Heuristic (Informed) Search

(Where we try to choose smartly)

Recall that the ordering of FRINGE defines the search strategy

SEARCH#2

1. INSERT(initial-node,OL)
2. Repeat:
   a. If empty(OL) then return failure
   b. N ← REMOVE(OL)
   c. s ← STATE(N)
   d. If GOAL?(s) then return path or goal state
   e. For every state s' in SUCCESSORS(s)
      i. Create a new node N' as a child of N
      ii. INSERT(N',OL)

Best-First Search

- It exploits state description to estimate how "good" each search node is
- An evaluation function f maps each node N of the search tree to a real number
  $f(N) \geq 0$
  [Traditionally, f(N) is an estimated cost; so, the smaller f(N), the more promising N]
- Best-first search sorts the OL in increasing f
  [Arbitrary order is assumed among nodes with equal f]

Best-First Search

- It exploits state description to estimate how "good" each search node is
- An evaluation function f maps each node N of the search tree to a real number
  f(N) ≥ 0
  [Traditionally, f(N) is a
  more promising N]
- Best-first search sorts the OL in increasing f
  [Arbitrary order is assumed among nodes with equal f]

"Best" does not refer to the quality of the generated path
Best-first search does not generate optimal paths in general

How to construct f?

- Typically, f(N) estimates:
  - either the cost of a solution path through N

    Then f(N) = g(N) + h(N), where
    - g(N) is the cost of the path from the initial node to N
    - h(N) is an estimate of the cost of a path from N to a goal node
  - or the cost of a path from N to a goal node

    Then f(N) = h(N)   Greedy best-search

- But there are no limitations on f.
  Any function of your choice is acceptable.
  But will it help the search algorithm?

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Heuristic function

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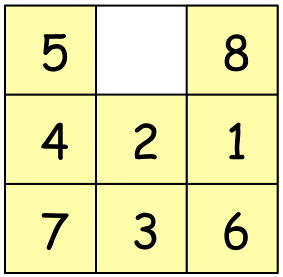
Heuristic Function

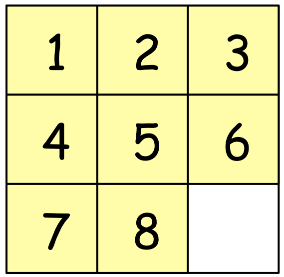
- The heuristic function $h(N) \geq 0$ estimates the cost to go from STATE(N) to a goal state

  Its value is *independent of the current search tree*; it depends only on STATE(N) and the goal test GOAL?

- Example:



STATE(N)



Goal state

$h_1(N)$ = number of misplaced numbered tiles = 6

[Why is it an estimate of the distance to the goal?]

Other Examples

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

Other Examples

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Other Examples

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STATE(N)

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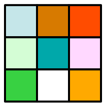
Goal state

- $h_1(N)$ = number of misplaced numbered tiles = 6
- $h_2(N)$ = sum of the (Manhattan) distance of every numbered tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
- $h_3(N)$ = sum of permutation inversions
  $= n_5 + n_8 + n_4 + n_2 + n_1 + n_7 + n_3 + n_6$
  $= 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0$
  $= 16$

8-Puzzle

f(N) = h(N) = number of misplaced numbered tiles



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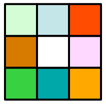

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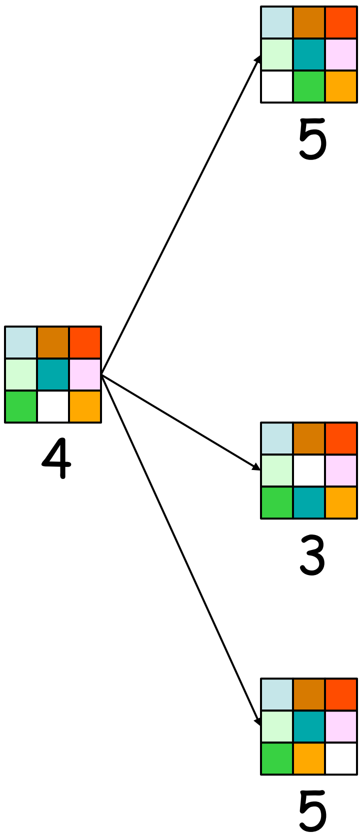

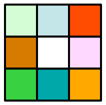

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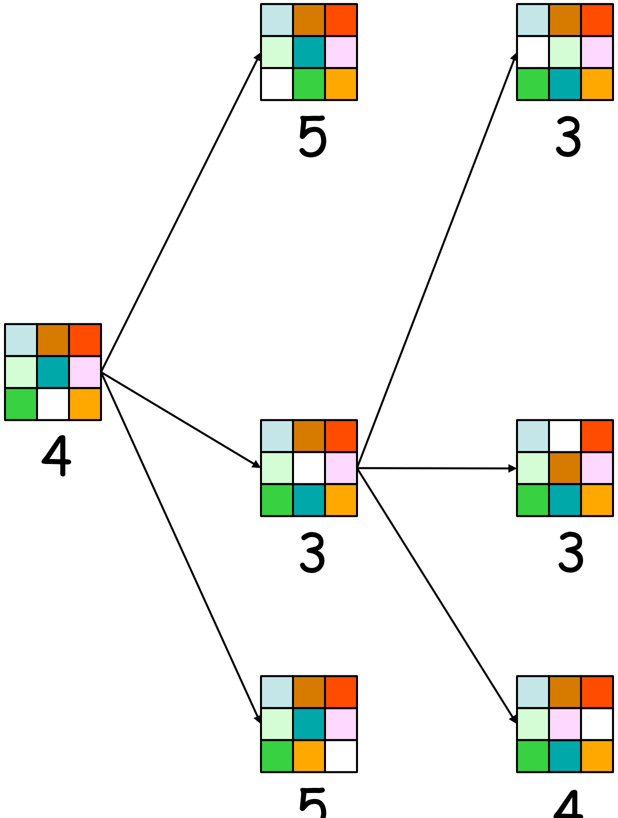

