Repeated Games

Branislav Bošanský

Artificial Intelligence Center,
Department of Computer Science,
Faculty of Electrical Engineering,
Czech Technical University in Prague

branislav.bosansky@agents.fel.cvut.cz

May 18, 2018

Repeated Games

Repeated Games are the simplest type of a dynamic game that evolves over time.

As such we can treat them as an extensive-form game (the finitely repeated case), or a stochastic game (the infinitely repeated case). However, such representations are very inefficient.

Repeated games can thus be seen as an example of a compact representation.

	C	D
C	$(1,1)$	$(-1,2)$
D	$(2,-1)$	$(0,0)$

Natural question: Is a NE of a single game the same as in the (in)finitely repeated game?

Repeated Games

Definition

Let $G' = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. An **infinitely repeated game** with discounted payoff is an extensive-form game with simultaneous moves $G^\infty = (\mathcal{N}, \mathcal{H}, \mathcal{A}, g, \delta)$, where

- $\mathcal{H} = \{\emptyset\} \cup \bigcup_{t=1}^{\infty} A^t \cup A^\infty$
- $\mathcal{S}_i : \mathcal{H} \to \mathcal{A}_i$
- $g_i(s_i, s_{-i}) = (1-\delta) \sum_{t=1}^{\infty} \delta^t \mathbb{E}_{a_i \sim s_i, a_{-i} \sim s_{-i}}(u_i(a_i, a_{-i}))$
- $\delta \in (0,1)$ is the discount factor

Repeated Games

We can define alternative utility functions in repeated games based on payoff vectors v_i^t for each:

- overtaking payoff: $\lim_{T\to\infty} \sum_{t=1}^{T} v_i^t$
- average payoff (or limit mean payoff): $\lim_{T\to\infty} \sum_{t=1}^{T} v_i^t / T$

Definition

Player i's min-max payoff is

$$\underline{v_i} = \min_{s_{-i}} \max_{s_i} g_i(s_i, s_{-i})$$

A strategy s is *individually rational* if $g_i(s) \geq \underline{v_i}$

Repeated Games

Theorem (Nash Folk Theorem)

If v_i is a feasible and an individually rational payoff, then there exists a discount factor $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$, there is a Nash equilibrium of G with payoff v_i.

Proof.

If v_i is feasible then there exist a strategy s such that $g_i(s) = v_i$ and let m_{-i} be the minmax strategy of other players to reach value $\underline{v_i}$ for player i. Let consider the following strategy:

1. play according to s_i as long as no one deviates
2. let $\overline{v_i}$ be the maximum value player i can get by a deviation in step t

$$(1-\delta)[v_i + \delta v_i + \ldots + \delta^t \overline{v_i} + \delta^{t+1} \underline{v_i} + \ldots] \leq$$
$$\leq (1-\delta)[v_i + \delta v_i + \ldots + \delta^t v_i + \quad \delta^{t+1} v_i + \ldots]$$

Repeated Games

(Proof cont.)

By setting $\underline{\delta}$ sufficiently large approaching 1 the above inequality holds. □

The Nash folk theorem says that essentially anything goes as a Nash equilibrium payoff in a discounted repeated game.

The players threat by playing *grim trigger* strategies, however, the threats might be considered non-credible:

	L	R
U	$(6, 6)$	$(0, -100)$
D	$(7, 1)$	$(0, -100)$

Repeated Games

Theorem (Perfect Folk Theorem)

Let V^* *is set of feasible and individually rational payoffs such that* $\dim V^* = |\mathcal{N}|$*. Then for any* $v \in V^*$ *such that* $v_i > \underline{v_i}$ *there exists* $\underline{\delta} < 1$ *such that for all* $\delta > \underline{\delta}$*, there is a Subgame Perfect Equilibrium of* G *with payoff* v_i*.*

Repeated Games

Proof (sketch).

Consider an outcome $v \in V^*$ that is reached via strategy profile $s \in \mathcal{S}$ such that $g_i(s) = v_i$. Now:

- Choose a feasible outcome $v' \in V^*$ such that $v'_i < v_i$ for all $i \in \mathcal{N}$.
- Choose T such that $\max_a g_i(a) + T\underline{v_i} < \min_a g_i(a) + Tv'_i$
- Choose $\varepsilon > 0$ and let

$$v^i(\varepsilon) = (v'_1 + \varepsilon, \ldots, v'_{i-1} + \varepsilon, v'_i, v'_{i+1} + \varepsilon, \ldots, v'_{|\mathcal{N}|} + \varepsilon)$$

- Let s^i be the strategy that achieves $v^i(\varepsilon)$ and m^i the minmax strategy to punish player i.

