Part II

Perspective Camera

1. Basic Entities: Points, Lines
2. Homography: Mapping Acting on Points and Lines
3. Canonical Perspective Camera
4. Changing the Outer and Inner Reference Frames
5. Projection Matrix Decomposition
6. Anatomy of Linear Perspective Camera
7. Vanishing Points and Lines
8. Real Camera with Radial Distortion

covered by

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, 7.4, Example: 2.19

▶Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	n	O
plane		π, φ

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^\top, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as $\mathbf{m} = (u, v)$, $\mathbf{X} = (x, y, z)$, etc.

- vectors are always meant to be columns $\mathbf{x} \in \mathbb{R}^{n,1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^\top, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^\top, \quad \underline{\mathbf{n}}$$

'in-line' forms: $\underline{\mathbf{m}} = (m_1, m_2, m_3)$, $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$, etc.

- matrices are $\mathbf{Q} \in \mathbb{R}^{m,n}$

▶Image Line

line in the plane $a\,u + b\,v + c = 0$

corresponds to (homogeneous) vector $\underline{\mathbf{n}} \simeq (a,\, b,\, c)$

and the equivalence class for $\lambda \in \mathbb{R},\, \lambda \neq 0$ $(\lambda a,\, \lambda b,\, \lambda c) \simeq (a,\, b,\, c)$

- the set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0, 0, 0)$ forms the projective space $\mathbb{P}^2$
  a set of rays
- standard representation for finite $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ is $\lambda \underline{\mathbf{n}}$, where $\lambda = \frac{\mathbf{1}}{\sqrt{n_1^2 + n_2^2}}$
  assuming $n_1^2 + n_2^2 \neq 0$; $\mathbf{1}$ is the unit, usually $\mathbf{1} = 1$
- naming convention: a special entity is the **Ideal Line** (line at infinity)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$$

- I may sometimes worngly use $=$ instead of $\simeq$, help me chase the mistakes down

▶Image Point

Point $\mathbf{m} = (u, v)$ is incident on the line $\underline{\mathbf{n}} = (a, b, c)$ iff — this works both ways!

$$a\,u + b\,v + c = 0$$

can be rewritten as (with scalar product): $(u, v, \mathbf{1}) \cdot (a, b, c) = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

point is also represented by a homogeneous vector $\underline{\mathbf{m}} \simeq (u, v, \mathbf{1})$

and the equivalence class for $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is $(m_1,\, m_2,\, m_3) = \lambda\, \underline{\mathbf{m}} \simeq \underline{\mathbf{m}}$

- standard representation for finite point $\underline{\mathbf{m}}$ is $\lambda\, \underline{\mathbf{m}}$, where $\lambda = \frac{\mathbf{1}}{m_3}$ — assuming $m_3 \neq 0$
- when $\mathbf{1} = 1$ then units are pixels and $\lambda \underline{\mathbf{m}} = (u, v, 1)$
- when $\mathbf{1} = f$ then all components have a similar magnitude, $f \sim$ image diagonal

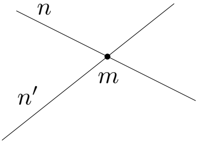
  use $\mathbf{1} = 1$ unless you know what you are doing;
  all entities participating in a formula must be expressed in the same units
- naming convention: **Ideal Point** (point at infinity) $\underline{\mathbf{m}}_\infty \simeq (m_1, m_2, 0)$

  a proper member of $\mathbb{P}^2$
- all such points lie on the ideal line $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$, ie. $\underline{\mathbf{m}}_\infty^\top\, \underline{\mathbf{n}}_\infty = 0$

►Line Intersection and Point Join

The point of **intersection** m of image lines n and n', $n \not\simeq n'$ is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$$



proof: If $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$ is the intersection point, it must be incident on both lines. Indeed,

$$\underline{\mathbf{n}}^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}}'^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} = 0$$