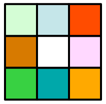

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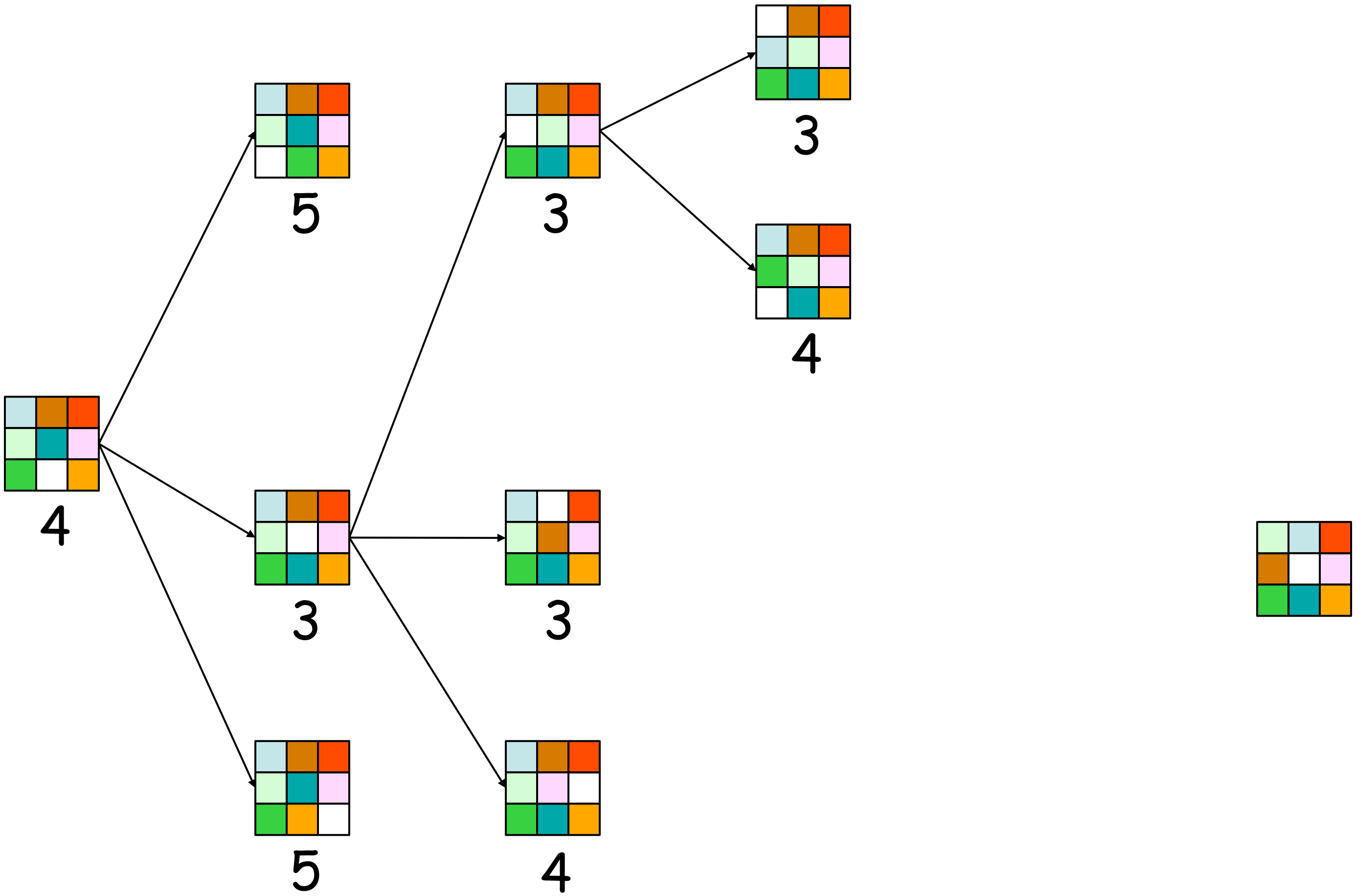

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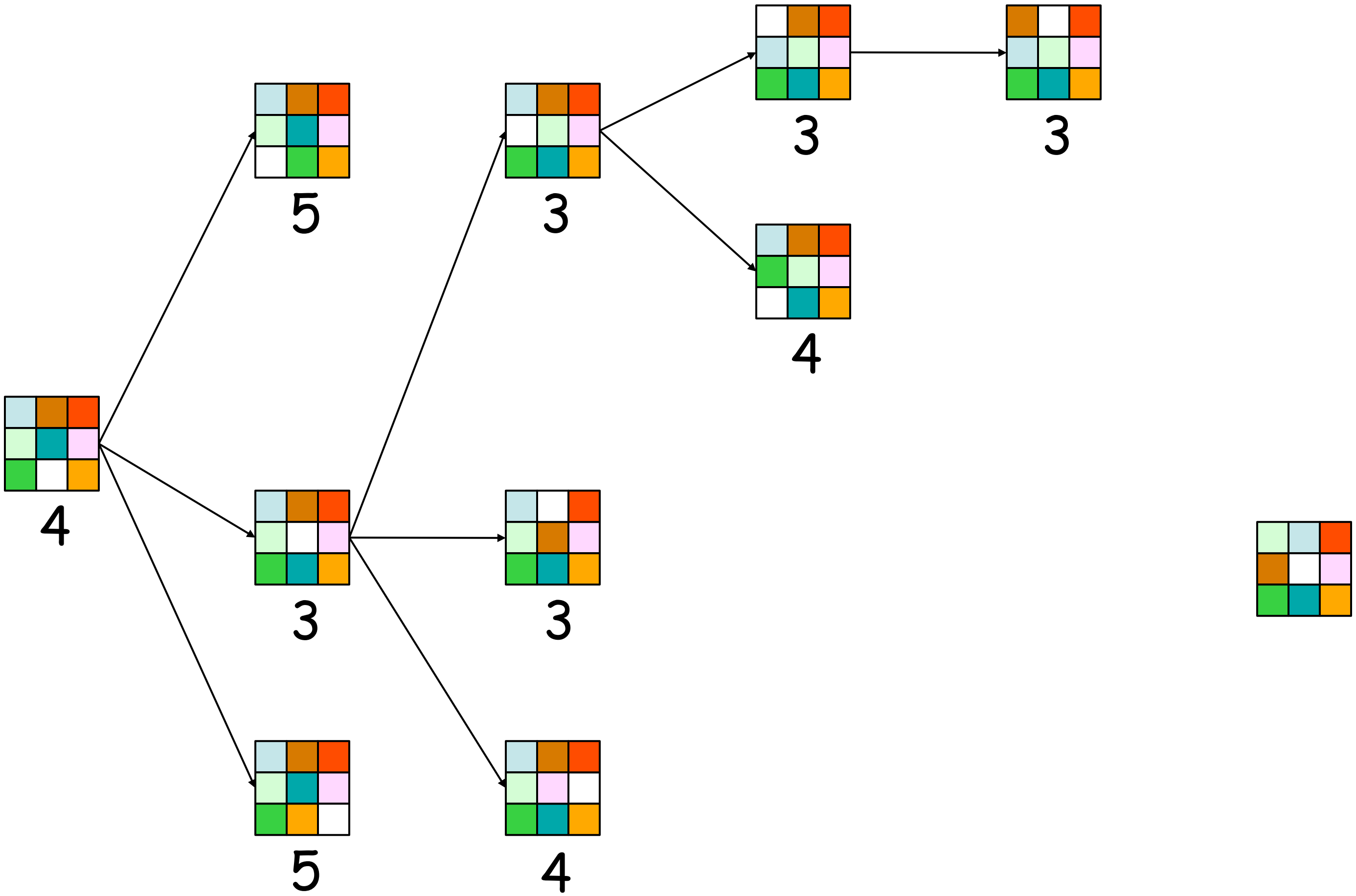

