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DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

GEOMETRIC SEARCHING PART 1: POINT LOCATION

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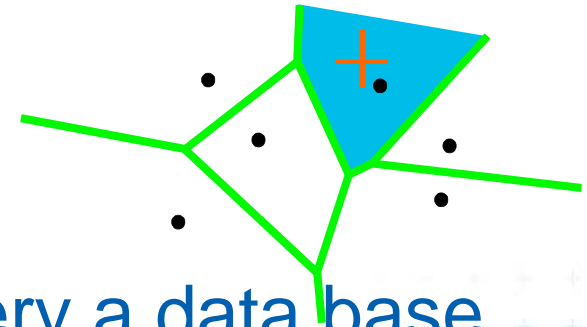
<https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start>

Based on [Berg] and [Mount]

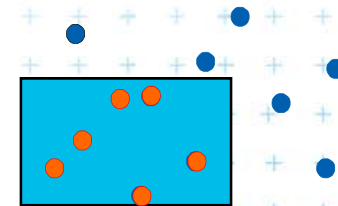
Version from 4.10.2012

Geometric searching problems

- Point location (static) – Where am I?
 - (Find the name of state, pointed by the mouse (cursor))
 - Search space S : a planar (spatial) subdivision
 - Query: point Q
 - Answer: region containing Q



- Orthogonal range searching – Query a data base
(Find points, located in d -dimensional axis-parallel box)
 - Search space S : a set of points
 - Query: set of orthogonal intervals q
 - Answer: subset of points in the box
 - (Was studied in DPG)



Point location

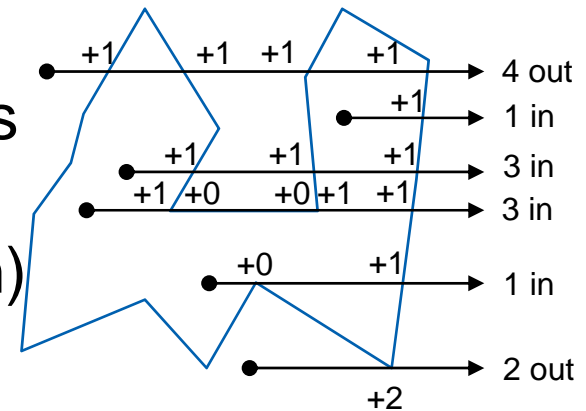
- Point location in polygon
- Planar subdivision
- DCEL data structure
- Point location in planar subdivision
 - slabs
 - monotone sequence
 - trapezoidal map



Point location in polygon by ray crossing

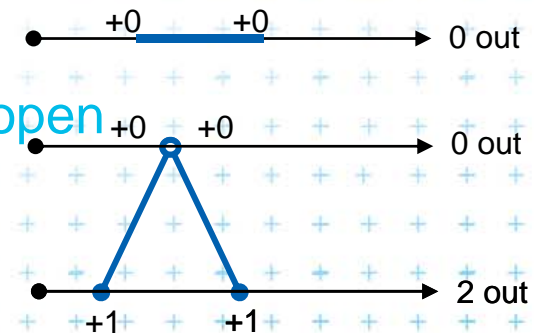
1. Ray crossing - $O(n)$

- Compute number t of intersections of ray with polygon edges (e.g., $X+$ after point move to origin)
- If $\text{odd}(t)$ then inside else out



- Singular cases must be handled!

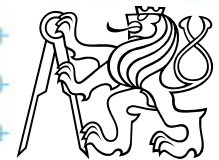
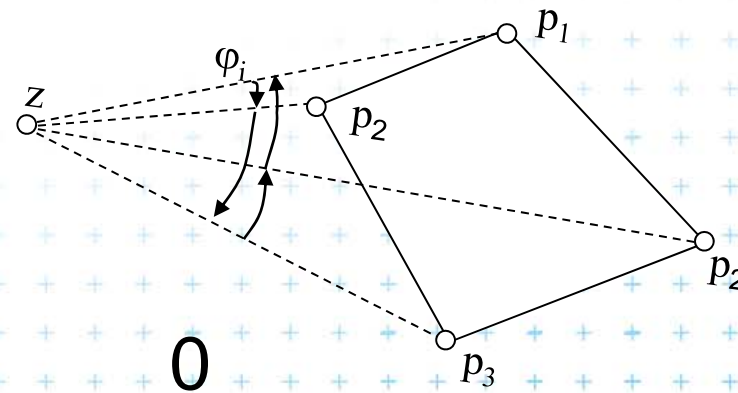
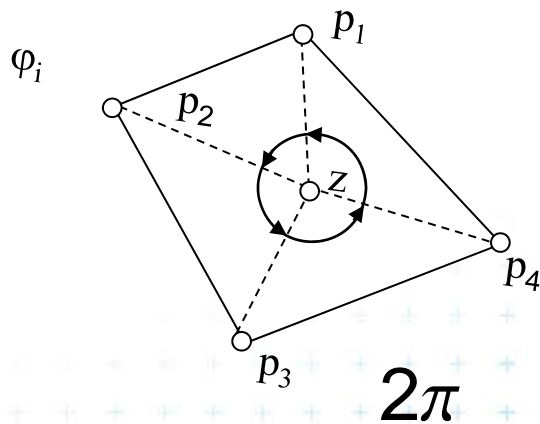
- Do not count horizontal line segments
- Take non-horizontal segments as **half-open** (upper point not part of the segment)



Point location in polygon

2. Winding number - $O(n)$ (number of turns around the point)

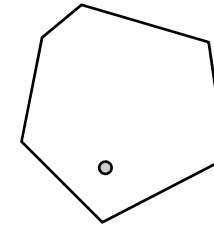
- Sum angles $\varphi_i = \angle(p_i, z, p_{i+1})$
- If (sum $\varphi_i = 2\pi$) then inside (1 turn)
- If (sum $\varphi_i = 0$) then outside (no turn)
- About 20-times slower than ray crossing



Point location in polygon

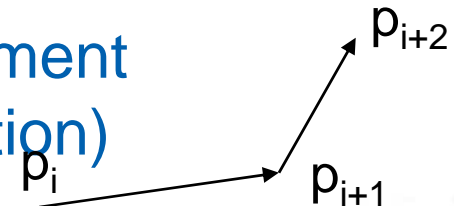
3. Position relative to all edges

- For convex polygons
- If (left from all edges) then inside



■ Position of point in relation to the line segment (Determination of convex polygon orientation)

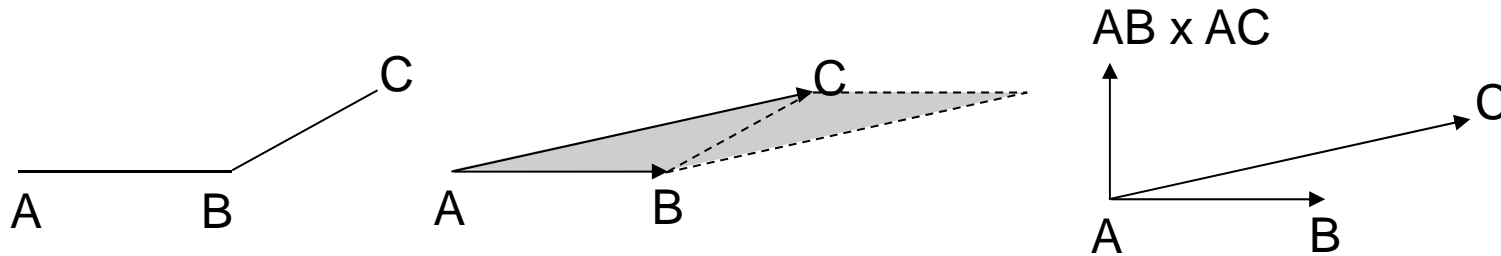
- Convex polygon,
noncollinear points $p_i = [x_i, y_i, 1]$, $p_{i+1} = [x_{i+1}, y_{i+1}, 1]$, $p_{i+2} = [x_{i+2}, y_{i+2}, 1]$



$$\begin{vmatrix} x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \\ x_{i+2} & y_{i+2} & 1 \end{vmatrix} > 0 \Rightarrow \text{point left from edge (CCW polygon)}$$
$$\begin{vmatrix} x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \\ x_{i+2} & y_{i+2} & 1 \end{vmatrix} < 0 \Rightarrow \text{point right from edge (CW polygon)}$$



Area of Triangle



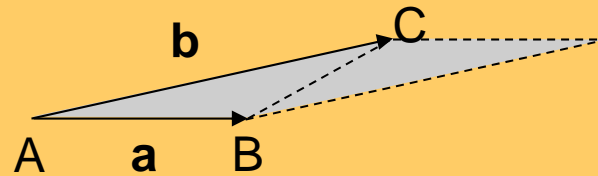
Vector product of vectors $AB \times AC$

- = Vector perpendicular to both vectors AB and AC
- For vectors in plane is perpendicular to the plane (normal)
- In 2D (plane xy) – has only z -coordinate is non-zero
- $|AB \times AC|$ = z -coordinate of the normal vector
- = area of parallelogram
- = $2 \times$ area T of triangle ABC



Area of Triangle

- $T = \frac{1}{2} |\mathbf{AB} \times \mathbf{AC}|$



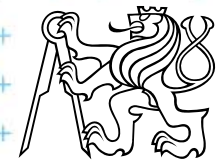
- $\mathbf{a} = \mathbf{B} - \mathbf{A}$

