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AE4M33RZN, Fuzzy logic: Introduction, Fuzzy operators

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Description logics

```
<Ontology ontologyIRI="http://example.com/tea.owl" ...>
  <Prefix name="owl" IRI="http://www.w3.org/2002/07/owl#"/>
  <Declaration>
    <Class IRI="Tea"/>
  </Declaration>
</Ontology>
```

- A *description logic* is a decideable fragment of *first order logic* (FOL).
- \+ Uses *concepts*, *roles* and *individuals* to capture **structured knowledge**.
- – An unexpected fact in the $\mathscr{A}$*Box* might lead to a *contradiction*, which is a pain. (See an example in a minute.)

Graphical probabilistic models

(Photo by ICMA Photos
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- GPM is an **efficient representation** of large probability distributions.
- \+ Captures *uncertainty* well.
- \+ Even unlikely events (tossing *head* 100 times in a row) can be processed.
- – Cannot formulate complex statements **explicitly**, such as “**Every** object in the database has **at least** one...”

Example: Smoking friends (1)

To illustrate the limitations of DL and GPM, consider an example from [Domingos and Lowd, 2009].

(Image: Matthew Romack under the CC-BY-SA 2.0.)

Obervation 1: High-school experience.

People start or stop smoking in groups of friends.

Obervation 2: Six degrees of separation.

Everyone is on average approximately six steps away, by way of introduction, from any other person in the world, so that **a chain of "a friend of a friend" statements can** be made, on average, **to connect any two people in six steps.** [Wikipedia, 2012]

Example: Smoking friends (2)

To formalize the example, let's use description logic $\mathscr{ALC}$:

Obervation 1: High-school experience.

If you have a friend, who is a smoker, you are a smoker as well:

$$\exists\, \text{friendOf} \cdot \text{Smoker} \sqsubseteq \text{Smoker}$$

Obervation 2: Six degrees of separation.

Joining the friendOf relation 6 times gives the ***top relation***.

$$\text{friendOf} \circ \ldots \circ \text{friendOf} \sqsubseteq \overline{\top}$$

What is wrong with this model?

Example: Smoking friends (3)

- If there is one smoker, the whole world starts smoking. (Formally, an interpretation $\mathscr{I}$ must satisfy $\mathsf{Smoker}^{\mathscr{I}} = \emptyset$ or $\mathsf{Smoker}^{\mathscr{I}} = \Delta$.)
- We start from **reasonable assumptions** and arrive at **counter-intuitive** conclusion. What's wrong with our reasoning?
- We would like to express something like

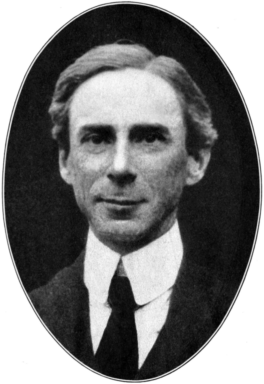
$$(\exists\, \mathsf{friendOf} \cdot \mathsf{Smoker} \sqsubseteq \mathsf{Smoker}) \text{ is “mostly” true.}$$

- Fuzzy logic can do that!

Conclusion

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

Bertrand Russel [Russell, 1923]



Crisp sets: Definition

- (Informally:) A *crisp set* („ostrá množina“) X is a collection of *objects* $x \in X$ that can be finite, countable or overcountable.
- We will speak about sets with in relation to a *universe* set („univerzum“).
- Let $\mathbb{P}(\Delta)$ be the *powerset* (a set of all subsets) of Δ (the universe). Then any crisp set is an element in the powerset of its universe: $A \in \mathbb{P}(\Delta)$.

Crisp sets: Example

Equivalent ways of describing a *crisp set* in $\mathbb{N}$:

$$A = \{1, 3, 5\} \tag{1}$$

$$A = \{x \in \mathbb{N} \mid x \leqslant 5 \text{ and } x \text{ is odd}\} \tag{2}$$

$$\mu_A(x) = \begin{cases} 0 & x > 5 \\ 0 & x \text{ is even} \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

μ_A is called the *membership function* („charakteristická funkce“, „funkce příslušnosti“).

