Computing Stackelberg Equilibrium

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Stackelberg Equilibrium

Players have different roles in the Stackelberg solution concept:

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

$$\arg\max_{\sigma \in \Sigma; \forall i \in \mathcal{N} \setminus \{1\} \sigma_i \in BR_i(\sigma_{-i})} u_1(\sigma)$$

There may be multiple Nash equilibria

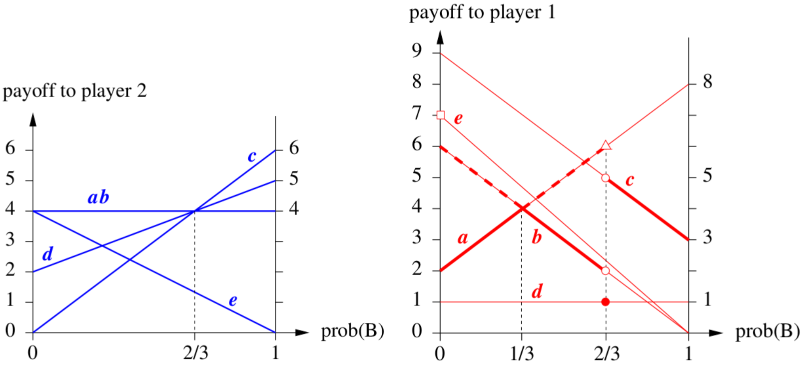
The followers need to break ties in case there are multiple NE:

- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),
- *Weak SE* – the followers select such NE that minimizes the outcome of the leader.

Exact Weak Stackelberg equilibrium does not have to exist.

$1 \setminus 2$	a	b	c	d	e
U	$(2, 4)$	$(6, 4)$	$(9, 0)$	$(1, 2)$	$(7, 4)$
D	$(8, 4)$	$(0, 4)$	$(3, 6)$	$(1, 5)$	$(0, 0)$

There may be multiple Nash equilibria



[1]

[1] Figure from [8].

Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower.

Baseline polynomial algorithm requires solving $|\mathcal{S}_2|$ linear programs:

$$\max_{\sigma_1 \in \Sigma_1} \sum_{s_1 \in \mathcal{S}_1} \sigma_1(s_1) u_1(s_1, s_2)$$

$$\sum_{s_1 \in \mathcal{S}_1} \sigma_1(s_1) u_2(s_1, s_2) \geq \sum_{s_1 \in \mathcal{S}_1} \sigma_1(s_1) u_2(s_1, s_2') \quad \forall s_2' \in \mathcal{S}_2$$

$$\sum_{s_1 \in \mathcal{S}_1} \sigma_1(s_1) = 1$$

one for each $s_2 \in \mathcal{S}_2$ assuming s_2 is the best response of the follower.

Computing a Stackelberg equilibrium in NFGs

We can reformulate the program as a mixed-integer linear program (MILP) that is a basis for the hard cases (e.g., computing a SE in Bayesian games):

$$\max_{\sigma \in \Sigma, y \in \{0,1\}^{|\mathcal{S}_2|}} \sum_{s \in \mathcal{S}} \sigma(s_1, s_2) u_1(s_1, s_2)$$

$$0 \leq \sigma(s_1, s_2) \leq y(s_2) \qquad \forall s_1, s_2 \in \mathcal{S}_{1,2}$$

$$\sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) u_2(s_1, s_2) \geq \sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) u_2(s_1, s_2') \quad \forall s_2' \in \mathcal{S}_2$$

$$\sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) = 1$$

$$\sum_{s_2 \in \mathcal{S}_2} y(s_2) = 1$$

Computing a Stackelberg equilibrium in EFGs

The problem is typically NP-hard [5, 2]:

- two-player EFGs with chance (there exists a FPTAS for this case [2]),
- two-player EFGs with imperfect information,
- two-player EFGs with perfect information but imperfect recall (games on DAGs).

Main algorithms are based on the sequence-form LCP for computing NE:

$$v_{\mathsf{inf}_i(\sigma_i)} = s_{\sigma_i} + \sum_{I_i' \in \mathcal{I}_i : \mathsf{seq}_i(I_i') = \sigma_i} v_{I_i'} + \sum_{\sigma_{-i} \in \Sigma_{-i}} g_i(\sigma_i, \sigma_{-i}) \cdot r_{-i}(\sigma_{-i}) \qquad \forall i, \sigma_i$$

$$r_i(\sigma_i) = \sum_{a \in A(I_i)} r_i(\sigma_i a) \qquad \forall i \in \mathcal{N}\ \forall I_i \in \mathcal{I}_i,\ \sigma_i = \mathsf{seq}_i(I_i)$$

$$r_i(\emptyset) = 1 \quad 0 = r_i(\sigma_i) \cdot s_{\sigma_i} \qquad \forall i \in \mathcal{N}\ \forall \sigma_i \in \Sigma_i$$

$$0 \le r_i(\sigma_i)\ ; \qquad 0 \le s_{\sigma_i} \quad \forall i \in \mathcal{N}\ \forall \sigma_i \in \Sigma_i$$