- $\mathbf{b} = \mathbf{C} - \mathbf{A}$

- $T = \frac{1}{2} (\mathbf{a}_x \mathbf{b}_y - \mathbf{a}_y \mathbf{b}_x)$

$$\Rightarrow 2T = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$

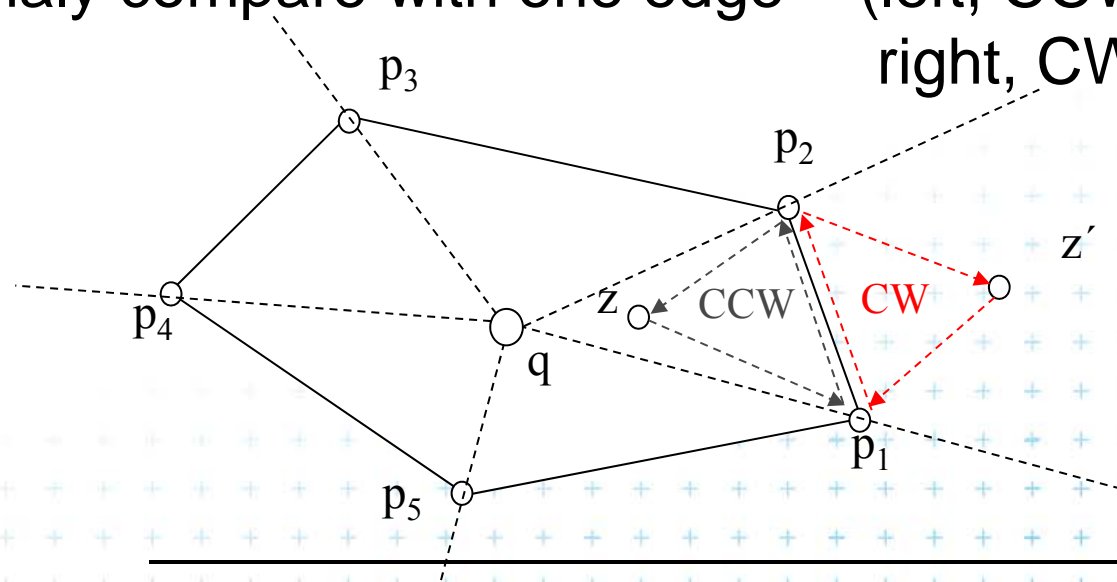
$$2T = \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$



Point location in polygon

4. Binary search in angles

- Works for convex and star-shaped polygons
- Choose any point q inside
- q forms wedges with polygon edges
- Binary search of **wedge** based on angle
- Finally compare with one edge (left, CCW \Rightarrow in, right, CW \Rightarrow out)

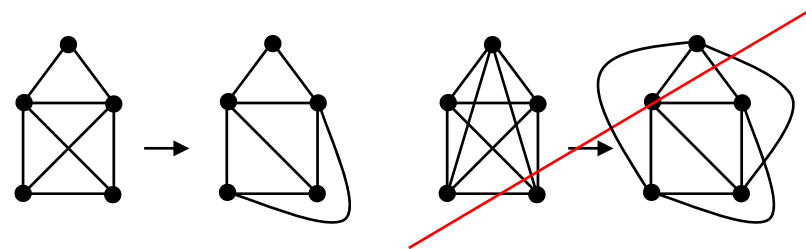


Planar graph

Planar graph

U=set of nodes, H=set of arcs

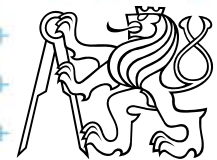
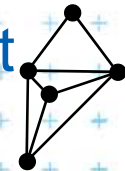
= Graph $G = (U,H)$ is planar, if it can be embedded into plane without crossings



Planar embedding of planar graph $G = (U,H)$

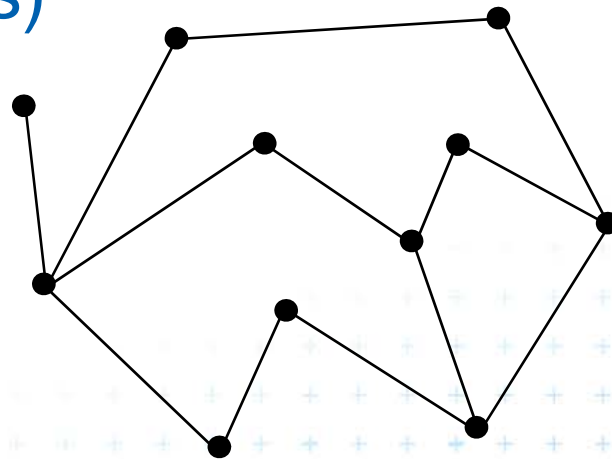
= mapping of each *node* in U to *vertex* in the plane and each *arc* in H into *simple curve (edge)* between the two images of extreme nodes of the arc, so that **no two images of arc intersect** except at their endpoints

Every planar graph can be embedded in such a way that arcs map to straight line segments [Fáry 1948]

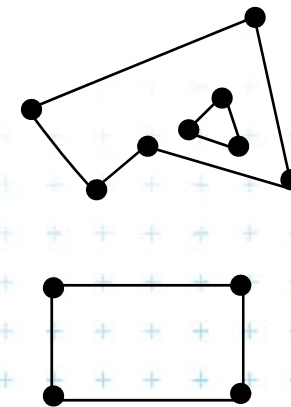


Planar subdivision

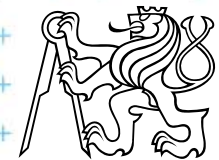
- = Partition of the plane determined by straight line planar embedding of a planar graph.
Also called PSLG – Planar straight line graph
- (embedding of a planar graph in the plane such that its arcs are mapped into straight line segments)



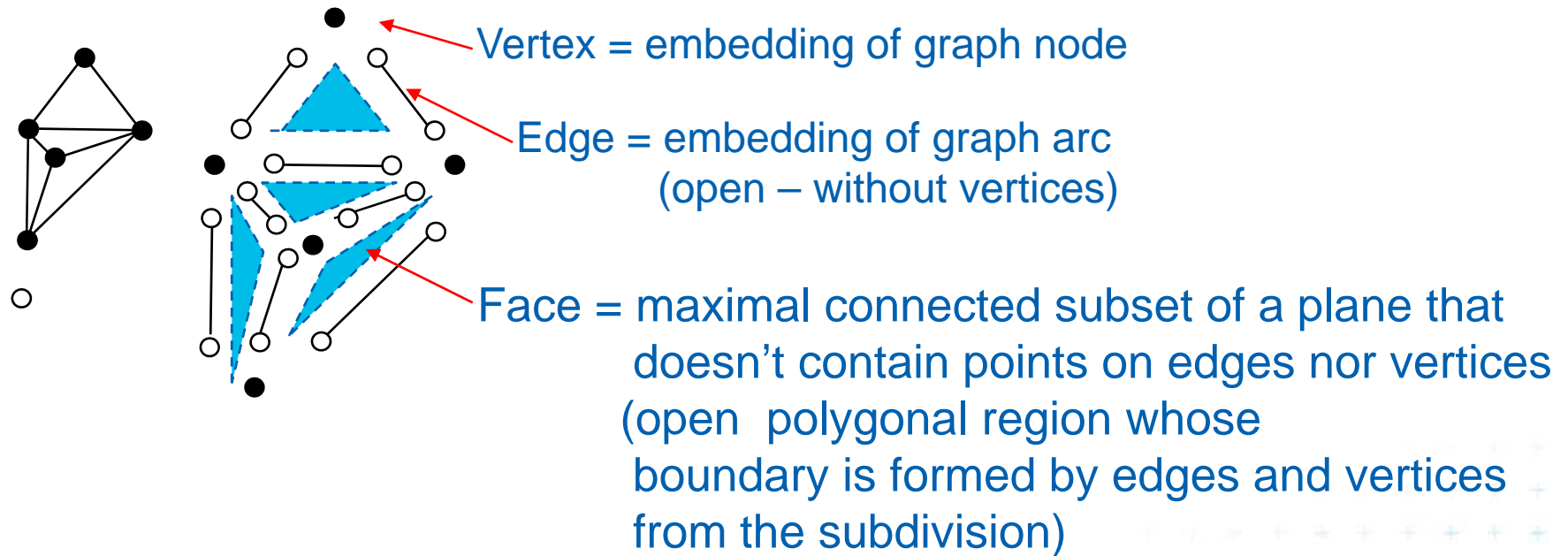
connected



disconnected



Planar subdivision



Complexity (size) of a subdivision = sum of number of vertices +
+ number of edges +
+ number of faces it consists of

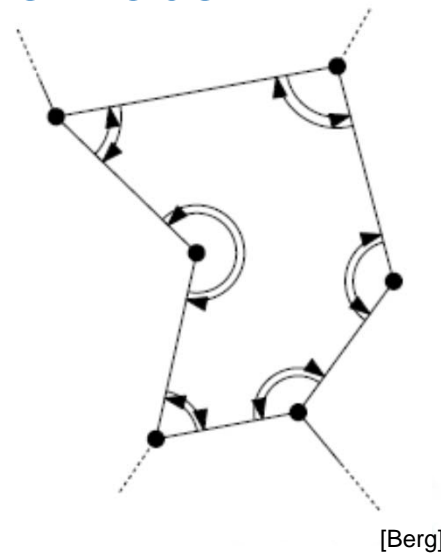
Euler's formula: $|V| - |E| + |F| \geq 2$



DCEL = Double Connected Edge List

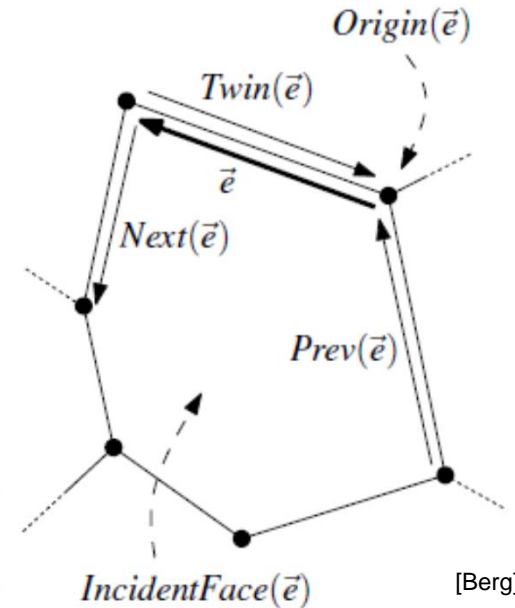
- A structure for storage of planar subdivision
- Operations like:

Walk around boundary of a given face



Pointers to next and prev edge

Get incident face

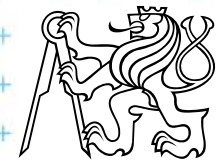
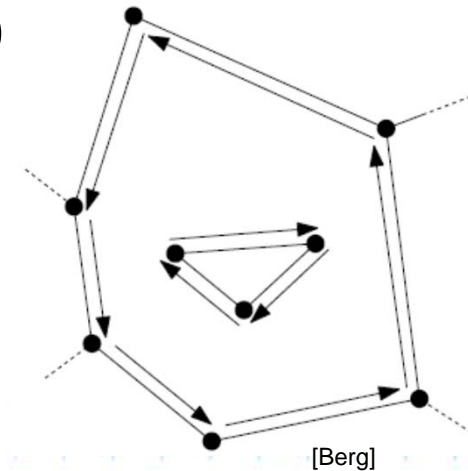