Membership function

If μ_A is a function $\Delta \to \{0, 1\}$, the *inverse membership function* μ_A^{-1} returns objects with the given membership degree:

$$\mu_A^{-1}(M) = \{x \in X \mid \mu_A(x) \in M\} \tag{4}$$

Example

$$\mu_A^{-1}(\{1\}) = \{1, 3, 5\} \tag{5}$$

Note

μ_A^{-1} is not an inverse in a strict mathematical sense. The inverse of $\Delta \to \{0, 1\}$ should be $\{0, 1\} \to \Delta$, but $\mu_A^{-1} : \mathbb{P}(\{0, 1\}) \to \mathbb{P}(\Delta)$.

Check your knowledge:

$$\mu_{\varnothing} = ?\quad \mathbf{0}$$

$$\mu_{\Delta} = ?\quad \mathbf{1}$$

$$\mu^{-1}(\{\mathbf{0},\mathbf{1}\}) = ?\quad \Delta$$

Fuzzy set

Definition

Fuzzy set („fuzzy množina“) is an object A described by a ***generalized membership function*** $\mu_A : \Delta \to [0, 1]$.

For better readability, $A(x) \equiv \mu_A(x)$.

The set of all fuzzy subsets of a crisp universe Δ will be denoted as $\mathbb{F}(\Delta)$.

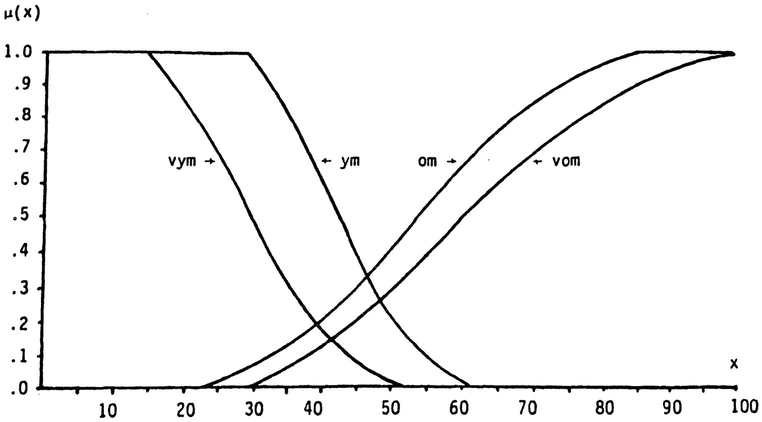


Figure 16–8. Empirical membership functions “Very Young Man,” “Young Man,” “Old Man,” “Very Old Man.”

Source: [Zimmermann, 2001]

Fuzzy set: Properties (1)

- *Cardinality* is the size of a fuzzy set.

$$|A| = \sum_{x \in \Delta} A(x) \tag{6}$$

- *Height* of a fuzzy set is the highest value of the membership function.

$$\text{Height}(A) = \sup \{\alpha \mid x \in \Delta, A(x) = \alpha\} \tag{7}$$

Fuzzy set: Properties (2)

- *Support* („nosič") is the set of objects contained in the fuzzy set "at least a bit".

$$\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\} = \mu_A^{-1}((0, 1]) \tag{8}$$

- *Core* („jádro") is the set of objects "fully contained" in the fuzzy set.

$$\mathrm{Core}(A) = \{x \in X \mid A(x) = 1\} = \mu_A^{-1}(\{1\}) \tag{9}$$

Horizontal representation

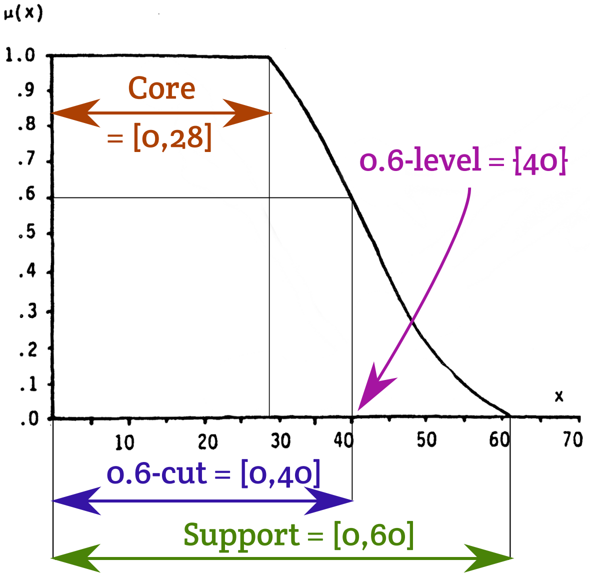
The α-***level*** („α-hladina“) of a fuzzy set A is a crisp set

$$\mu_A^{-1}(M) = \{x \in X \mid A(x) \in M\} \tag{10}$$

The α-***cut*** („α-řez“) of a fuzzy set A is a crisp set

$$\mathrm{R}_A(\alpha) = \{x \in X \mid A(x) \geqslant \alpha\} = \bigcup_{\beta \in [\alpha, 1]} \mu_A^{-1}(\beta) \tag{11}$$