Repeated Games

Proof (sketch cont.)

Now the following strategy achieves a SPE:

- I Play according to s as long as no one deviates. If j deviates, go to strategy II_j.
- II_j Play according to m^j for T periods then go to III_j if no one deviates. If k deviates, go to II_k.
- III_j Play according to s^j as long as no one deviates. If k deviates, go to II_k.

□

Why do we need the full dimensionality assumption?

Repeated Games

Strategies in repeated games can be represented as machines (or automata).

Definition

A *machine* (or an automaton) for player i is a tuple $(Q_i, q_i^0, f_i, \tau_i)$, where

- Q_i is the set of states
- q_i^0 is the initial state
- f_i is the output function that assigns an action to each state $f_i : Q_i \to \mathcal{A}_i$
- τ_i is the transition function that assigns a state to every pair of current state and a played action profile $\tau_i : Q_i \times \mathcal{A} \to Q_i$.

Repeated Games

How does the situation change in the finitely repeated games?

There is a known horizon (say T turns).

What the players have to play in the last turn?

How this affects the set of Nash equilibria and folk theorems?

Repeated Games

Theorem (Nash Folk theorem for finitely repeated games)

If $G = (\mathcal{N}, \mathcal{A}, u_i)$ has a Nash equilibrium s^* in which the payoff of every player i exceeds his minmax payoff $\underline{v_i}$ then for any strictly individually rational outcome s' of G and any $\epsilon > 0$ there exists an integer T' such that if $T > T'$ the T-period repeated game of G has a Nash equilibrium in which the payoff of each player i is within of ϵ of $u_i(s)$.

How do the machines look like in this case?

Repeated Games

How about Subgame Perfect Equilibrium in finitely repeated games?

	C	D	E
C	$(3,3)$	$(0,4)$	$(0,0)$
D	$(4,0)$	$(1,1)$	$(0,0)$
E	$(0,0)$	$(0,0)$	$(\frac{1}{2},\frac{1}{2})$

Extending Repeated Games

If the machines and repeated games are seen as compact representation of strategies in sequential games, we can reason about modifications of strategy representation beyond mixed/behavioral strategies.

Consider an EFG with imperfect recall that is created as an abstraction (or basically any dynamic game) and a strategy can be represented as a machine, where mixed and behavioral strategies are two extremes.

Or we can seek an optimal machine to commit to in repeated games [2].

Algorithmic Rationality: Game Theory with Costly Computation

Joseph Halpern and Rafael Pass introduced game-theoretic framework for reasoning about strategic agents performing possibly costly computation.

Costly computations.

Consider the following game. You are given a random odd n-bit number x and you are supposed to decide whether x is prime or composite. If you guess correctly you receive \$2, if you guess incorrectly you instead have to pay a penalty of \$1000. Additionally you have the choice of "playing safe" by giving up, in which case you receive \$1.

- traditional game theory (computation is considered "costless") – compute whether x is prime or composite and output the correct answer; this is the only Nash equilibrium of the one-person game, no matter what n (the size of the prime) is;
- when n grows larger most people would probably decide to "play safe"; eventually the cost of computing the answer (e.g., by buying powerful enough computers) outweighs the possible gain of \$1.

Algorithmic Rationality: Game Theory with Costly Computation

Costly computations can cause non-existence of Nash equilibria:

Consider rock-paper-scissors game and machines that play this game. Suppose that we take the complexity of a deterministic strategy to be 1, and the complexity of a strategy that uses randomization to be 2. The utility is reward in the game (from the set $\{-1, 0, 1\}$) minus the cost for complexity.

From any randomized strategy, a player wants to deviate to a pure strategy, but there is no pure equilibrium stable strategy.

References I

(besides the books)

[1] M. Osborne and A. Rubinstein, *A course in game theory*.
MIT press, 1994.

[2] S. Zuo and P. Tang, "Optimal Machine Strategy to Commit to in Two-Person Repeated Games," in *In Proceedings of AAAI Conference on Artificial Intelligence (AAAI)*, 2015.