Half-edge, op. $Twin(e)$, unique $Next(e)$, $Prev(e)$

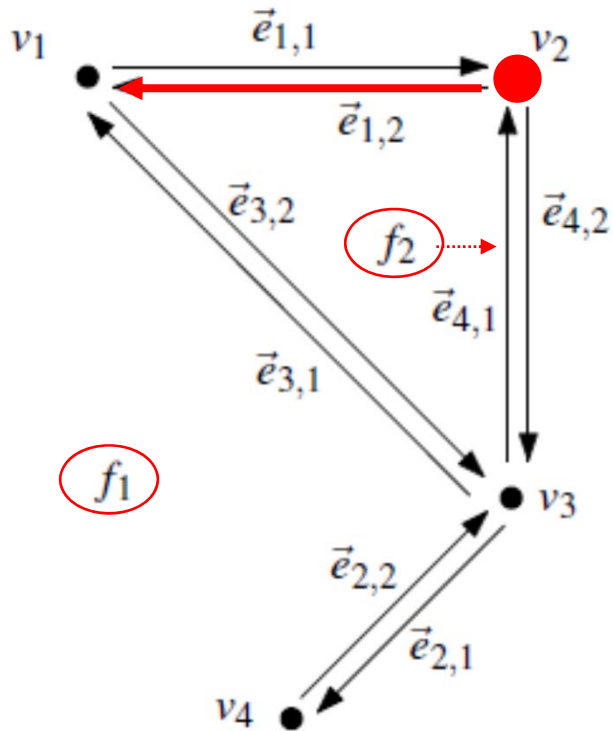


DCEL = Double Connected Edge List

- Vertex record v
 - $\text{Coordinates}(v)$ and pointer to $\text{IncidentEdge}(v)$
- Face record f
 - $\text{OuterComponent}(f)$ pointer (boundary)
 - List of holes – $\text{InnerComponent}(f)$
- Half-edge record e
 - $\text{Origin}(e)$, $\text{Twin}(e)$, $\text{IncidentFace}(e)$
 - $\text{Next}(e)$, $\text{Prev}(e)$
 - [$\text{Dest}(e) = \text{Origin}(\text{Twin}(e))$]
- Possible attribute data for each



DCEL = Double Connected Edge List



Vertex	Coordinates	IncidentEdge
v_1	(0,4)	$\vec{e}_{1,1}$
v_2	(2,4)	$\vec{e}_{4,2}$
v_3	(2,2)	$\vec{e}_{2,1}$
v_4	(1,1)	$\vec{e}_{2,2}$

One of edges

List of holes

Face	OuterComponent	InnerComponents
f_1	nil	$\vec{e}_{1,1}$
f_2	$\vec{e}_{4,1}$	nil

Half-edge	Origin	Twin	IncidentFace	Next	Prev
$\vec{e}_{1,1}$	v_1	$\vec{e}_{1,2}$	f_1	$\vec{e}_{4,2}$	$\vec{e}_{3,1}$
$\vec{e}_{1,2}$	v_2	$\vec{e}_{1,1}$	f_2	$\vec{e}_{3,2}$	$\vec{e}_{4,1}$
$\vec{e}_{2,1}$	v_3	$\vec{e}_{2,2}$	f_1	$\vec{e}_{2,2}$	$\vec{e}_{4,2}$
$\vec{e}_{2,2}$	v_4	$\vec{e}_{2,1}$	f_1	$\vec{e}_{3,1}$	$\vec{e}_{2,1}$
$\vec{e}_{3,1}$	v_3	$\vec{e}_{3,2}$	f_1	$\vec{e}_{1,1}$	$\vec{e}_{2,2}$
$\vec{e}_{3,2}$	v_1	$\vec{e}_{3,1}$	f_2	$\vec{e}_{4,1}$	$\vec{e}_{1,2}$
$\vec{e}_{4,1}$	v_3	$\vec{e}_{4,2}$	f_2	$\vec{e}_{1,2}$	$\vec{e}_{3,2}$
$\vec{e}_{4,2}$	v_2	$\vec{e}_{4,1}$	f_1	$\vec{e}_{2,1}$	$\vec{e}_{1,1}$

[Berg]



DCEL simplifications

- If no operations with vertices and no attributes
 - Store vertex coords in origin
 - No separate vertex record
- If no need for faces (e.g. river network)
 - No face record and no IncidentFace() field
- If only connected subdivision allowed
 - Join holes with rest by dummy edges
 - Visit all half-edges by simple graph traversal
 - No InnerComponent() list for faces



Point location in planar subdivision

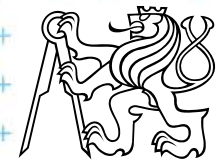
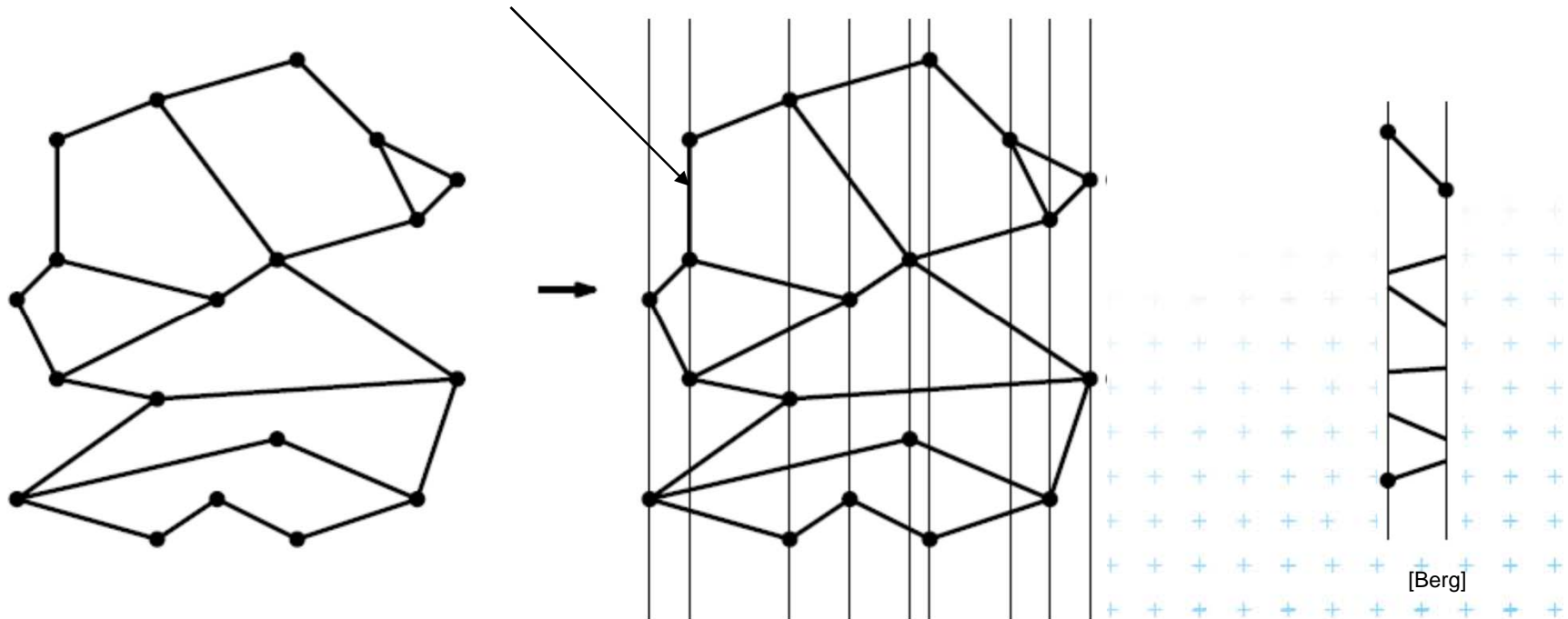
- Using special search structures
an optimal algorithm can be made with
 - $O(n)$ preprocessing,
 - $O(n)$ memory and
 - $O(\log n)$ query time.
- Simpler methods
 1. Slabs $O(\log n)$ query
 2. monotone chain tree $O(\log^2 n)$ query
 3. trapezoidal map $O(\log n)$ query expected time