Sometimes we speak about a ***strong*** α-***cut*** („ostrý α-řez“), where $\geqslant$ in the definition is replaced by $>$.
For better readability $A^{-1}(x) \equiv \mu_A^{-1}(x)$.



The set “Age of Young Men” with its properties.

Check your knowledge:

$$\mathrm{R}_A(0) = ? \quad \Delta$$

$$\mathrm{Core}(A) = ? \quad \mathrm{R}_A(1)$$

$$\mathrm{Height}(A) = \sup\{? \mid \mathrm{R}_A \ ?\} \quad \sup\{\alpha \in [0,1] \mid \mathrm{R}_A(\alpha) \neq \emptyset\}$$

Converting vertical and horizontal representation

- Horizontal representation $\sim$ the α-cuts $\mathtt{R}$.
- Vertical representation $\sim$ the characteristic function μ.

$1 \Rightarrow 2$: From the definition on the previous slide.

$2 \Rightarrow 1$: By taking the "highest" α-level containing x:

$$A(x) = \max\{\alpha \in [0,1] \mid x \in \mathtt{R}_A(\alpha)\} \tag{12}$$

Special cases of fuzzy sets

Definition

Fuzzy interval A is a fuzzy set on $\Delta = \mathbb{R}$ s.t.

- $\mathrm{R}_A(\alpha)$ is a **closed interval** for all $\alpha \in [0, 1]$
- $\mathrm{R}_A(1)$ is **not empty.**
- $\left|\mathrm{Supp}(A)\right|$ is **finite.**

Special cases of fuzzy intervals

- *Fuzzy number* A is a fuzzy interval s.t. $|\mathrm{Core}(A)| = 1$
- *Trapezoidal interval* will be denoted by $\langle a, b, c, d\rangle$.
- *Triangular number* will be denoted by $\langle a, b, c\rangle = \langle a, b, b, c\rangle$.
- A *crisp interval* $[a, b]$ is also $\langle a, a, b, b\rangle$.

Operations on fuzzy sets

set operation	propositional operation
$\overline{\cdot} : \mathbb{P}(\Delta) \Rightarrow \mathbb{P}(\Delta)$	$\neg \cdot : \{0,1\} \Rightarrow \{0,1\}$
$\cdot \cap \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \Rightarrow \mathbb{P}(\Delta)$	$\cdot \wedge \cdot : \{0,1\}^2 \Rightarrow \{0,1\}$
$\cdot \cup \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \Rightarrow \mathbb{P}(\Delta)$	$\cdot \vee \cdot : \{0,1\}^2 \Rightarrow \{0,1\}$

We can use the **logical operators** to define the **set operators**:

$$\overline{A} = \{x \in \Delta \mid \neg(x \in A)\} \tag{LS1}$$

$$A \bigcap B = \{x \in \Delta \mid (x \in A) \wedge (x \in B)\} \tag{LS2}$$

$$A \cup B = \{x \in \Delta \mid (x \in A) \vee (x \in B)\} \tag{LS3}$$

Therefore we will cover the logical negation, conjunction and disjunction. We get the set operations “for free”.

Fuzzy negation

Fuzzy negation is a *non-increasing*, *involutive*, *unary* function $\underset{\circ}{\neg} : [0, 1] \to [0, 1]$ s.t.

$$\text{if } \alpha \leqslant \beta \text{ then } \underset{\circ}{\neg}\beta \leqslant \underset{\circ}{\neg}\alpha \tag{N1}$$

$$\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \alpha \tag{N2}$$

Example

Standard („standardní“), Łukasiewicz negation

$$\underset{S}{\neg}\alpha = 1 - \alpha \tag{13}$$

The fuzzy set *complement* is a defined using (LS1).