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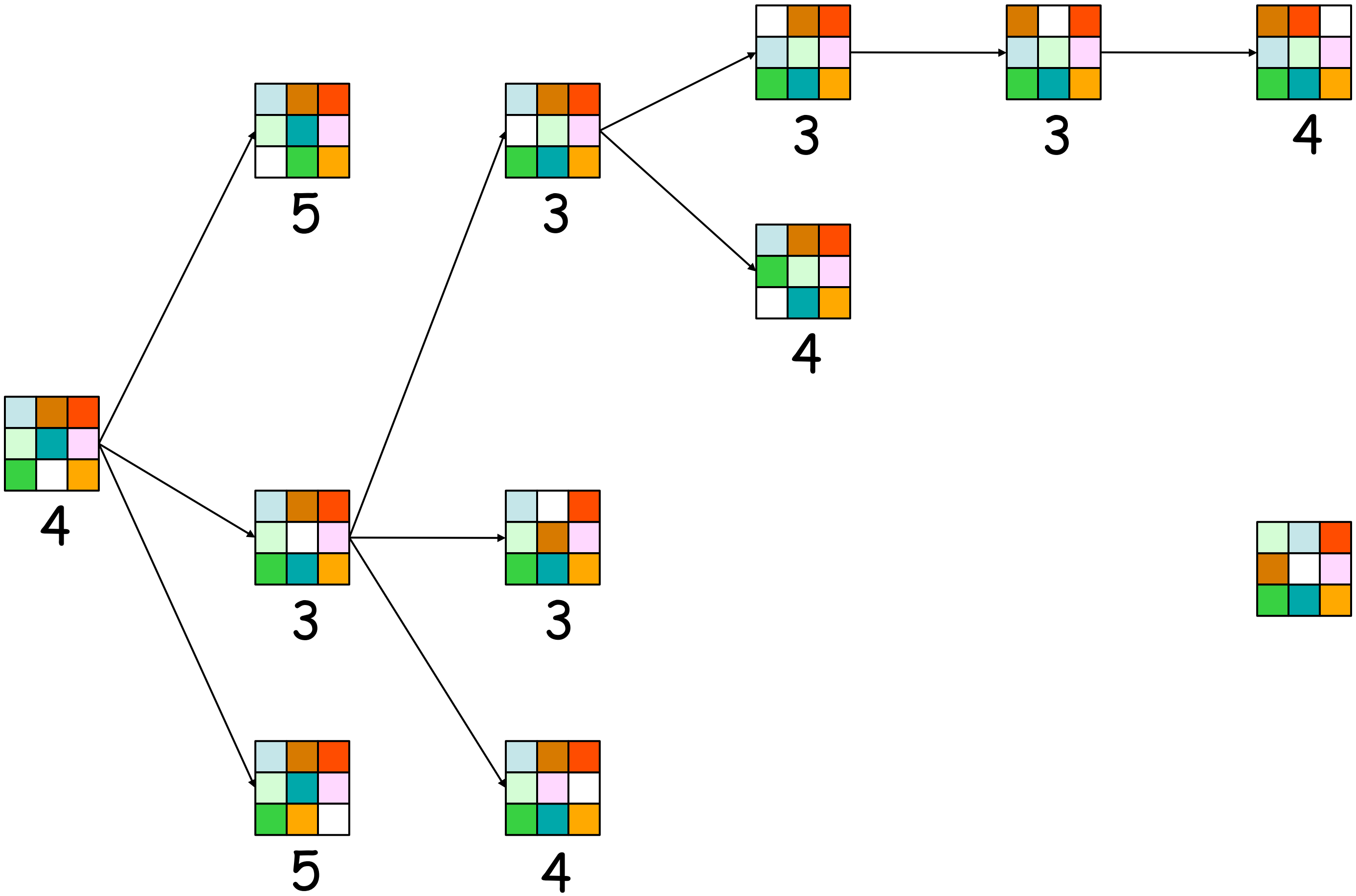

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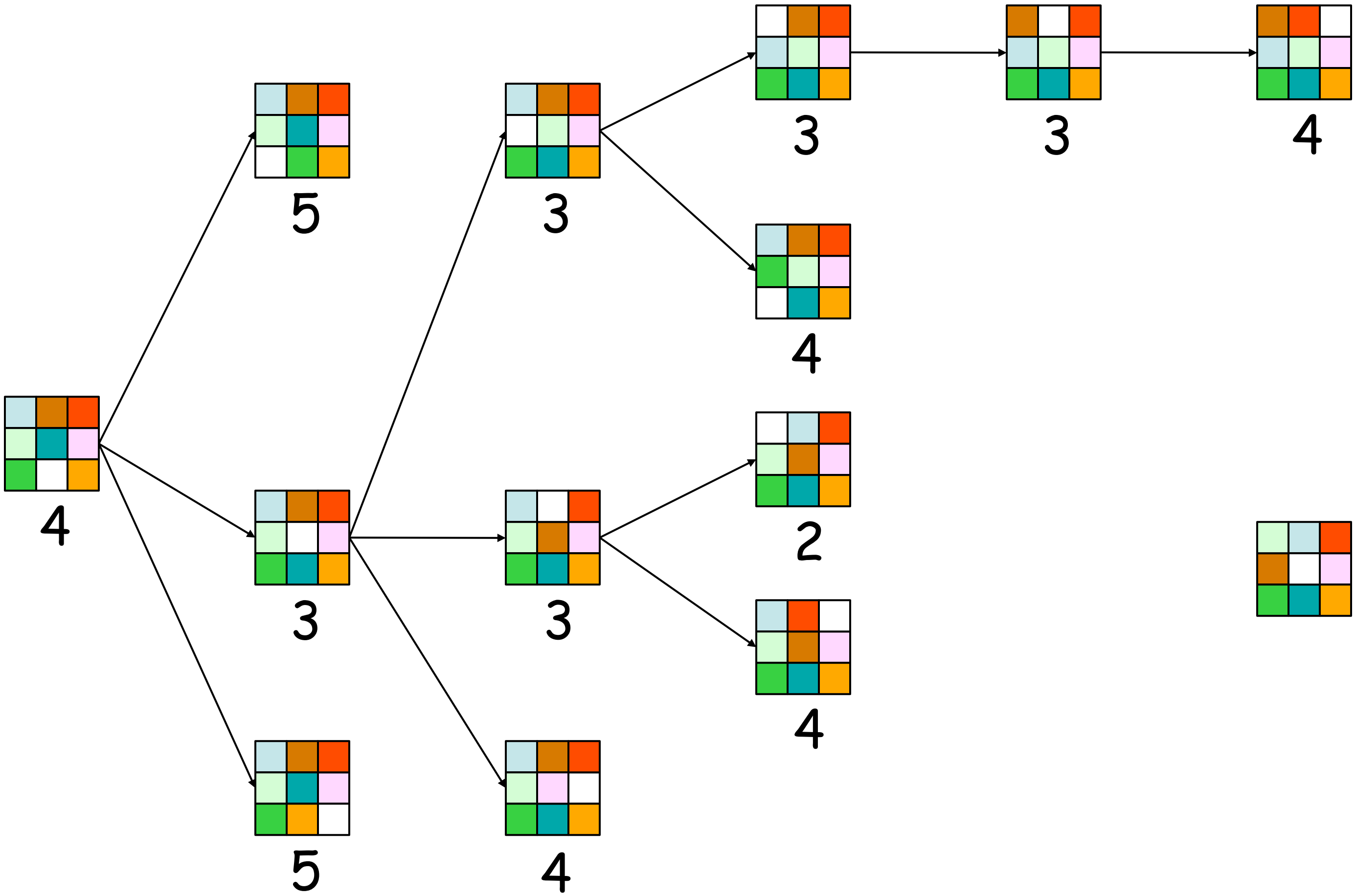