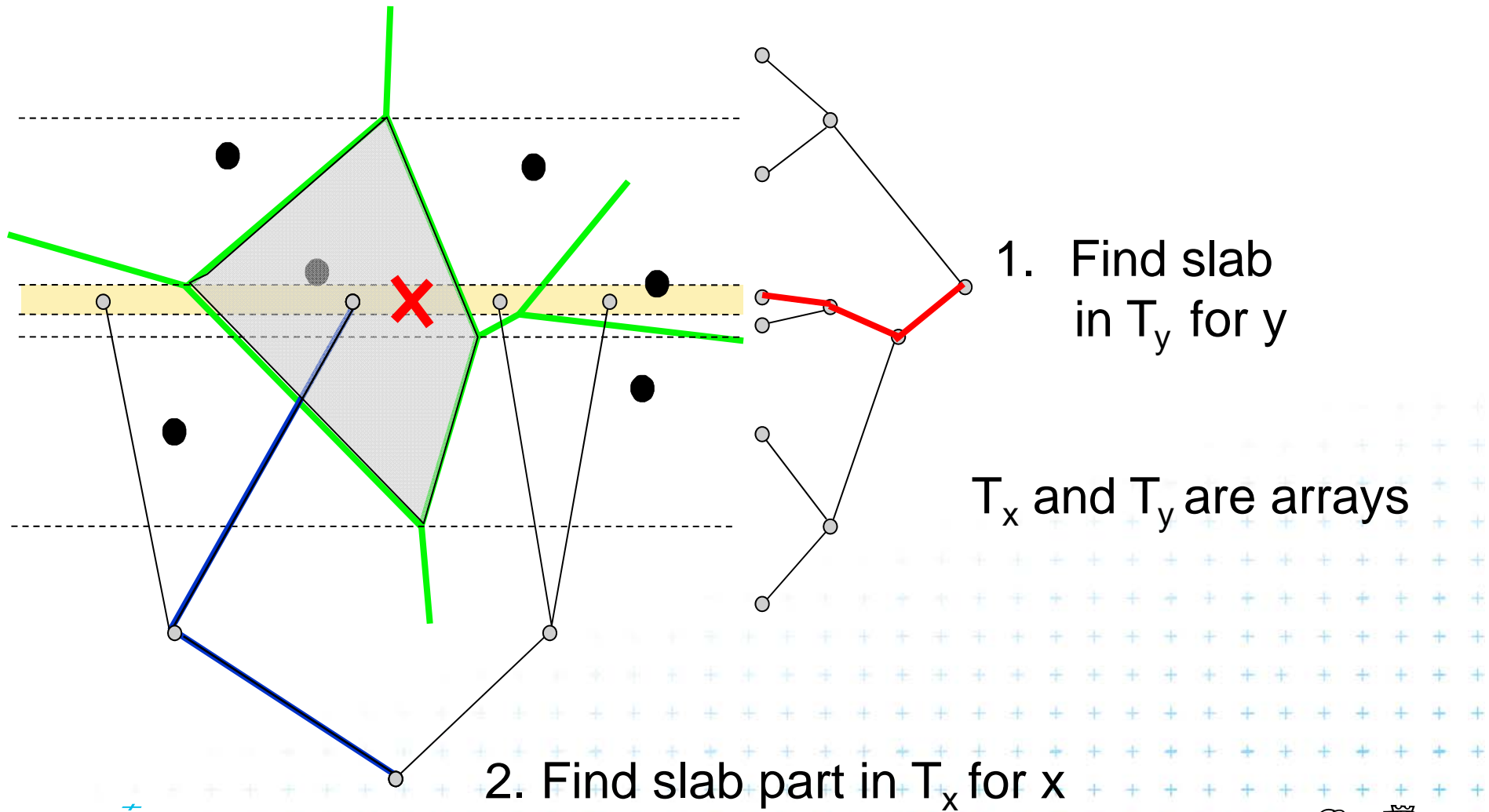
1. Vertical (horizontal) slabs

[Dobkin and Lipton, 1976]

- Draw vertical or horizontal lines through vertices
- It partitions the plane into vertical slabs
 - Avoid points with same x coordinate (to be solved later)

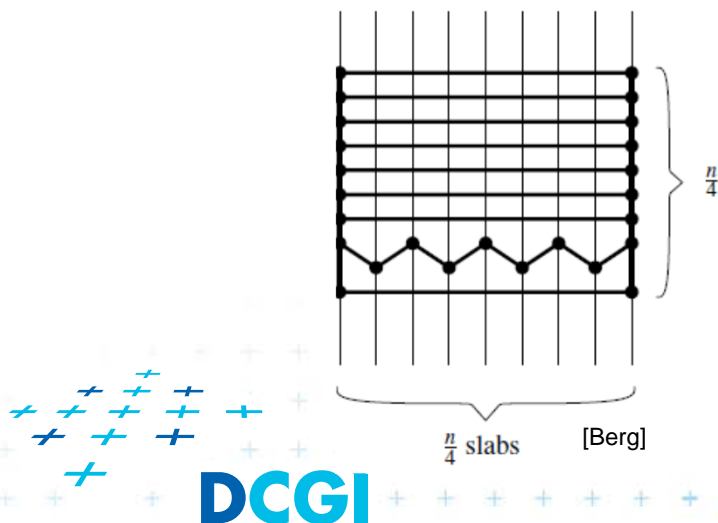


Horizontal slabs example



Horizontal slabs complexity

- Query time $O(\log n)$
 - $O(\log n)$ time in slab array T_y (size max $2n$ endpoints)
 - + $O(\log n)$ time in slab array T_x (slab crossed max by n edges)
- Memory $O(n^2)$
 - Slabs: Array with y-coordinates of vertices ... $O(n)$
 - For each slab $O(n)$ edges intersecting the slab



$O(n \log n)$ construction

$O(\log n)$ query

$O(n^2)$ memory



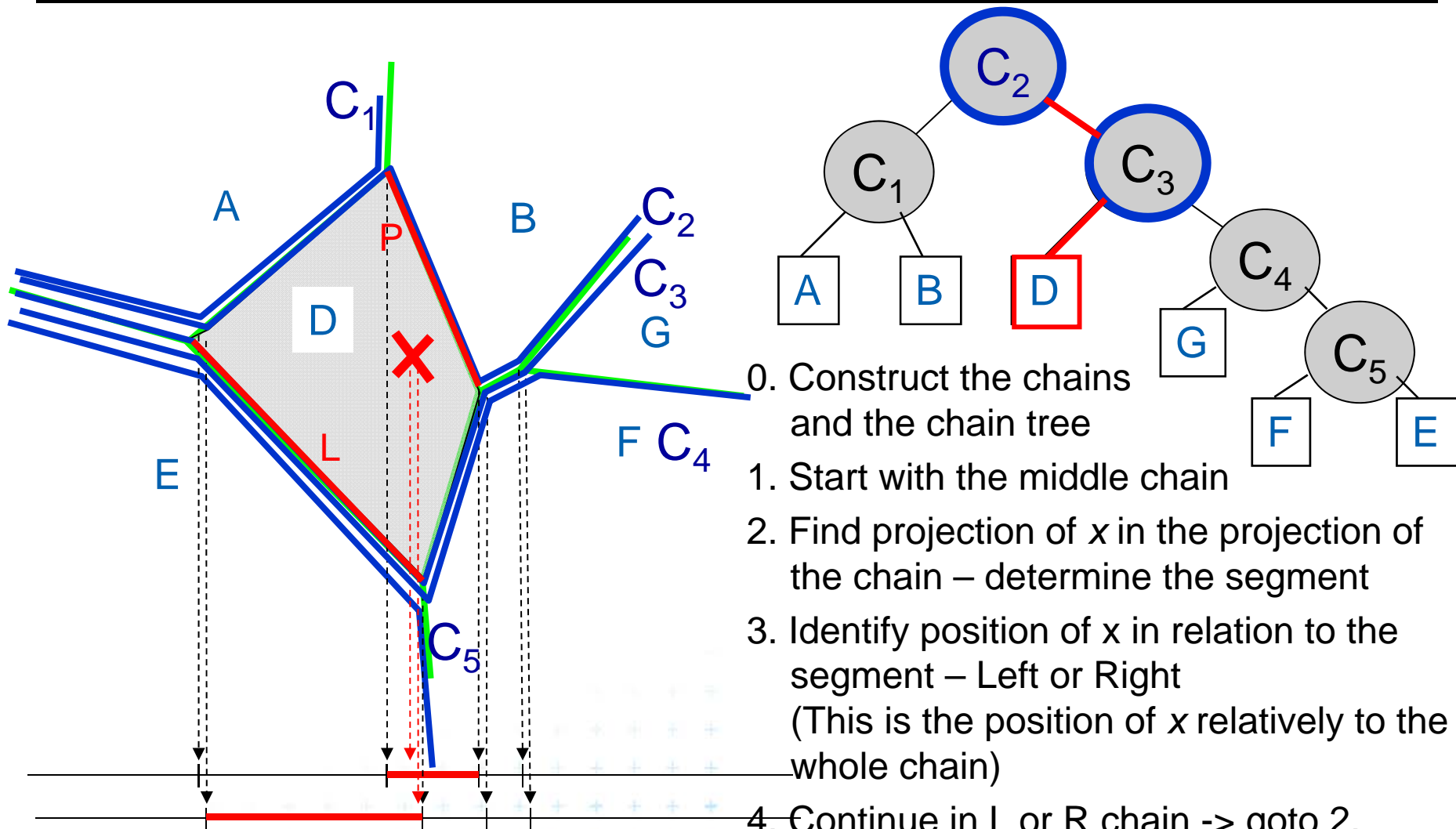
2. Monotone chain tree

[Lee and Preparata, 1977]

- Construct monotone planar subdivision
 - The edges are monotone in the same direction
- Each separator chain
 - is monotone (can be projected to line and searched)
 - splits the plane into two parts – allows binary search
- Algorithm
 - Preprocess: Find the separators (e.g., horizontal)
 - Search:
 - Binary search among separators (Y) ... $O(\log n)$
 - Binary search along the separator (X) ... $O(\log n)$
 - Not optimal, but simple $O(\log^2 n)$ query
 - Can be made optimal, but the algorithm and data structures are complicated $O(n^2)$ memory

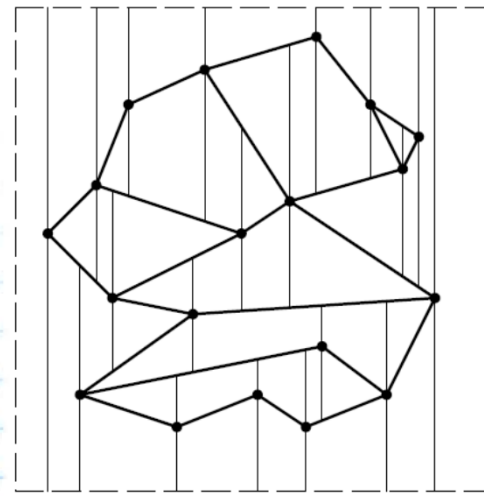


Monotone chain tree example



3. Trapezoidal map (TM) search

- The simplest and most practical known optimal algorithm
- Randomized algorithm with $O(n)$ expected storage and $O(\log n)$ expected query time
- Expectation depends on the random order of segments during construction, not on the position of the segments
- TM is refinement of original subdivision
- Converts complex shapes into simple ones
- Weaker assumption on input:
 - Input individual segments, not polygons
 - $S = \{s_1, s_2, \dots, s_n\}$
 - S_i subset of first i segments
 - Answer: segment below the pointed trapezoid (Δ)

