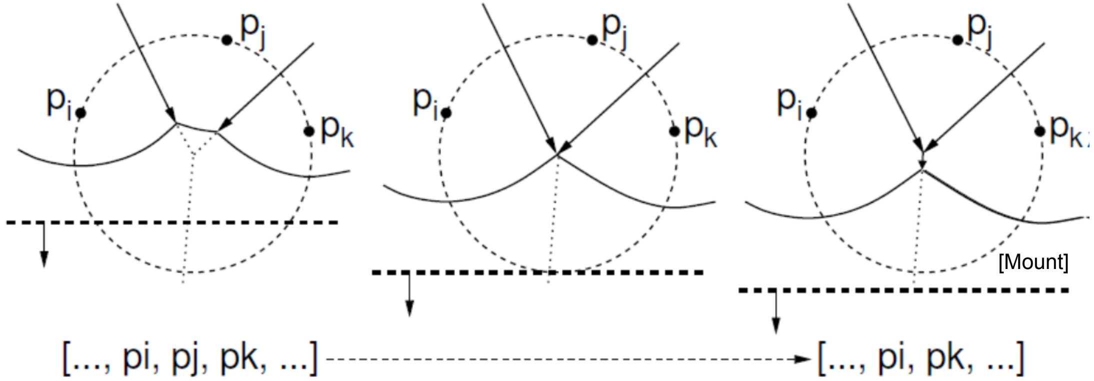
HandleSiteEvent(p_i)
Input: event site p_i
Output: updated DCEL



1. Search in *T* for arc d vertically above p_i. Let p_j be the correspond. site
2. Apply insert-and-split operation, inserting a new entry of p_i to the beach line *T* (new arc), thus replacing û…, p_j ,…ë with û…, p_j , p_i , p_j ,…ë
3. Create a new (dangling) edge in the Voronoi diagram, which lies on the bisector between p_i and p_j
4. Neighbors on the beach line changed -> check the neighboring triples of arcs and *insert or delete Voronoi vertex events* (insert only if the circle intersects the sweep line and it is not present yet).
   Note: Newly created triple p_j , p_i , p_j cannot generate an event because it only involves two distinct sites.

Handle Voronoi vertex (circle) event

HandleVoroVertexEvent(p_j)
Input: event site p_j
Output: updated DCEL



Let p_i , p_j , p_k be the sites that generated this event (from left to right).

1

Q: Struktura pobřežní čáry obsahuje: abcdef

Když vymažu d, které trojice zmizí a které přibudou?

– $\langle p_i, p_j, p_k \rangle$ and join the two Voronoi edges for the bisectors $\langle p_i, p_j \rangle$ and $\langle p_j, p_k \rangle$ to this vertex (dangling edges – created in step 3 above).

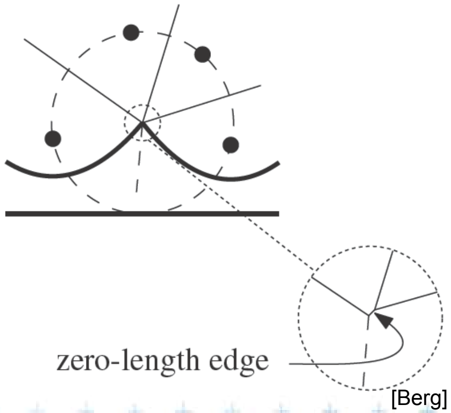
– Create a new (dangling) edge for the bisector between $\langle p_j, p_k \rangle$

– Delete any Voronoi vertex events (max. three) from Q that arose from triples involving the arc d of p_j and generate (two) new events corresponding to consecutive triples involving p_i, and p_k.

Handling degeneracies

Algorithm handles degeneracies correctly

- 2 or more events with the same y
  - if x coords are different, process them in any order
  - if x coords are the same (cocircular sites)
    process them in any order,
    it creates duplicated vertices with
    zero-length edges,
    remove them in post processing step



- degeneracies while handling an event
  - Site below a beach line breakpoint
  - Creates circle event of zero diameter,
    remove zero-length edges in post processing step

References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapter 7, http://www.cs.uu.nl/geobook/

[Mount] David Mount, - *CMSC 754: Computational Geometry, Lecture Notes for Spring 2007*, University of Maryland, Lectures 12 and 29. http://www.cs.umd.edu/class/spring2007/cmsc754/lectures.shtml

[Preparata] Preperata, F.P., Shamos, M.I.: *Computational Geometry. An Introduction.* Berlin, Springer-Verlag,1985. Chapter 5

[VoroGlide] VoroGlide applet: http://www.pi6.fernuni-hagen.de/GeomLab/VoroGlide/

[Fortune] Fortune's algorithm applet: http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/Fortune/fortune.htm

[Muhama] http://www.personal.kent.edu/~rmuhamma/Compgeometry/compgeom.html