The **join** n of two image points m and m', $m \not\simeq m'$ is

$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}}'$$

Paralel lines intersect at the line at infinity $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$

$$a\,u + b\,v + c = 0,$$
$$a\,u + b\,v + d = 0, \qquad d \neq c$$
$$(a, b, c) \times (a, b, d) \simeq (b, -a, 0)$$

- all such intersections lie on the ideal line $\underline{\mathbf{n}}_\infty$
- line at infinity represents a set of directions in plane

▶Homography

Projective space $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$

but including 'points at infinity' and the 'line at infinity'

Collineation: Let $\underline{\mathbf{x}}_1$, $\underline{\mathbf{x}}_2$, $\underline{\mathbf{x}}_3$ be collinear points in $\mathbb{P}^2$. Bijection (1:1, onto) $h\colon \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a collineation iff $h(\underline{\mathbf{x}}_1)$, $h(\underline{\mathbf{x}}_2)$, $h(\underline{\mathbf{x}}_3)$ are collinear.

i.e.

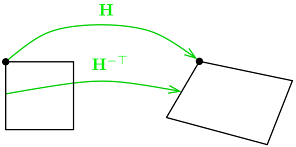
- collinear image points are mapped to collinear image points — lines are mapped to lines
- concurrent image lines are mapped to concurrent image lines — bijection!
  concurrent = intersecting at the same point
- point-line incidence is preserved

- a mapping $h : \mathbb{P}^2 \to \mathbb{P}^2$ is a collineation iff there exists a non-singular 3×3 matrix $\mathbf{H}$ such that

$$h(\underline{\mathbf{x}}) \simeq \mathbf{H}\,\underline{\mathbf{x}} \quad \text{for all } \underline{\mathbf{x}} \in \mathbb{P}^2$$

- homogeneous matrix representant: $\det \mathbf{H} = 1$
- collineations form a group isomorphic to $SO(3)$
  group of 3×3 matrices with unit determinant and with matrix multiplication
- in this course we will use the term **homography** but mean collineation

►Mapping Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{image line}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^\top \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^\top \mathbf{H}^\top \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

1. collineation has 8 DOF; it is given by 4 correspondences (points, lines) in a general position
2. extending pixel coordinates to homogeneous coordinates $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
3. mapping by homography, eg. $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
4. conversion of the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ to canonical coordinates (pixels):

$$u' = \frac{m_1'}{m_3'}\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\mathbf{1}$$

5. can use the unity for the homogeneous coordinate on one side of the equation only!

Elementary Decomposition of a Homography

Unique decompositions: $\mathbf{A} = \mathbf{A}_S\,\mathbf{A}_A\,\mathbf{A}_P \quad (= \mathbf{A}'_P\,\mathbf{A}'_A\,\mathbf{A}'_S)$

$$\mathbf{A}_S = \begin{bmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{similarity}$$

$$\mathbf{A}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{special affine}$$

$$\mathbf{A}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix} \qquad \text{special projective}$$

$\mathbf{K}$ – upper triangular matrix with positive diagonal entries
$\mathbf{R}$ – orthogonal, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = 1$
$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

$$\mathbf{A} = \begin{bmatrix} s\mathbf{R}\mathbf{K} + \mathbf{t}\,\mathbf{v}^\top & w\,\mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

- must use 'skinny' QR decomposition, which is unique [Golub & van Loan 1996, Sec. 5.2.6]
- $\mathbf{A}_S$, $\mathbf{A}_A$, $\mathbf{A}_P$ are collineation subgroups
  (eg. $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$, $\mathbf{K}^{-1}$, $\mathbf{I}$ are all upper triangular with unit determinant, associativity holds)