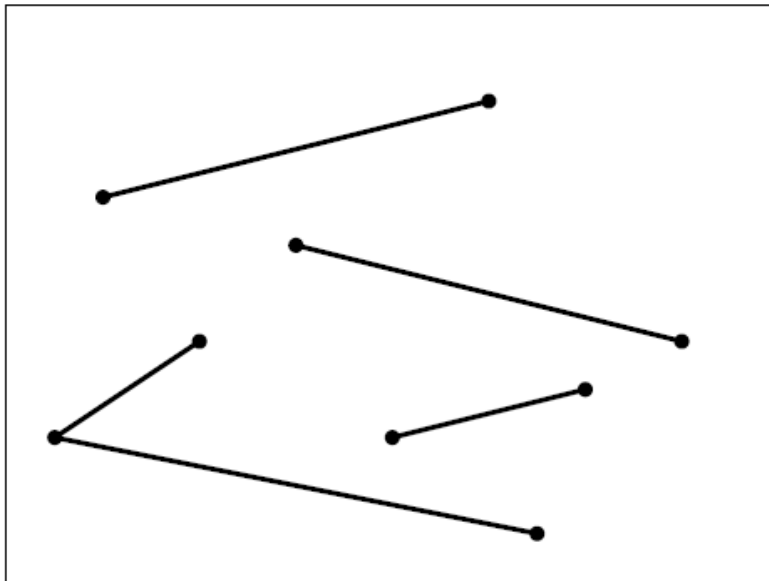


R
[Berg]



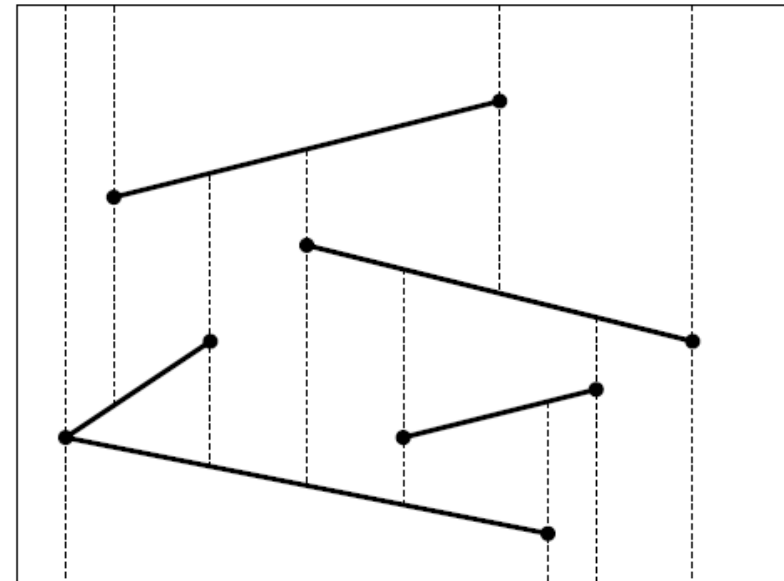
Trapezoidal map of line segments in general position

Input: individual segments S



Construction →

Trapezoidal map T



[Mount]

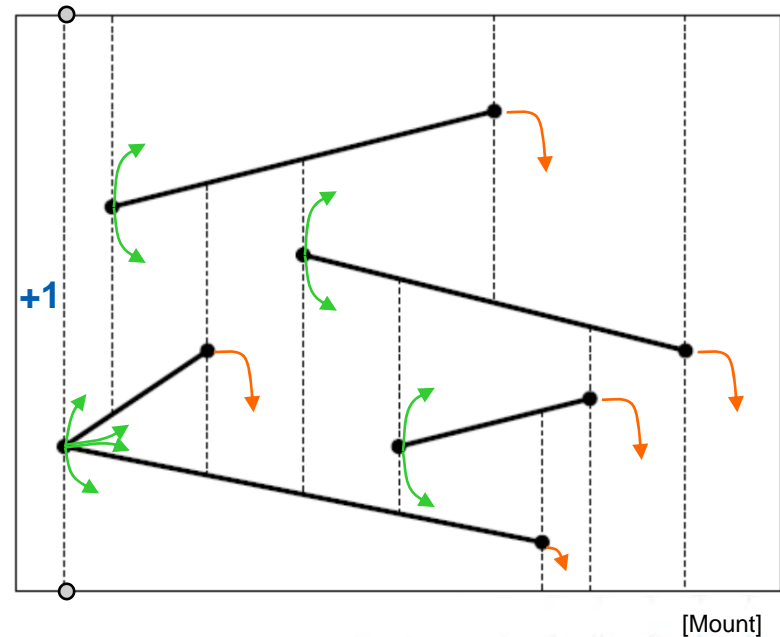
- They do not intersect, except in endpoints
- No vertical segments
- No 2 distinct endpoints with the same x-coordinate

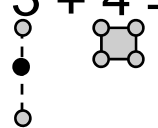
- Bounding rectangle
- 4 Bullets up and down
- Stop on input segment or on bounding rectangle

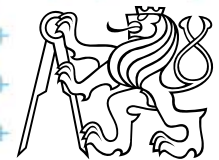


Trapezoidal map of line segments in general position

- Faces are trapezoids Δ with vertical sides
- Given n segments, TM has
 - at most $6n+4$ vertices
 - at most $3n+1$ trapezoids



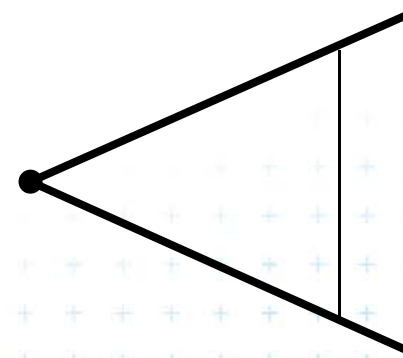
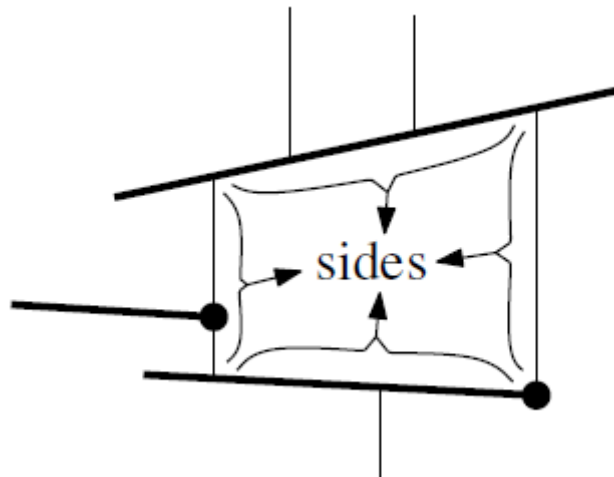
- Proof:
 - each point 2 bullets $\rightarrow 1+2$ points
 - $2n$ endpoints $\cdot 3 + 4 = 6n+4$ vertices
 - 
 - start point \rightarrow max 2 trapezoids
 - end point \rightarrow 1 trapezoid
 - $3 \cdot (n \text{ segments}) + 1 \text{ left } \Delta \Rightarrow \text{max } 3n+1 \Delta$



Trapezoidal map of line segments in general position

Each face has

- one or two **vertical sides** (trapezoid or triangle) and
- exactly two **non-vertical sides**



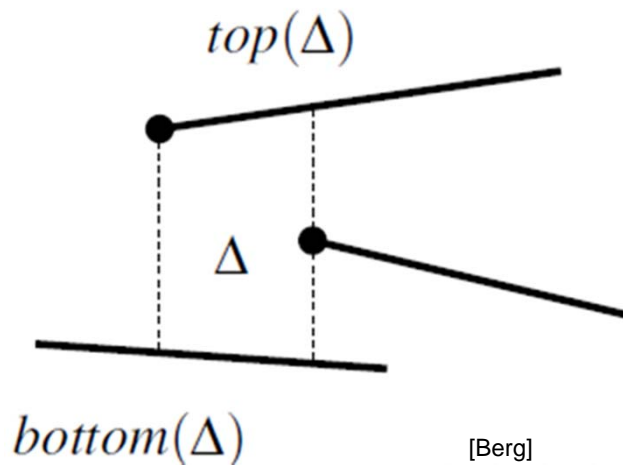
[Berg]



Two non-vertical sides

Non-vertical side

- is contained in a segment of S
- or in the horizontal edge of bounding rectangle R

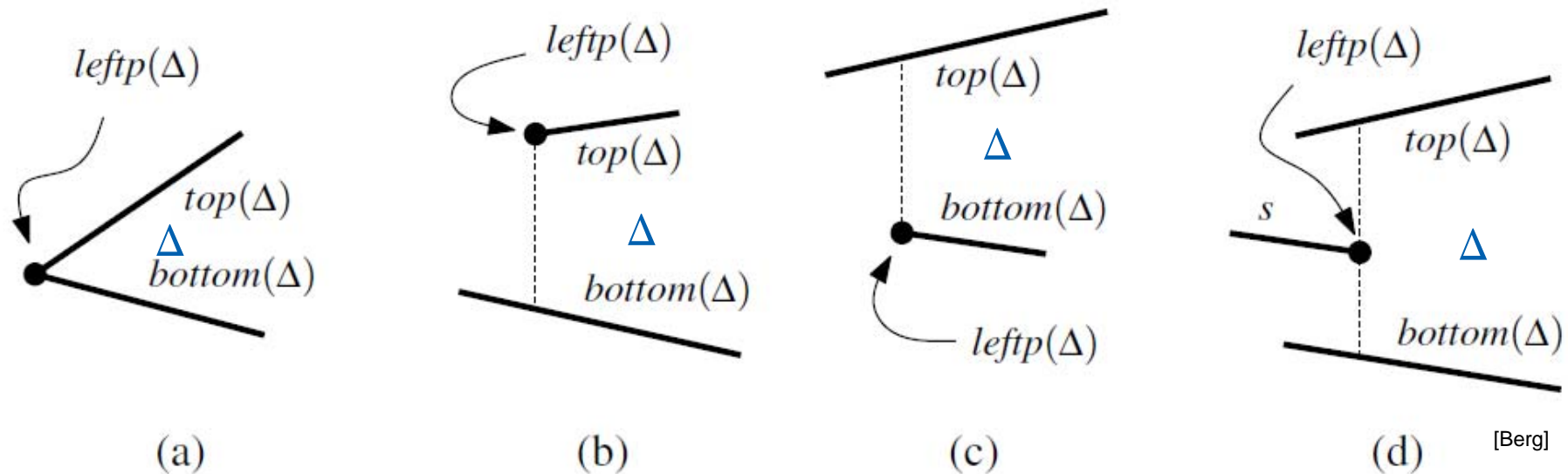


$top(\Delta)$ - bounds from above

$bottom(\Delta)$ - bounds from below



Vertical sides – left vertical side of Δ



Left vertical side is defined by the segment end-point $p = \text{leftp}(\Delta)$