Fuzzy negation: More examples

- Cosine negation

$$\underset{\cos}{\neg}\alpha = (\cos(\pi\alpha) + 1)/2 \tag{14}$$

- Sugeno negation

$$\underset{S\lambda}{\neg}\alpha = \frac{1-\alpha}{1+\lambda a} \tag{15}$$

- Yager negation

$$\underset{Y\lambda}{\neg}\alpha = (1 - a^{\lambda})^{1/\lambda} \tag{16}$$

Fuzzy negation: Properties

The axioms (N1) and (N2) imply more properties of fuzzy negations:

Theorem 24

Every fuzzy negation $\underset{\circ}{\neg}$ is a

- continuous
- decreasing
- bijective
- generalization of the propositional negation $\neg$

Fuzzy negation: Proof of 24

- **Injective ($f(a) = f(b) \Rightarrow a = b$):** Take 2 values, whose negations are equal: $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$. By (N2) $\alpha = \underset{\circ}{\neg}\overbrace{\underset{\circ}{\neg}\alpha}^{\square}$. The $\square$ can be substituted using the assumption: $\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\underset{\circ}{\neg}\beta$. Using (N1) gives $\underset{\circ}{\neg}\underset{\circ}{\neg}\beta = \beta$. Therefore $\alpha = \beta$.
- Every non-increasing function (N1) which is injective, must be **decreasing.** If $\alpha < \beta$ WLOG, then $\underset{\circ}{\neg}\alpha \geqslant \underset{\circ}{\neg}\beta$. Then either $\underset{\circ}{\neg}\alpha > \underset{\circ}{\neg}\beta$ and $\underset{\circ}{\neg}$ is decreasing, or $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$, which contradicts the injectivity.

Fuzzy negation: Proof of 24

- **Surjective** $\forall y \exists x. f(x) = y$**:** We seek a value of β for each α s.t. $\alpha = \underset{\circ}{\neg}\beta$. Using injectivity, the condition is equivalent to $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\underset{\circ}{\neg}\beta$. Using (N2), we find the value of β for any α: $\beta = \underset{\circ}{\neg}\alpha$.
- **Bijection** is an injective and surjective function (by definition).
- **Continuous:** Every decreasing bijection is continuous.
- **Boundary values:** Let $\underset{\circ}{\neg}0 = \alpha$ and suppose that $\alpha < 1$. Then from surjectivity, there must be some other $\beta > 0$ s.t. $\underset{\circ}{\neg}\beta = 1$. This contracits monotonicity, because $\underset{\circ}{\neg}0 < \underset{\circ}{\neg}\beta$. The other boundary value is proven similarly.

Fuzzy conjunctions (t-norms)

Fuzzy t-norm (triangluar norm, conjunction) is a binary, *comutative*, operation $\underset{\circ}{\wedge}$ s.t.

$$\alpha \underset{\circ}{\wedge} \beta = \beta \underset{\circ}{\wedge} \alpha \tag{T1}$$

$$\alpha \underset{\circ}{\wedge} (\beta \underset{\circ}{\wedge} \gamma) = (\alpha \underset{\circ}{\wedge} \beta) \underset{\circ}{\wedge} \gamma \tag{T2}$$

$$\text{if } \beta \leqslant \gamma \text{ then } (\alpha \underset{\circ}{\wedge} \beta) \leqslant (\alpha \underset{\circ}{\wedge} \gamma) \tag{T3}$$

$$(\alpha \underset{\circ}{\wedge} \mathbf{1}) = \alpha \tag{T4}$$

The fuzzy set *intersection* is a defined using (LS2).

Fuzzy conjunctions: Examples

- Standard (Gödel, Zadeh)

$$\alpha \underset{S}{\wedge} \beta = \mathbf{min}\,(\alpha, \beta) \tag{17}$$

- Łukasiewicz

$$\alpha \underset{L}{\wedge} \beta = \mathbf{max}\,(\alpha + \beta - 1, 0) \tag{18}$$

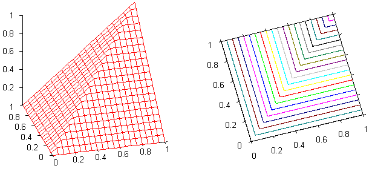
- Algebraic product („součinová“)

$$\alpha \underset{A}{\wedge} \beta = \alpha \cdot \beta \tag{19}$$

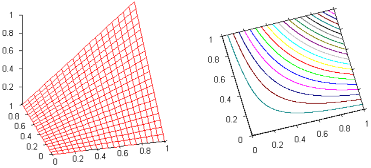
- Weak („drastická“)

$$\alpha \underset{W}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

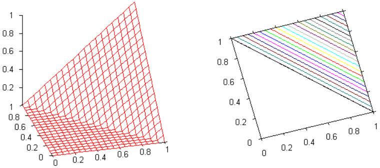
Fuzzy conjunctions: Visualization [Wikipedia]