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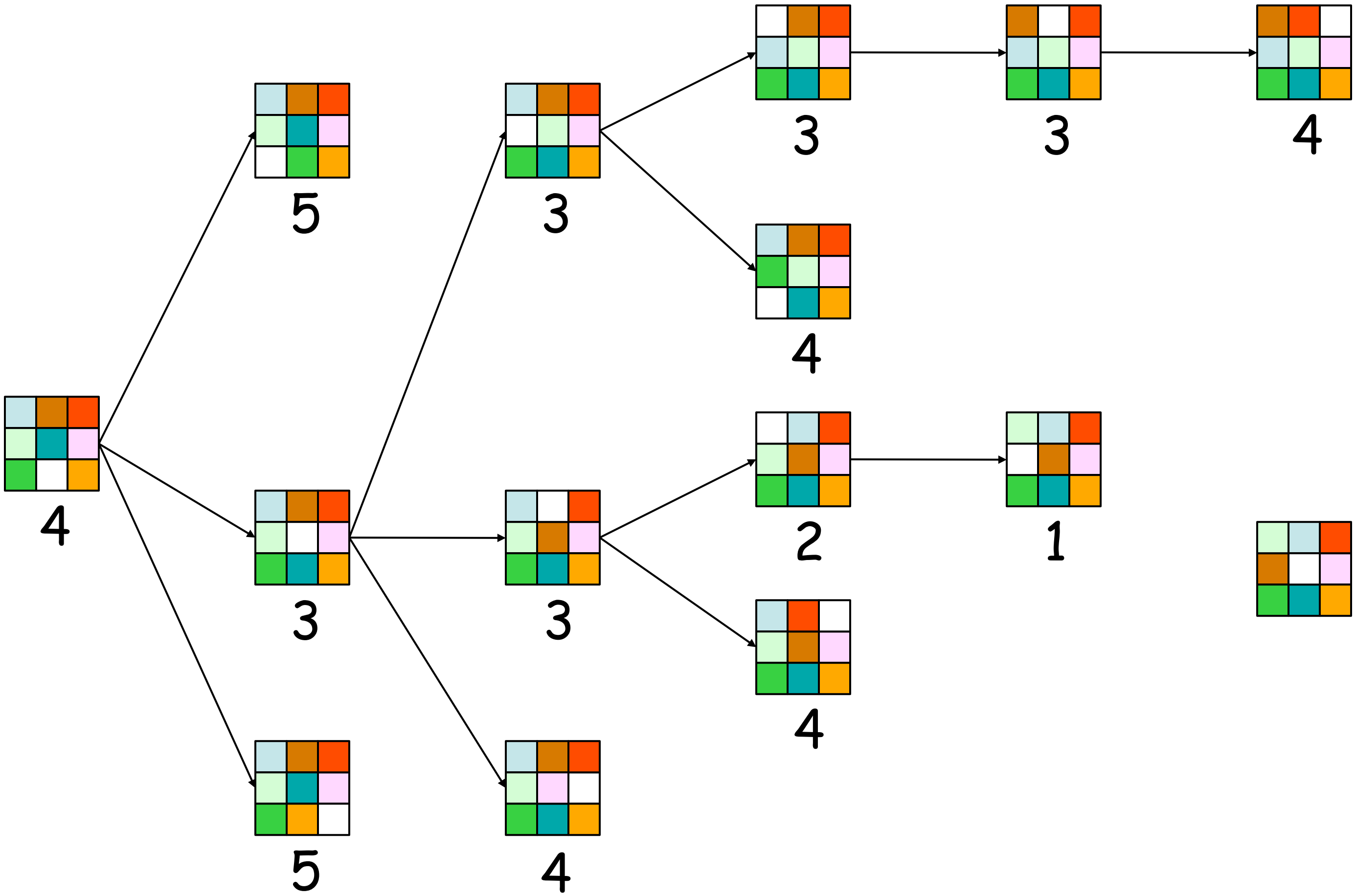

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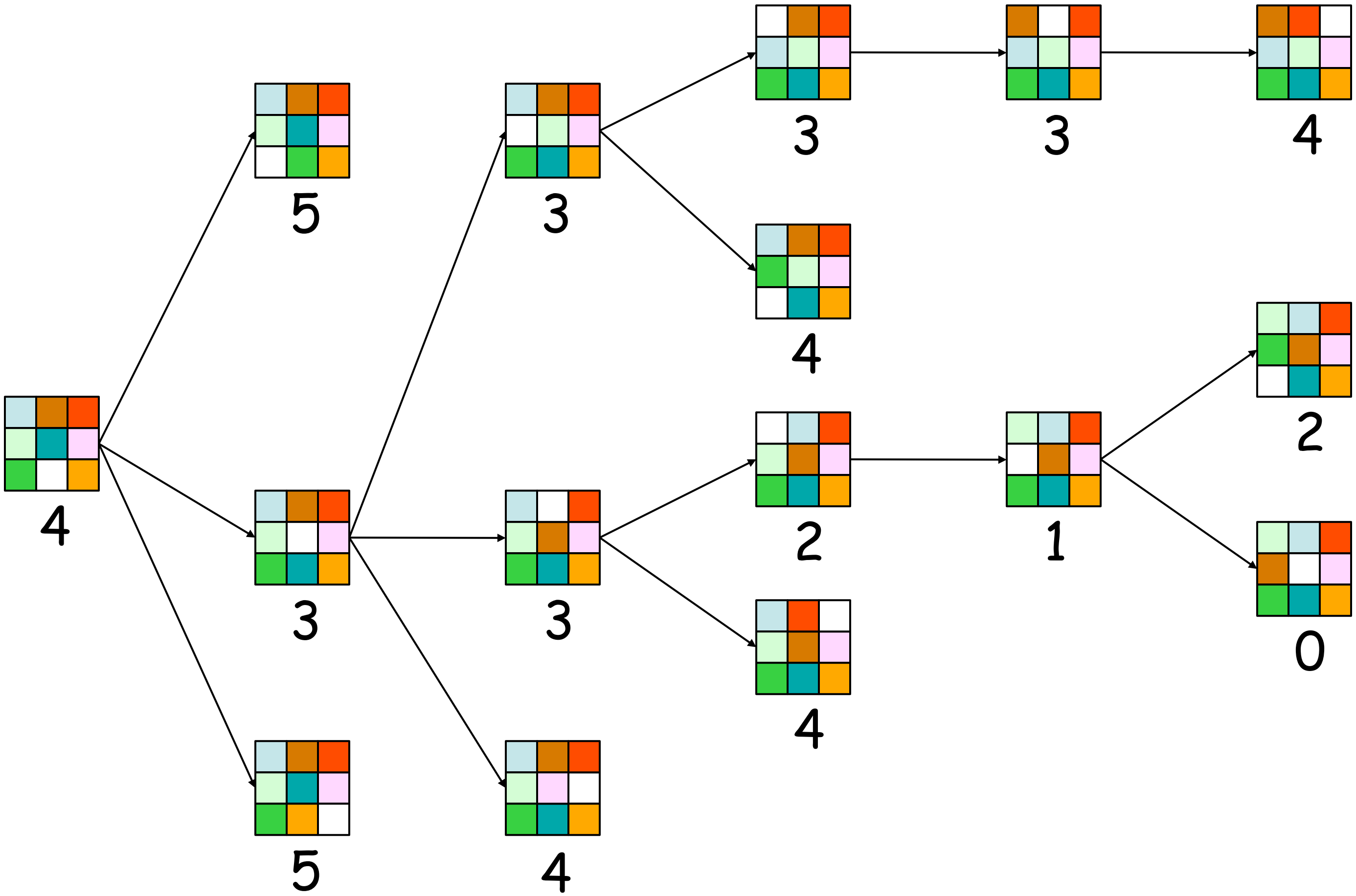

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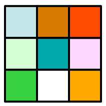
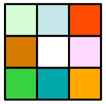