(a) common left point p itself

(b) by the lower vert. extension of left point p ending at $\text{bottom}(\Delta)$

(c) by the upper vert. extension of left point p ending at $\text{top}(\Delta)$

(d) by both vert. extensions of the right point p

(e) the left edge of the bounding rectangle R (leftmost Δ only)



Vertical sides - summary

Vertical edges are defined by segment endpoints

- $leftp(\Delta)$ = the end point defining the left edge of Δ
- $rightp(\Delta)$ = the end point defining the right edge of Δ

$leftp(\Delta)$ is

- the **left endpoint** of $top()$ or $bottom()$ (a,b,c)
- the **right point** of a third segment (d)
- the **lower left corner** of R (e)



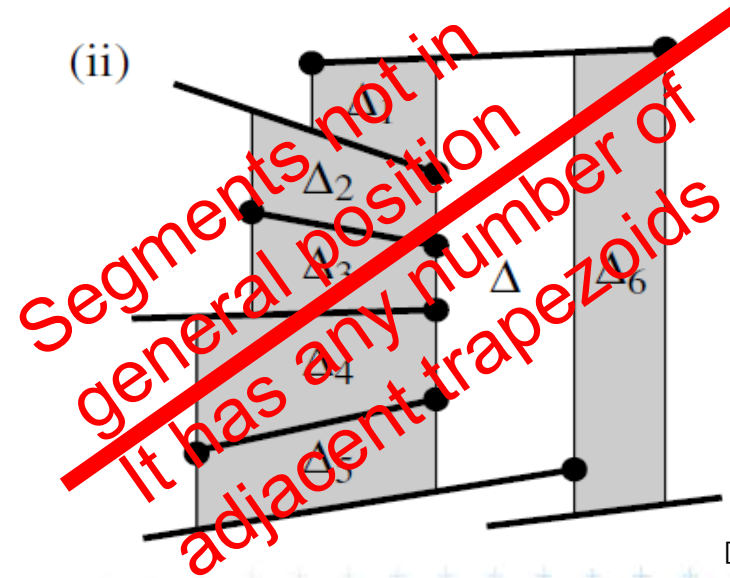
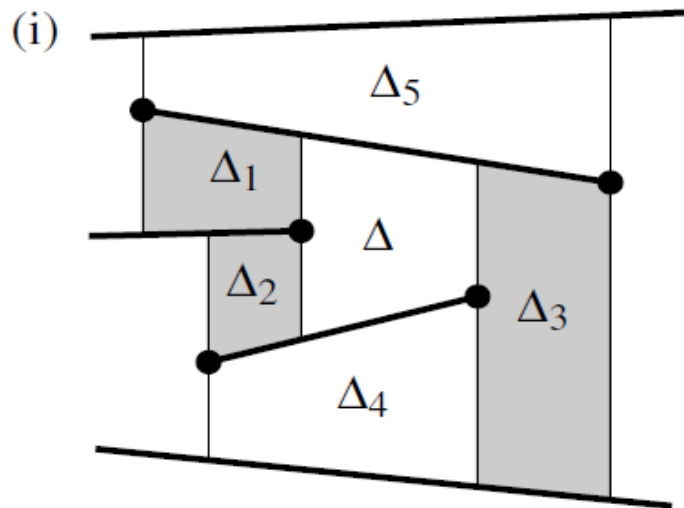
Trapezoid Δ

- Trapezoid Δ is uniquely defined by the segments $top(\Delta)$, $bottom(\Delta)$
- And by the endpoints $leftp(\Delta)$, $rightp(\Delta)$



Adjacency of trapezoids segments in general position

- Trapezoids Δ and Δ' are **adjacent**, if they meet along a vertical edge



- Δ_1 = upper left neighbor of Δ (common $top(\Delta)$ edge)
- Δ_2 = lower left neighbor of Δ (common $bottom(\Delta)$)
- Δ_3 is a right neighbor of Δ (common $top(\Delta)$ & $bottom(\Delta)$)



Representation of the trapezoidal map T

Special trapezoidal map structure $T(S)$ stores:

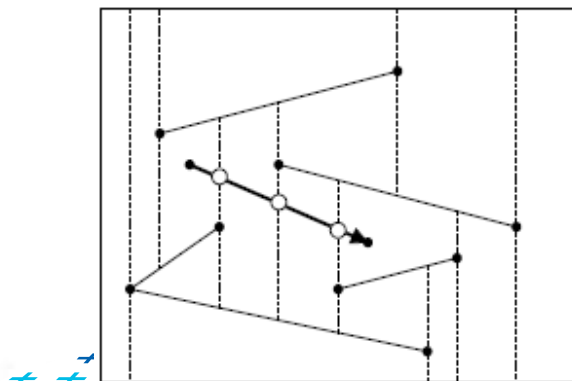
- Records for all **line segments** and **end points**
- Records for each **trapezoid** $\Delta \in T(S)$
 - Definition of Δ - pointers to segments $top(\Delta)$, $bottom(\Delta)$,
- pointers to points $leftp(\Delta)$, $rightp(\Delta)$
 - Pointers to its max **four neighboring trapezoids**
 - Pointer to the **leaf** \square in the search structure D (see below)
- Does not store the geometry explicitly!
- Geometry of trapezoids is computed in $O(1)$



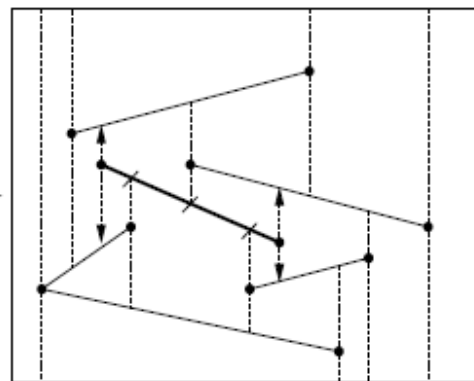
Construction of trapezoidal map

■ Randomized incremental algorithm

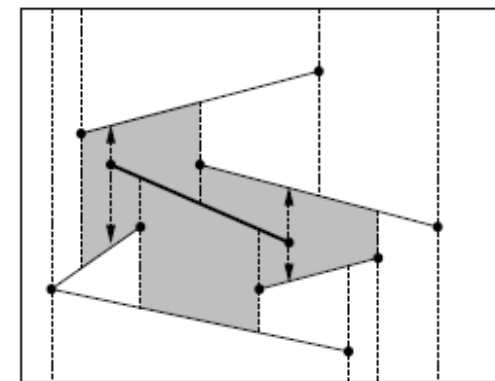
1. Create the initial bounding rectangle ($T_0 = 1\Delta$) ... $O(n)$
2. Randomize the order of segments in S
3. for $i = 1$ to n do
4. Add segment S_i to trapezoidal map T_i
5. locate left endpoint of S_i in T_{i-1}
6. find intersected trapezoids
7. shoot 4 bullets from endpoints of S_i
8. trim intersected vertical bullet paths



Locate left endpoint and determine intersections



Shoot new bullet paths and trim intersecting rays

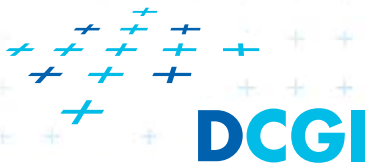


Newly created trapezoids

[Mount]

Trapezoidal map point location

- While creating the trapezoidal map T construct the *Point location data structure* D
- Query this data structure



Point location data structure D

- Rooted directed acyclic graph (not a tree!!)

- Leaves \square – trapezoids, each appears exactly once

- Internal nodes – 2 outgoing edges, guide the search

$\circ p_1$ x-node – x-coord x_0 of segment start- or end-point

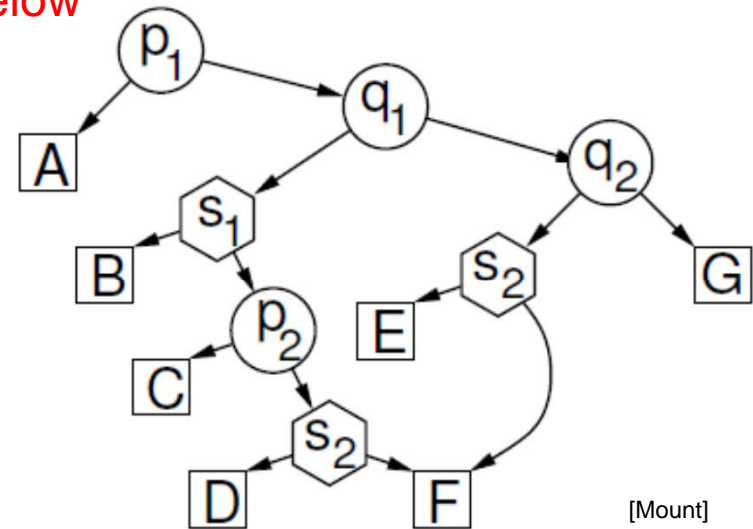
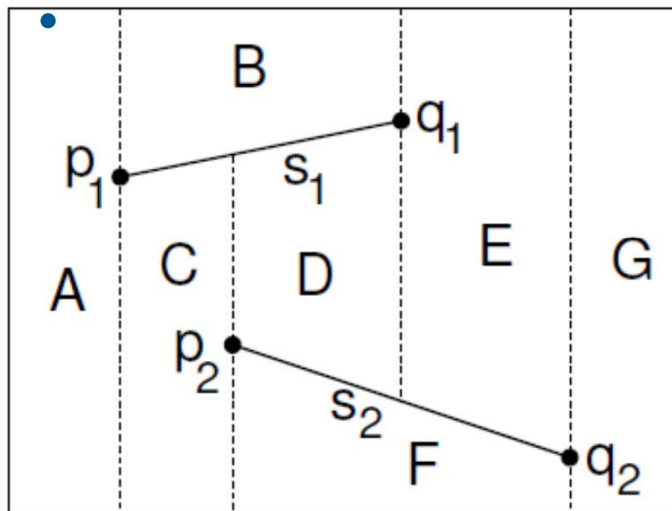
left child lies left of vertical line $x=x_0$

right child lies right of vertical line $x=x_0$

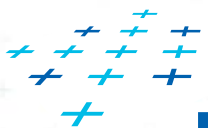
- used first to detect the vertical slab

$\hexagon s_1$ y-node – pointer to the line segment of the subdivision (not only its y!!!)