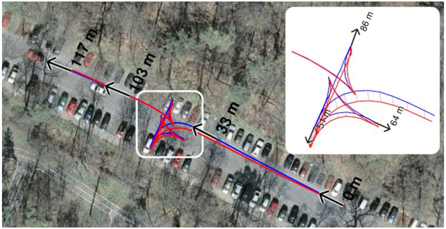
Homography Subgroups

group	DOF	matrix	invariant properties
projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	incidence, concurrency, colinearity, cross-ratio, convex hull, order of contact (intersection, tangency, inflection), tangent discontinuities and cusps.
affine	6	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: parallelism, ratio of areas, ratio of lengths on parallel lines, linear combinations of vectors (e.g. midpoints), line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)
similarity	4	$\begin{bmatrix} s\cos\phi & s\sin\phi & t_x \\ -s\sin\phi & s\cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: ratio of lengths, angle, the circular points $I = (1, i, 0)$, $J = (1, -i, 0)$.
Euclidean	3	$\begin{bmatrix} \cos\phi & \sin\phi & t_x \\ -\sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: length, area

Some Homographic Tasters

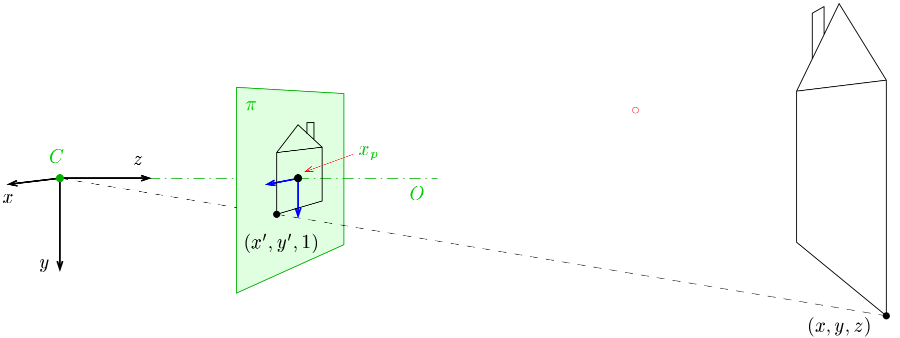
Rectification of camera rotation: Slides 60 (geometry), 122 (homography estimation)


Homographic Mouse for Visual Odometry: Slide TBD

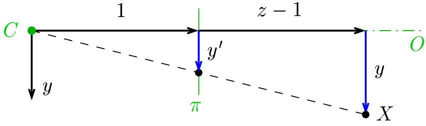



illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

►Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. right-handed canonical coordinate system (x, y, z)
2. origin = center of projection C
3. image plane π at <u>unit</u> distance from C
4. optical axis O is perpendicular to π
5. principal point x_p: intersection of O and π
6. in this picture we are looking 'down the street'
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

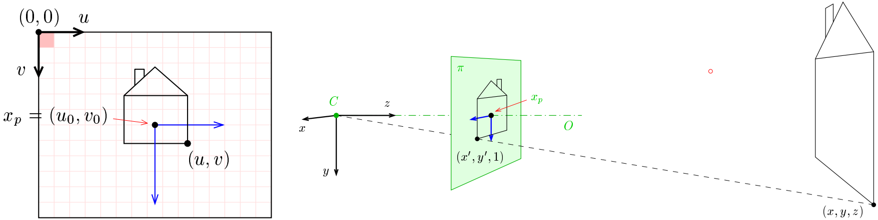
$$\frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

▶Natural and Canonical Image Coordinate Systems

projected point **in canonical camera**

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix}^\top = \begin{bmatrix} \frac{x}{z}, & \frac{y}{z}, & 1 \end{bmatrix}^\top = \frac{1}{z} \begin{bmatrix} x, & y, & z \end{bmatrix}^\top \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \, \underline{\mathbf{X}}$$

projected point **in scanned image** notice the chimney!



$$u = f\,\frac{x}{z} + u_0 \qquad v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0\,\underline{\mathbf{X}} = \mathbf{P}\,\underline{\mathbf{X}}$$

- ‘calibration’ matrix $\mathbf{K}$ transforms canonical camera $\mathbf{P}_0$ to standard projective camera $\mathbf{P}$

Thank You