0+4



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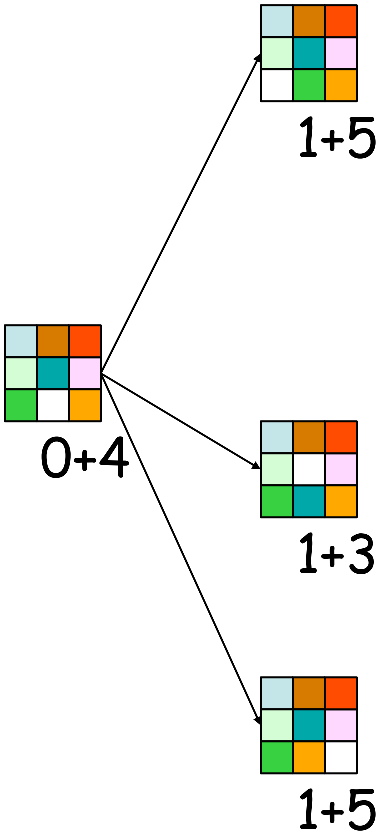

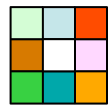

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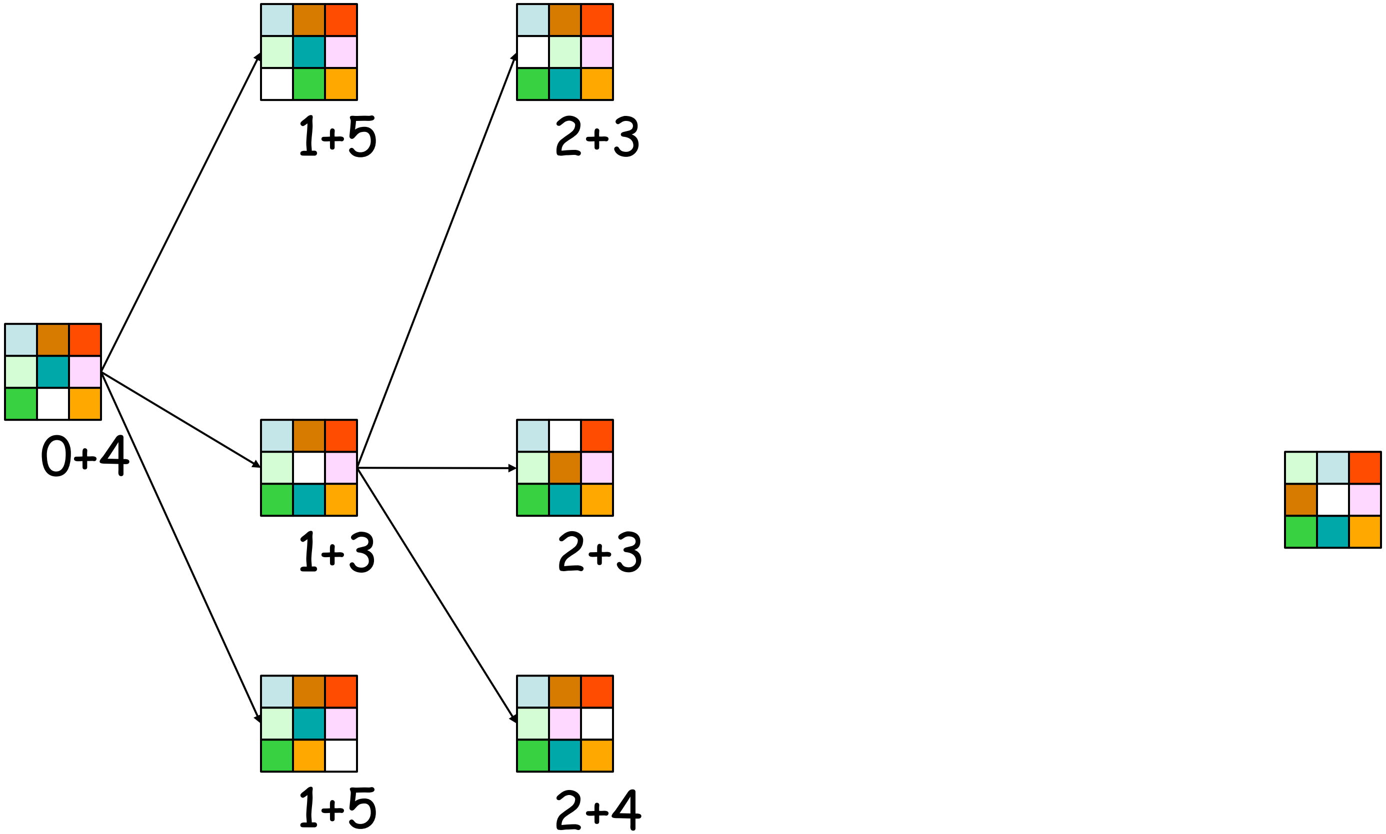

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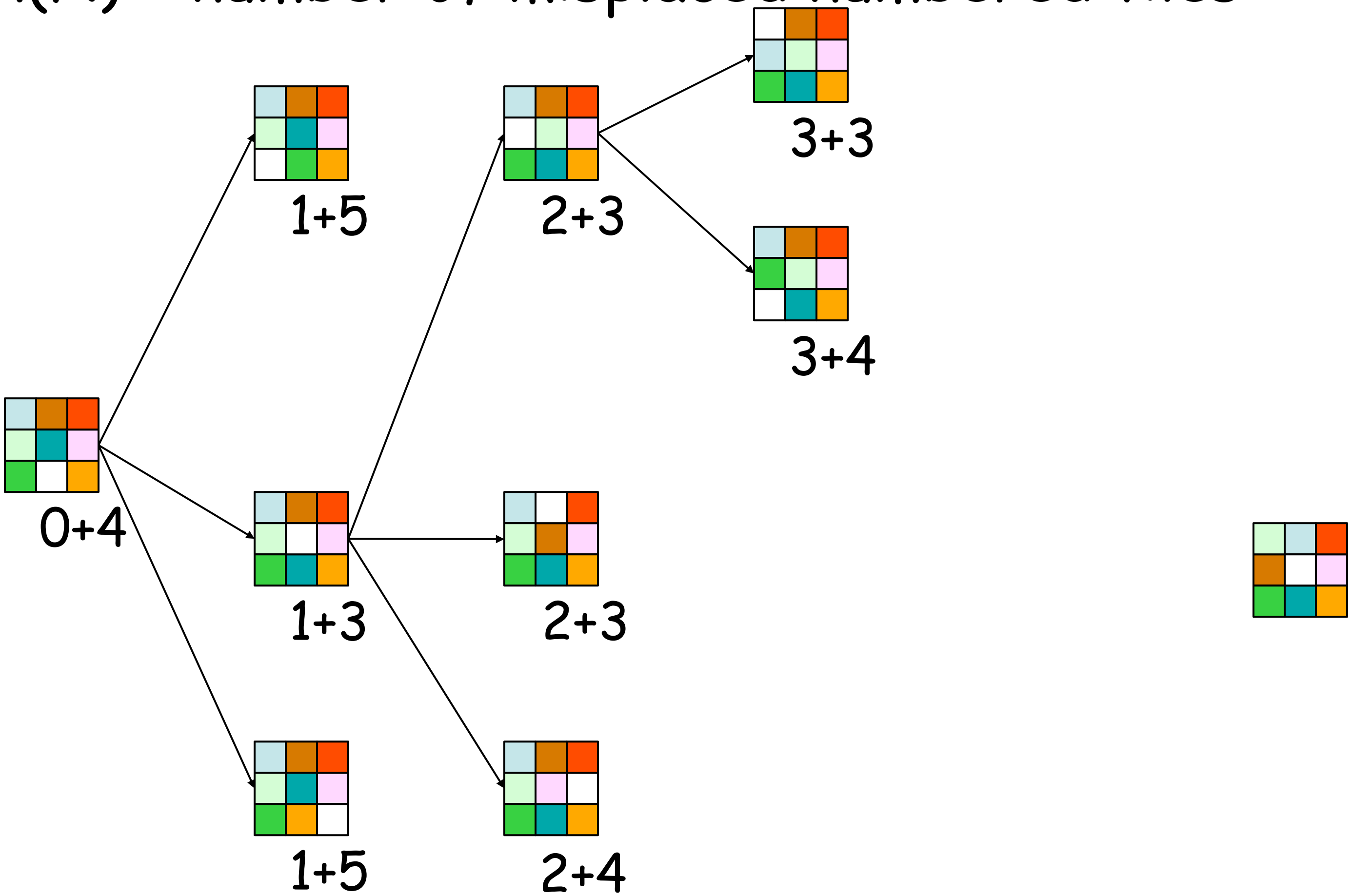