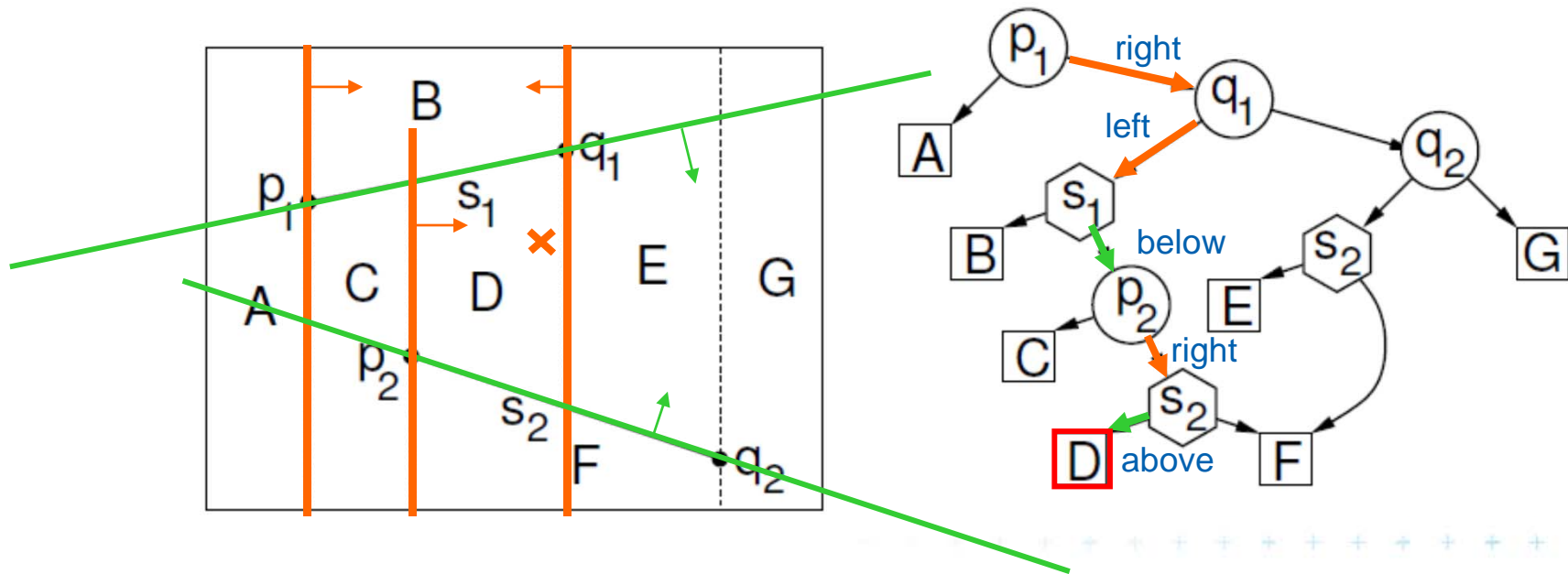
left – above, right – below



[Mount]

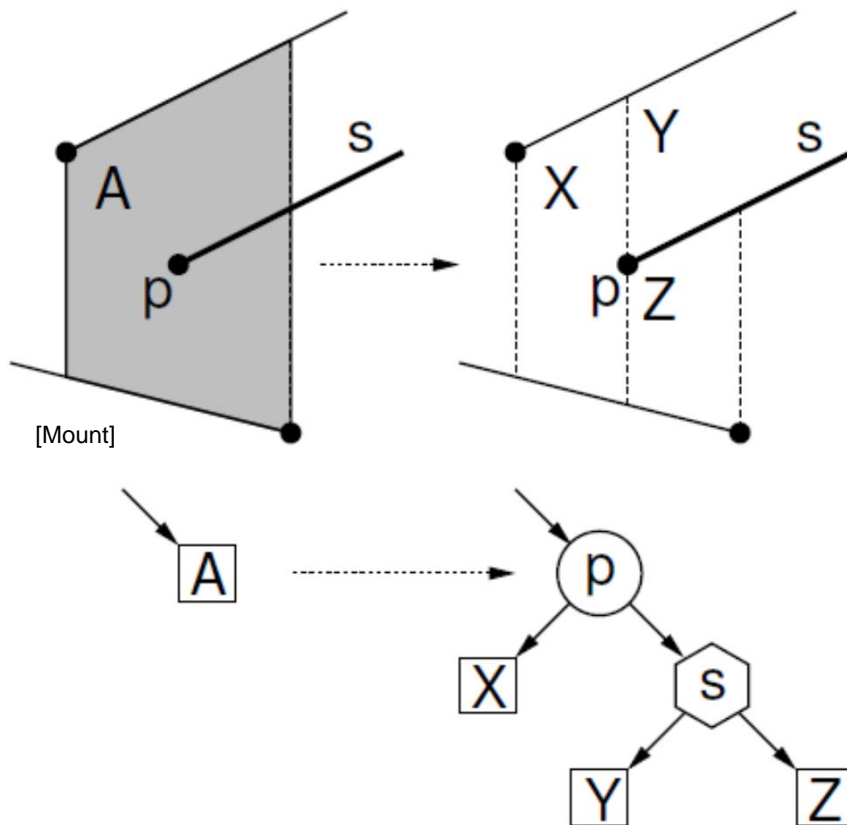


TM search example



Construction – addition of a segment

a) Single (left or right) endpoint - 3 new trapezoids



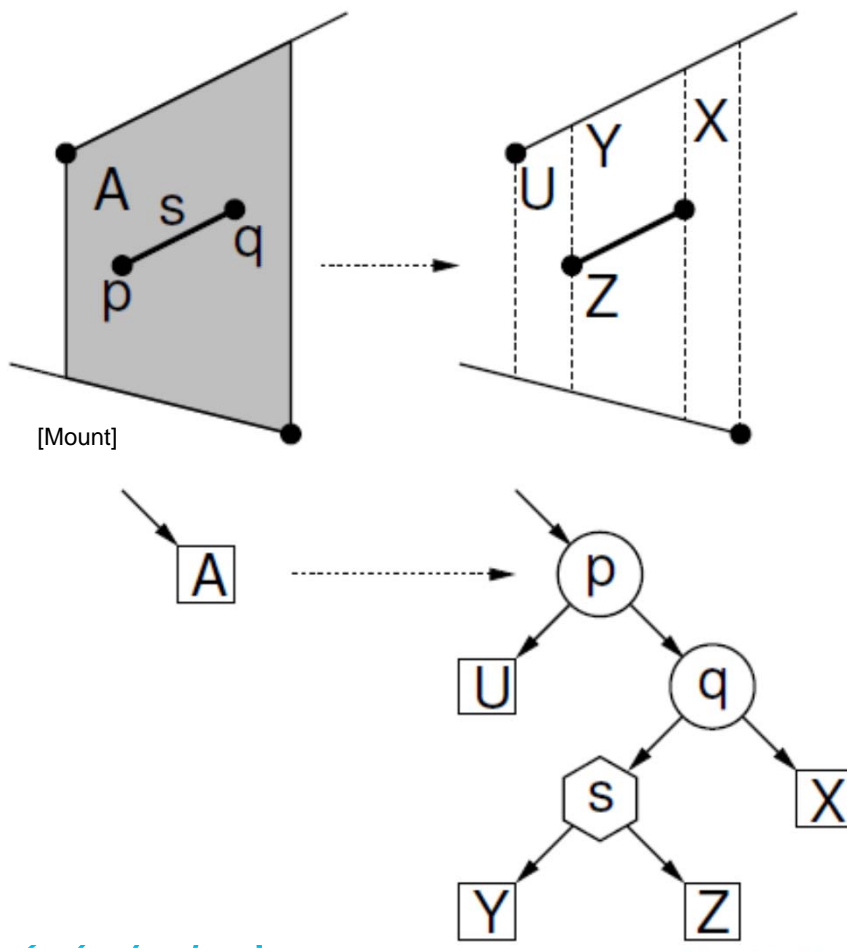
Trapezoid A replaced by

- * x-node for point p
- add left leaf for $X \Delta$
- add right subtree
- * y-node for segment s
- add left leaf for $Y \Delta$ above
- add right leaf $Z \Delta$ below



Construction – addition of a segment

b) Two segment endpoints – 4 new trapezoids



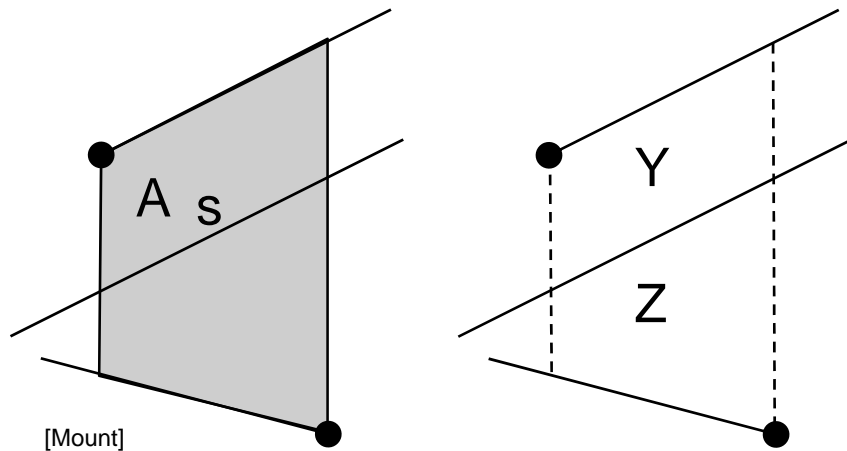
Trapezoid A replaced by

- * x-node for point p
- * x-node for point q
- * y-node for segment s
- add leaves for U, X, Y, Z