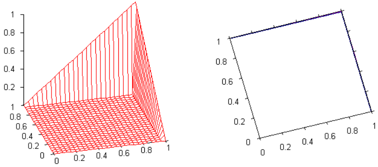
Standard



Algebraic



Łukasiewicz



Drastic

Fuzzy conjunctions: Properties (1)

Theorem 30

The weak and standard conjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \underset{W}{\wedge} \beta) \leqslant (\alpha \underset{\circ}{\wedge} \beta) \leqslant (\alpha \underset{S}{\wedge} \beta) \tag{21}$$

Proof

Assume WLOG $\alpha \leqslant \beta$.

$\beta = 1$ The condition (T4) gives the same result for all conjunctions.

$\beta < 1$ $\alpha \underset{W}{\wedge} \beta = 0$, which gives the lower bound. The upper bound is rewritten using the definition of standard conjunction (17): $\alpha \underset{S}{\wedge} \beta = \alpha$. From (T4) follows that $\alpha = \alpha \underset{\circ}{\wedge} 1 \leqslant \alpha \underset{\circ}{\wedge} \beta$.

Fuzzy conjunctions: Properties (2)

Theorem 31

The standard conjunction is the only *idempotent* conjunction:

$$\alpha \underset{\circ}{\wedge} \alpha = \alpha \tag{22}$$

Proof

Assume WLOG $\alpha \leqslant \beta$.

$$\alpha = \alpha \underset{\circ}{\wedge} \alpha \overset{(T3)}{=} \alpha \underset{\circ}{\wedge} \beta \overset{(T3)}{=} \alpha \underset{\circ}{\wedge} \mathbf{1} \overset{(T4)}{=} \alpha \tag{23}$$

Therefore $\alpha \underset{\circ}{\wedge} \beta = \alpha$. There is only one such conjunction: $\underset{S}{\wedge}$.

Fuzzy disjunctions (s-norm)

Fuzzy s-norm (t-conorm, disjunction) is a binary operation $\mathring{\vee}$ s.t.

$$\alpha \mathring{\vee} \beta = \beta \mathring{\vee} \alpha \tag{S1}$$

$$\alpha \mathring{\vee} (\beta \mathring{\vee} \gamma) = (\alpha \mathring{\vee} \beta) \mathring{\vee} \gamma \tag{S2}$$

$$\text{if } \beta \leqslant \gamma \text{ then } (\alpha \mathring{\vee} \beta) \leqslant (\alpha \mathring{\vee} \gamma) \tag{S3}$$

$$(\alpha \mathring{\vee} 0) = \alpha \tag{S4}$$

Union

The fuzzy set *union* is a defined using the disjunction:

$$\mu_{A \cup B}(x) = \mu_A(x) \mathring{\vee} \mu_B(x) \tag{24}$$

Fuzzy disjunctions: Examples (1)

- Standard (Gödel, Zadeh)

$$\alpha \overset{S}{\vee} \beta = \mathbf{max}(\alpha, \beta) \tag{25}$$

- Łukasiewicz

$$\alpha \overset{L}{\vee} \beta = \mathbf{min}(\alpha + \beta, 1) \tag{26}$$

- Algebraic sum („součinová“)

$$\alpha \overset{A}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta \tag{27}$$

Fuzzy disjunctions: Examples (2)

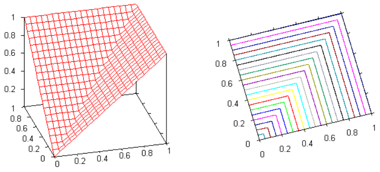
- Weak („drastická“)

$$\alpha \overset{W}{\vee} \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ \beta & \text{if } \alpha = 0 \\ 1 & \text{otherwise} \end{cases} \tag{28}$$

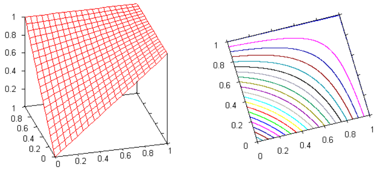
- Einstein

$$\alpha \overset{E}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta} \tag{29}$$