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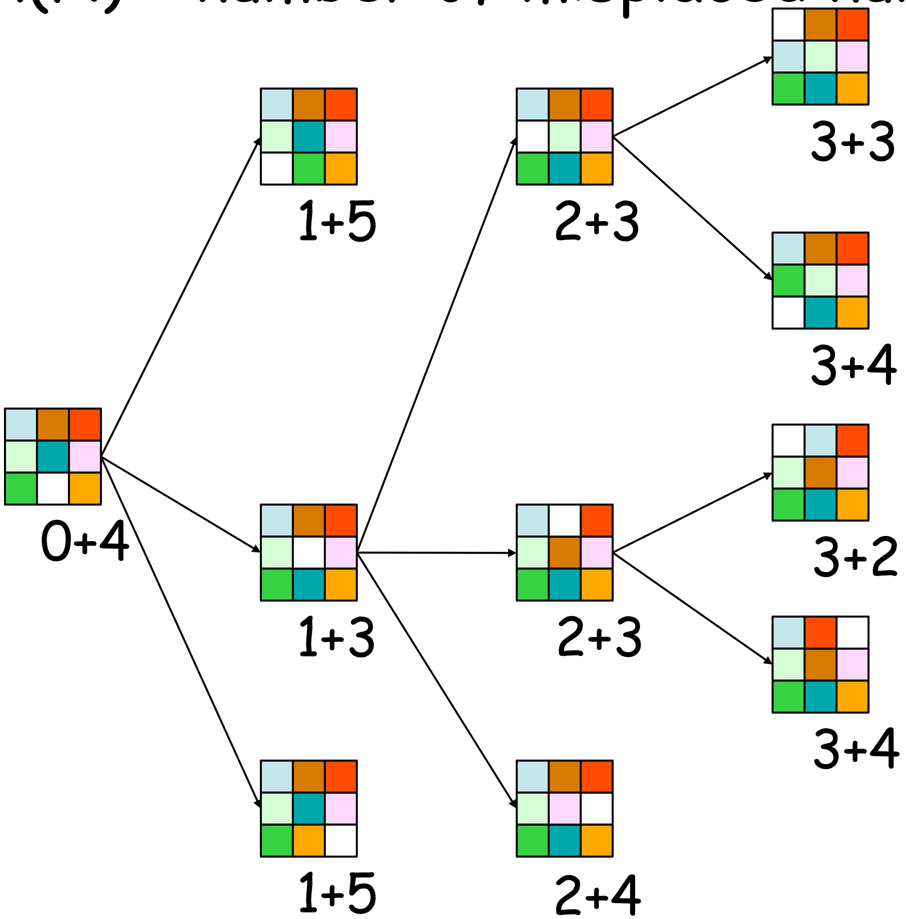

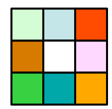

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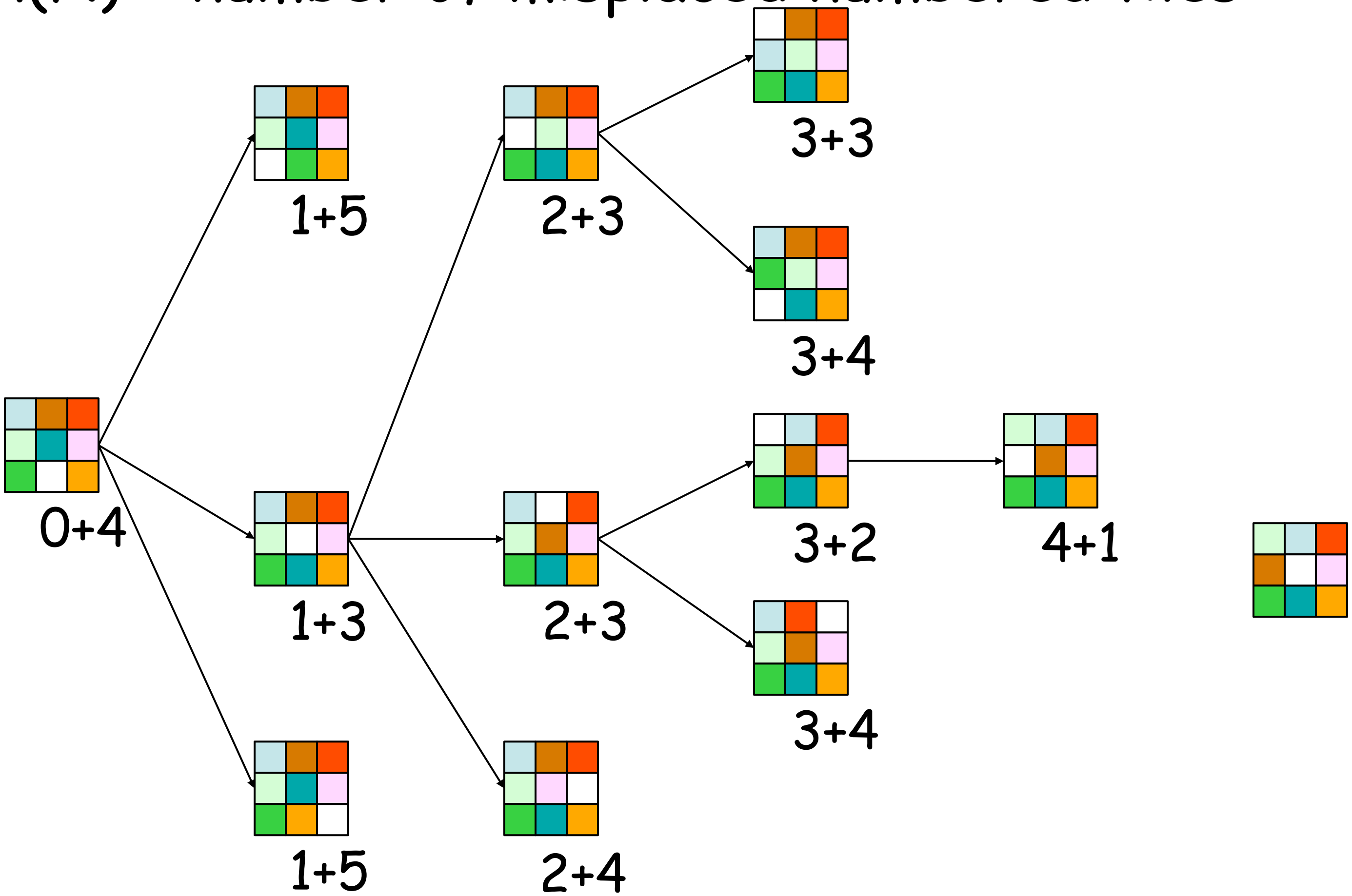