Computing a Stackelberg equilibrium in EFGs

MILP for computing SE for two-player extensive-form game with perfect recall:

$$\max_{p,r,v,s} \sum_{z \in \mathcal{Z}} p(z) u_1(z) \mathcal{C}(z)$$

$$v_{\mathsf{inf}_2(\sigma_2)} = s_{\sigma_2} + \sum_{I' \in \mathcal{I}_2 : \mathsf{seq}_2(I') = \sigma_2} v_{I'} + \sum_{\sigma_1 \in \Sigma_1} r_1(\sigma_1) g_2(\sigma_1, \sigma_2) \qquad \forall \sigma_2 \in \Sigma_2$$

$$r_i(\emptyset) = 1 \qquad r_i(\sigma_i) = \sum_{a \in A_i(I_i)} r_i(\sigma_i a) \quad \forall i \in \mathcal{N}\ \forall I_i \in \mathcal{I}_i, \sigma_i = \mathsf{seq}_i(I_i)$$

$$0 \le s_{\sigma_2} \le (1 - r_2(\sigma_2)) \cdot M \qquad \forall \sigma_2 \in \Sigma_2$$

$$0 \le p(z) \le r_2(\mathsf{seq}_2(z)) \qquad \forall z \in \mathcal{Z}$$

$$0 \le p(z) \le r_1(\mathsf{seq}_1(z)) \qquad \forall z \in \mathcal{Z}$$

$$1 = \sum_{z \in \mathcal{Z}} p(z) \mathcal{C}(z)$$

$$r_2(\sigma_2) \in \{0, 1\} \qquad \forall \sigma_2 \in \Sigma_2$$

$$0 \le r_1(\sigma_1) \le 1 \qquad \forall \sigma_1 \in \Sigma_1$$

Stackelberg and Correlated Equilibrium

Recall the MILP program Stackelberg equilibrium and compare it to the LP for correlated equilibrium:

- we maximize the expected utility of the leader
- we restrict the joint probability distribution so that the follower plays a pure strategy
- there are no incentive constraints of the leader

We can compute a Stackelberg equilibrium if we modify an algorithm for computing an optimal correlated equilibrium.

Computing a Stackelberg equilibrium in NFGs (2)

We can reformulate the MILP program as a single LP:

$$\max_{\sigma \in \Sigma} \sum_{s \in \mathcal{S}} \sigma(s_1, s_2) u_1(s_1, s_2)$$

$$\sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) u_2(s_1, s_2) \geq \sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) u_2(s_1, s_2') \quad \forall s_2' \in \mathcal{S}_2$$

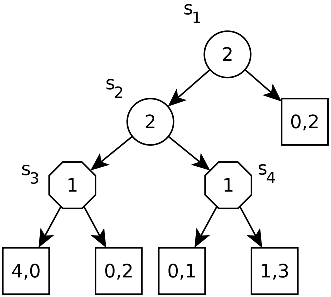
$$\sum_{s_1, s_2 \in \mathcal{S}_{1,2}} \sigma(s_1, s_2) = 1$$

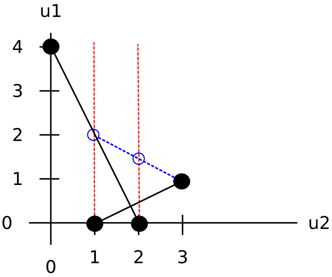
Properties:

- the objective is the same as in the MILP case (or multiple LPs) case,
- strategy σ does not necessarily corresponds to Stackelberg equilibrium (the follower can receive multiple recommendations that are best responses).

Computing a Stackelberg equilibrium in EFGs (2)

How does it work in EFGs?



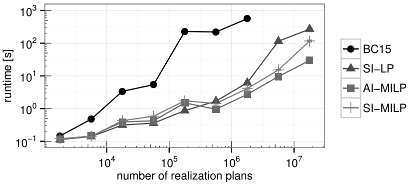


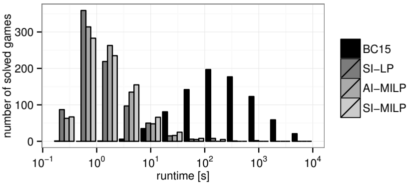
We can define a Stackelberg extension of EFCE [2] – the leader (1) controls the correlation device, (2) sends signals to the follower, (3) maximizes her expected utility.

Computing a Stackelberg equilibrium in EFGs (2)

We can follow the same steps [3]:

- consider an algorithm for computing an optimal EFCE in an EFGs
- remove the incentives constraints of the leader
- add objective to maximize the expected value of the leader
- restrict the recommendations to the follower so that only a unique action in an information set





References I

(besides the books)

[1] B. Bošanský and J. Čermák, “Sequence-Form Algorithm for Computing Stackelberg Equilibria in Extensive-Form Games,” in *AAAI Conference on Artificial Intelligence*, 2015.

[2] B. Bošanský, S. Branzei, K. A. Hansen, P. B. Miltersen, and T. B. Sørensen, “Computation of stackelberg equilibria of finite sequential games,” in *Proceedings of Web and Internet Economics: 11th International Conference (WINE)*, pp. 201–215, 2015.

[3] J. Čermák, B. Bošanský, K. Durkota, V. Lisý, and C. Kiekintveld, “Using correlated strategies for computing stackelberg equilibria in extensive-form games,” in *Proceedings of AAAI Conference on Artificial Intelligence (to appear)*, 2016.

[4] V. Conitzer and D. Korzhyk, “Commitment to Correlated Strategies,” in *Proceedings of AAAI Conference on Artificial Intelligence*, 2011.

References II

[5] J. Letchford and V. Conitzer, "Computing optimal strategies to commit to in extensive-form games," in *Proceedings of the 11th ACM conference on Electronic commerce*, (New York, NY, USA), pp. 83–92, ACM, 2010.

[6] J. Letchford, L. MacDermed, V. Conitzer, R. Parr, and C. L. Isbell, "Computing optimal strategies to commit to in stochastic games," in *AAAI*, 2012.

[7] von Stengel, Bernhard and Zamir, Shmuel, "Leadership with commitment to mixed strategies," 2004.

[8] von Stengel, Bernhard and Zamir, Shmuel, "Leadership games with convex strategy sets," 2010.