Construction – addition of a segment

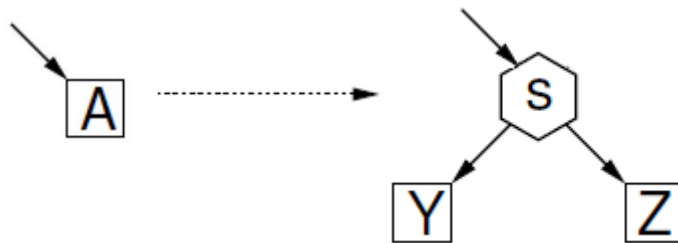
c) No segment endpoint – create 2 trapezoids



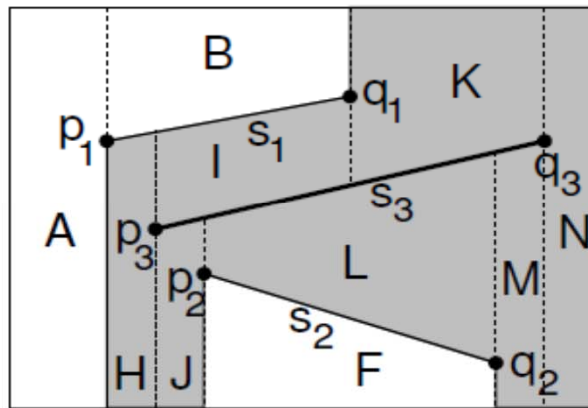
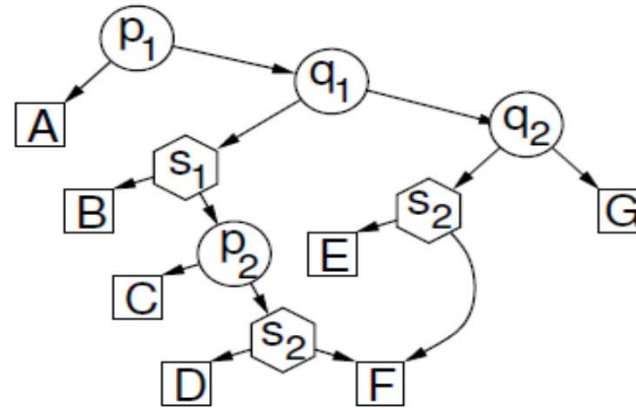
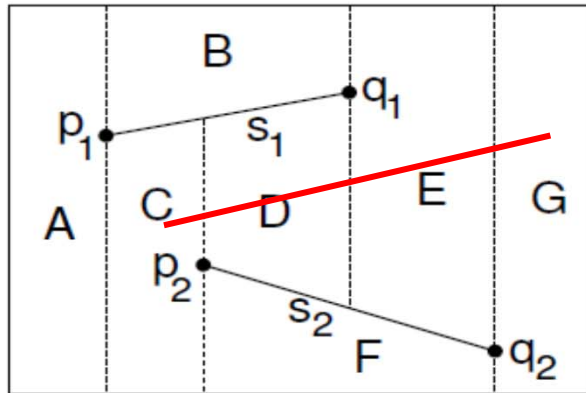
Trapezoid A replaced by

- * y-node for segment s
- add leaves for Y, Z

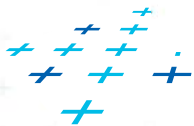
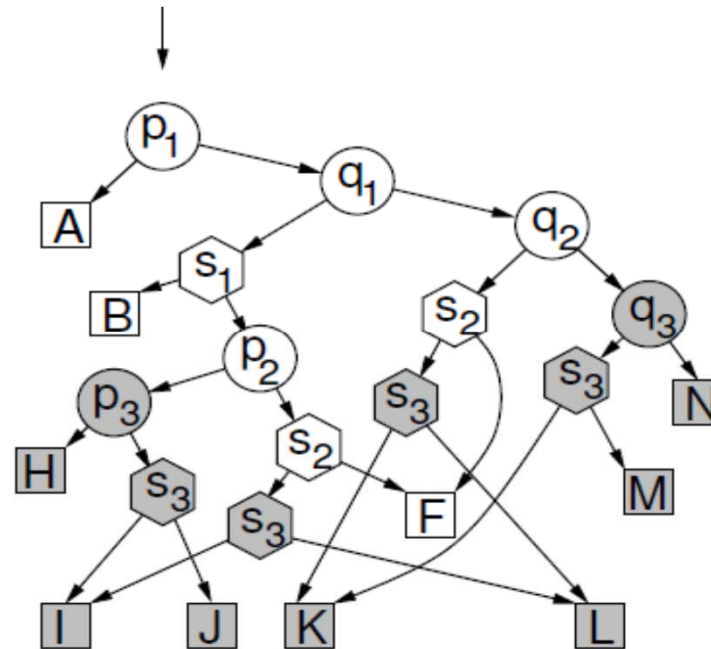
[Mount]



Segment insertion example



[Mount]



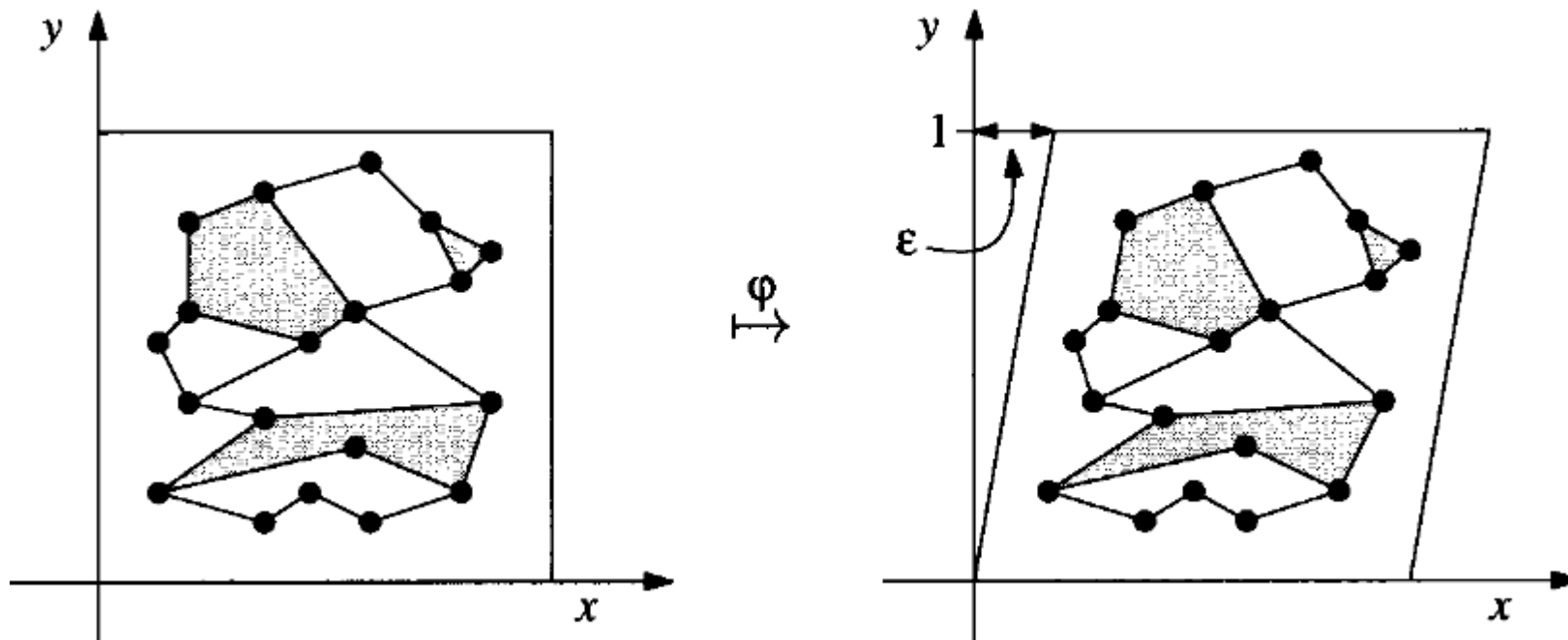
Analysis and proofs

- This holds:
 - Number of newly created Δ for inserted segment:
 $k_i = K+4 \Rightarrow O(k_i) = O(1)$ for K trimmed bullet paths
 - Search point $O(\log n)$ in average
 \Rightarrow Expected construction $O(n(1 + \log n)) = O(n \log n)$
- For detailed analysis and proofs see
 - [Berg] or [Mount]

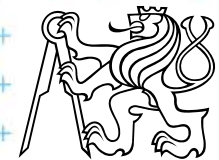


Handling of degenerate cases - principle

- No distinct endpoints lie on common vertical line
 - Rotate or shear the coordinates $x' = x + \epsilon y$, $y' = y$



[Berg]



Handling of degenerate cases - realization

■ Trick

- store original (x,y) , we need to perform just 2 operations
1. For two points p,q determine if transformed point q is to the **left**, to the **right** or **on** vertical line through point p
 - If $x_p = x_q$ then compare y_p and y_q (on only for $y_p = y_q$)
 - \Rightarrow use the original coords (x, y) and **lexicographic order**
 2. For segment given by two points decide if **3rd point q lies above, below or on the segment $p_1 p_2$**
 - Mapping preserves this relation
 - \Rightarrow use the original coords (x, y)



Point location summary

- **Slab method** [Dobkin and Lipton, 1976]
 - $O(n^2)$ memory $O(\log n)$ time
- **Monotone chain tree in planar subdivision** [Lee and Preparata, 77]
 - $O(n^2)$ memory $O(\log^2 n)$ time
- **Layered directed acyclic graph (Layered DAG) in planar subdivision** [Chazelle, Guibas, 1986] [Edelsbrunner, Guibas, and Stolfi, 1986]
 - $O(n)$ memory $O(\log n)$ time => optimal algorithm of planar subdivision search (optimal but complex alg. => see elsewhere)
- **Trapeziodal map**
 - $O(n)$ expected memory $O(\log n)$ expected time
 - $O(n \log n)$ expected preprocessing (simple alg.)



References

- **[Berg]** Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5 <http://www.cs.uu.nl/geobook/>
- **[Mount]** David Mount, - CMSC 754: Computational Geometry, Lecture Notes for Spring 2007, University of Maryland <http://www.cs.umd.edu/class/spring2007/cmsc754/lectures.shtml>





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