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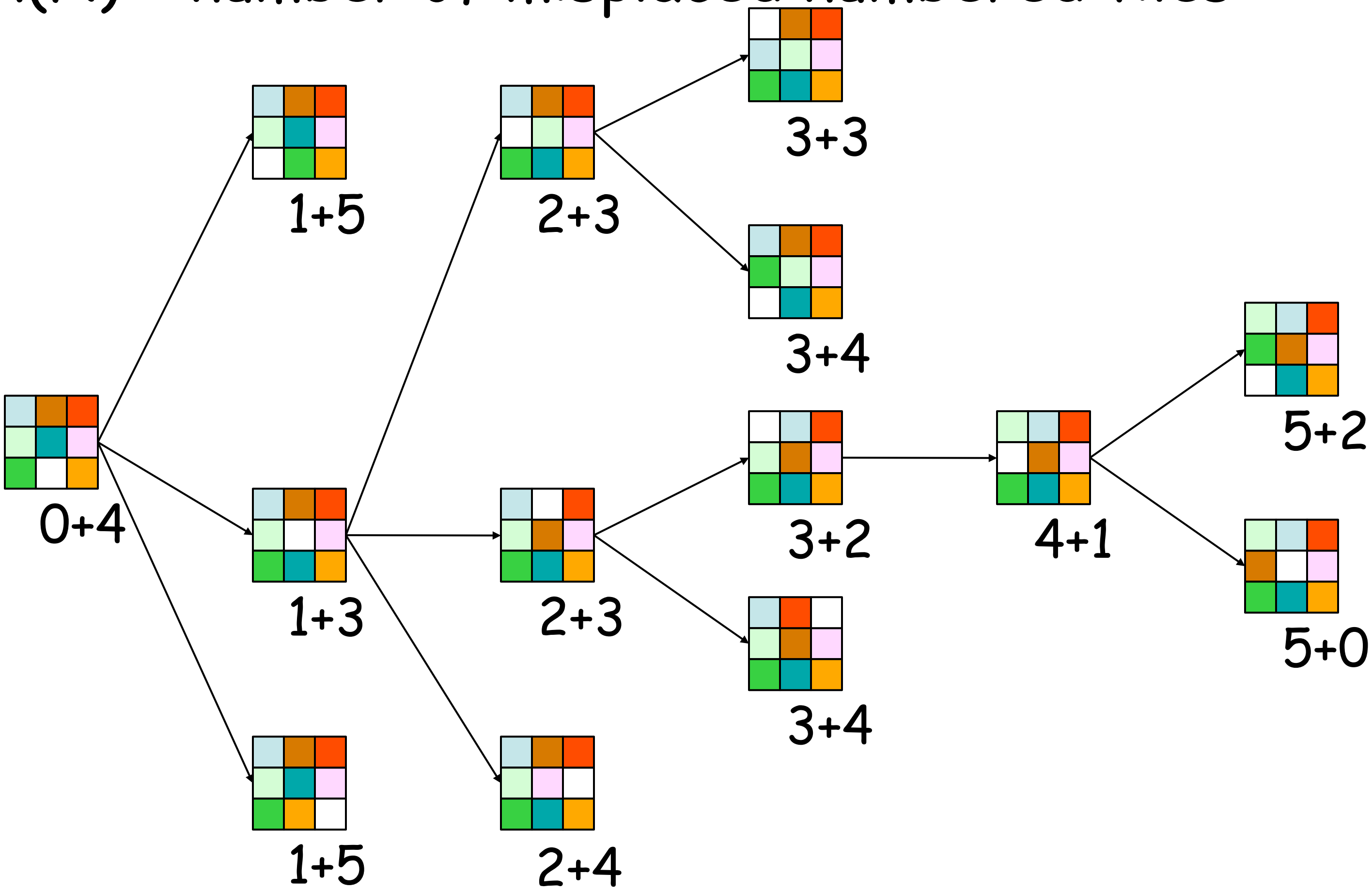

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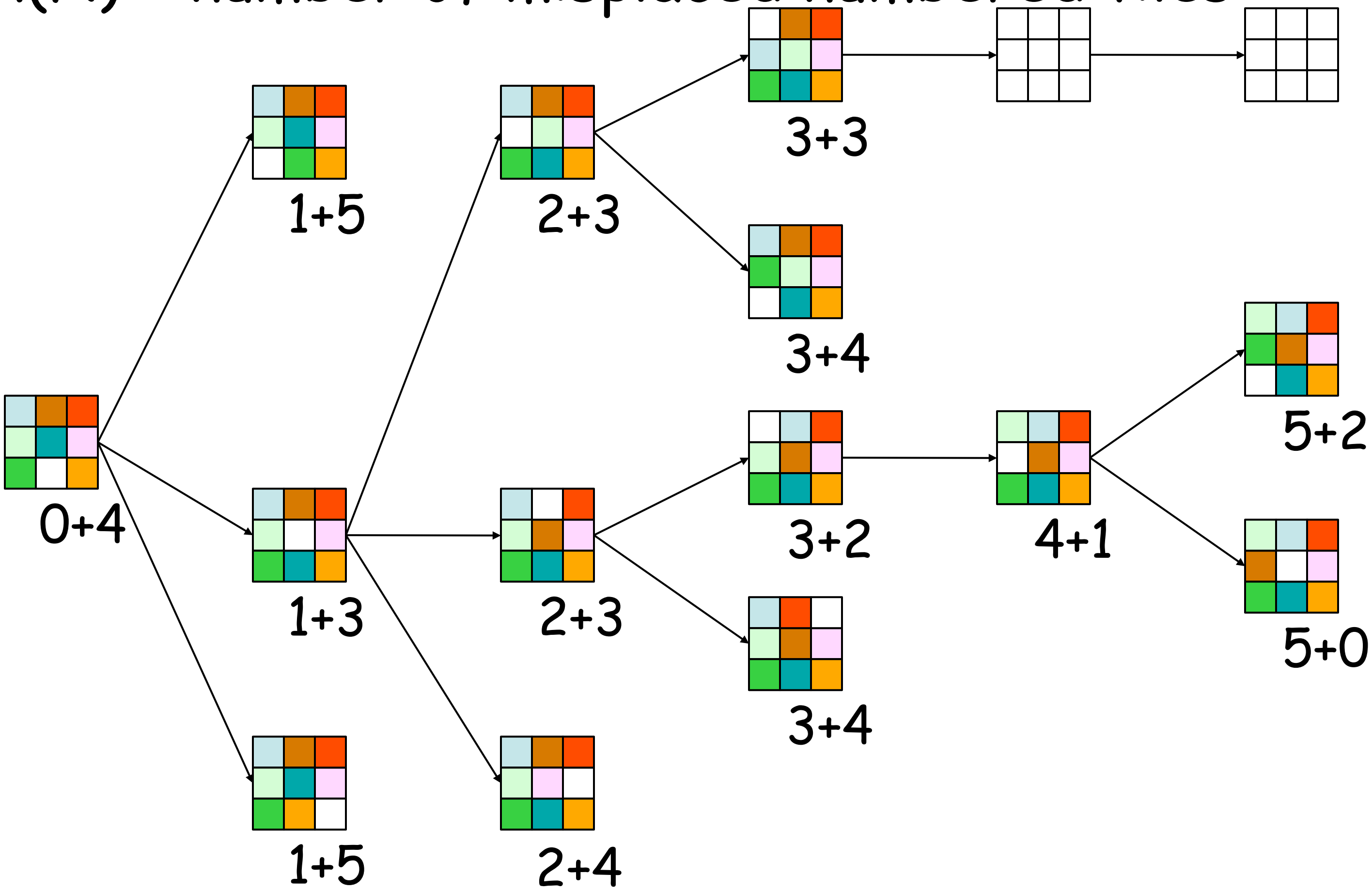