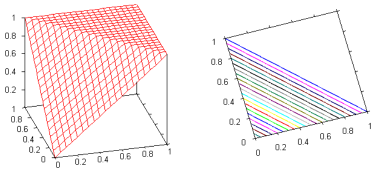
Fuzzy disjunctions: Visualization [Wikipedia]



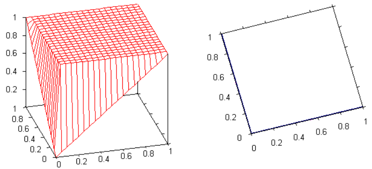
Standard



Algebraic



Łukasiewicz



Drastic

Fuzzy disjunctions: Properties

- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{S}{\vee} \beta) \leqslant (\alpha \overset{\circ}{\vee} \beta) \leqslant (\alpha \overset{W}{\vee} \beta) \tag{30}$$

- The standard disjunctions is the only *idempotent* conjunction:

$$\alpha \overset{\circ}{\vee} \alpha = \alpha \tag{31}$$

Conjunction - disjunction duality

A If $\underset{\circ}{\wedge}$ is a fuzzy conjunction, then $\alpha \overset{\circ}{\vee} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg}\alpha \underset{\circ}{\wedge} \underset{\circ}{\neg}\beta)$ is a fuzzy disjunction (dual to $\underset{\circ}{\wedge}$ w.r.t. $\underset{\circ}{\neg}$).

B If $\overset{\circ}{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\circ}{\wedge} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg}\alpha \overset{\circ}{\vee} \underset{\circ}{\neg}\beta)$ is a fuzzy conjunction (dual to $\overset{\circ}{\vee}$ w.r.t. $\underset{\circ}{\neg}$).

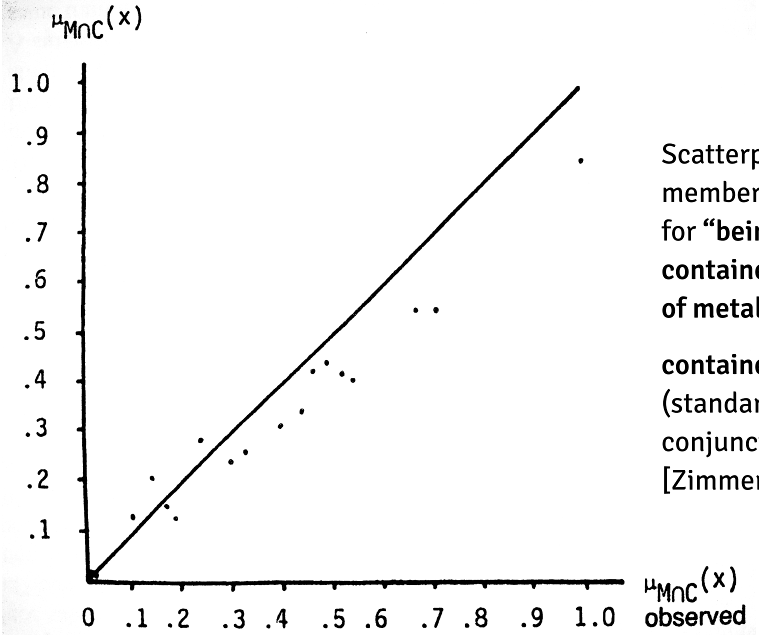
Theorems

- **Łukasiewicz** operations $\underset{\mathrm{L}}{\wedge}, \overset{\mathrm{L}}{\vee}$ are dual w.r.t. **standard** negation.
- **Algebraic** operations $\underset{\mathrm{A}}{\wedge}, \overset{\mathrm{A}}{\vee}$ are dual w.r.t. **standard** negation.
- **Standard** operations $\underset{\mathrm{S}}{\wedge}, \overset{\mathrm{S}}{\vee}$ are dual w.r.t. **any** negation.
- **Weak** operations $\underset{\mathrm{W}}{\wedge}, \overset{\mathrm{W}}{\vee}$ are dual w.r.t. **any** negation.

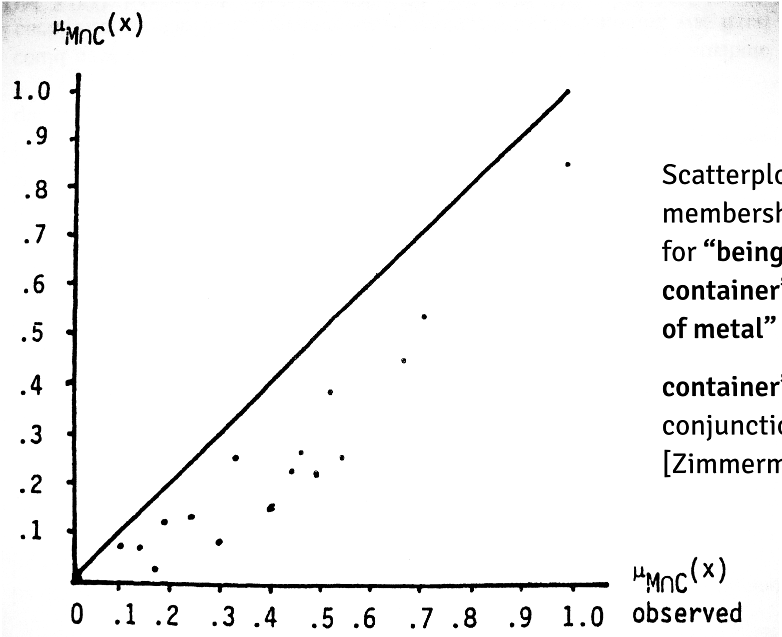
Table 16–2. Empirically determined grades of membership.

Stimulus x		$\mu_M(x)$	$\mu_C(x)$	$\mu_{M\cap C}(x)$
1.	bag	0.000	0.985	0.007
2.	baking tin	0.908	0.419	0.517
3.	ballpoint pen	0.215	0.149	0.170
4.	bathtub	0.552	0.804	0.674
5.	book wrapper	0.023	0.454	0.007
6.	car	0.501	0.437	0.493
7.	cash register	0.692	0.400	0.537
8.	container	0.847	1.000	1.000
9.	fridge	0.424	0.623	0.460
10.	Hollywood swing	0.318	0.212	0.142
11.	kerosene lamp	0.481	0.310	0.401
12.	nail	1.000	0.000	0.000
13.	parkometer	0.663	0.335	0.437
14.	pram	0.283	0.448	0.239
15.	press	0.130	0.512	0.101
16.	shovel	0.325	0.239	0.301
17.	silver spoon	0.969	0.256	0.330
18.	sledgehammer	0.480	0.012	0.023
19.	water bottle	0.564	0.961	0.714
20.	wine barrel	0.127	0.980	0.185

Degree of membership for 20 items into the sets **“make of metal”**, **“being a container”** and **“being a metalic container”**. [Zimmermann, 2001]



Scatterplot of membership degree for **"being a metalic container"** vs. **"make of metal"** $\underset{S}{\wedge}$ **"being a container"**. (standard conjunction). [Zimmermann, 2001]



Scatterplot of membership degree for "**being a metalic container**" vs. "**make of metal**" $\wedge_A$ "**being a container**" (algebraic conjunction). [Zimmermann, 2001]

Criteria for selecting operators (1)

- **Axiomatic strength:** The set of valid theorems may differ based on the choice of t-norms and s-norms (see tutorials).
- **Empirical fit:** Using fuzzy theory for a model of the real world, the chosen operator should match the real behavior of the system.
- **Adaptability:** Operators in a generic system should be able to fit several use cases. One way of increasing adaptibility is to use operators with parameters (e.g. Yager and Sugeno negations).

Criteria for selecting operators (2)

4. **Computational efficiency:** Evaluating e.g. the standard negation is usually faster than the Yager negation, which contains the power.
5. **Aggregating behavior:** When the operators combines a large number of operands, does the value tends to go to 0 (conjunction) or 1 (disjunction). The standard operators behave differently than the algebraic ones.

Bibliography

Domingos, P. and Lowd, D. (2009).
Markov Logic: An Interface Layer for Artificial Intelligence.
Morgan & Claypool.

Russell, B. (1923).
Vagueness.
In Australasian Journal of Psychology and Philosophy, page 84–92.

Wikipedia (2012).
Six degrees of separation — Wikipedia, the free encyclopedia.
[Online; accessed 15-October-2012].

Zimmermann, H.-J. (2001).
Fuzzy Set Theory and its Applications.
Springer.

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