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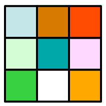
$f(N) = h(N) = \Sigma$ distances of numbered tiles to their goals



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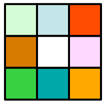

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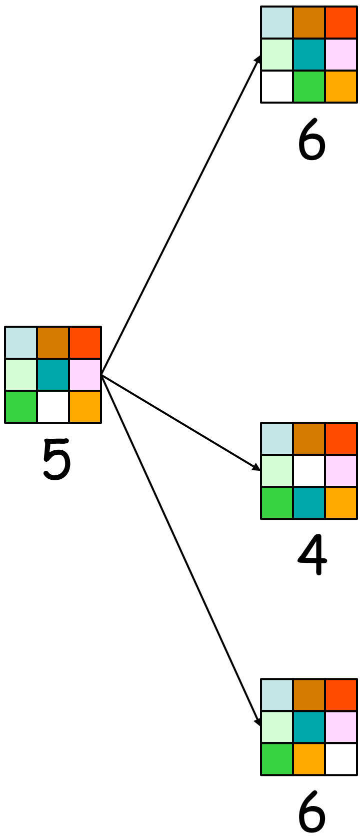

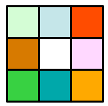

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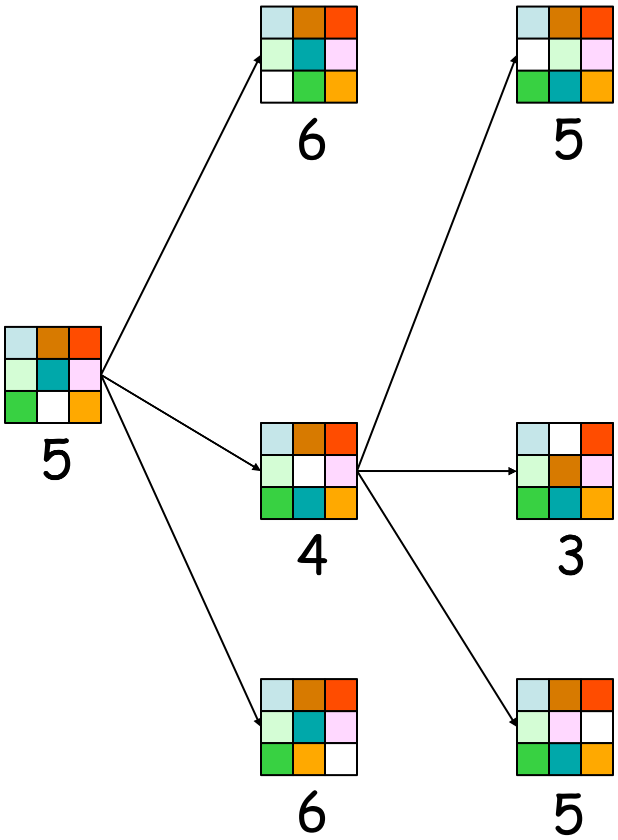

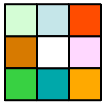

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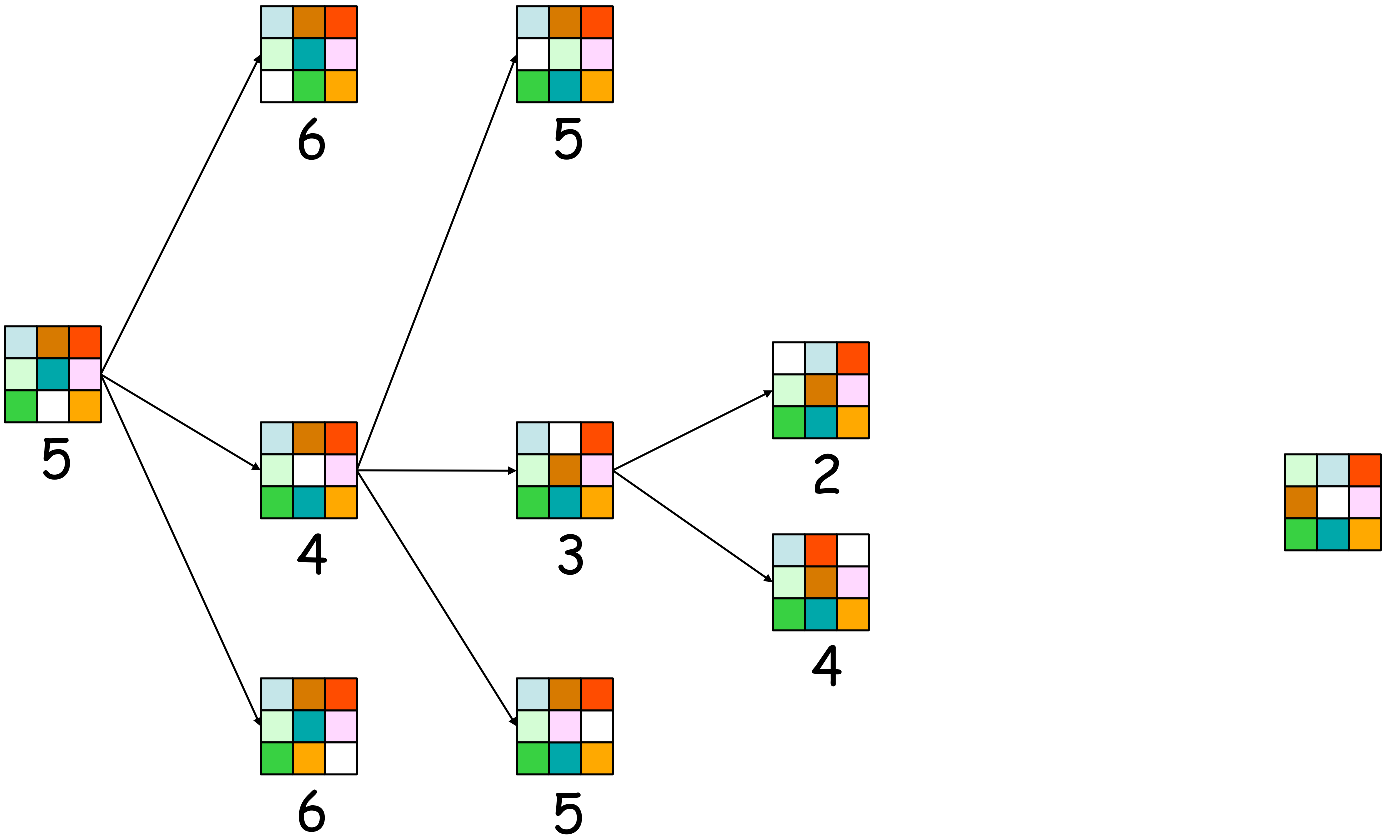

Sunday, February 26, 12

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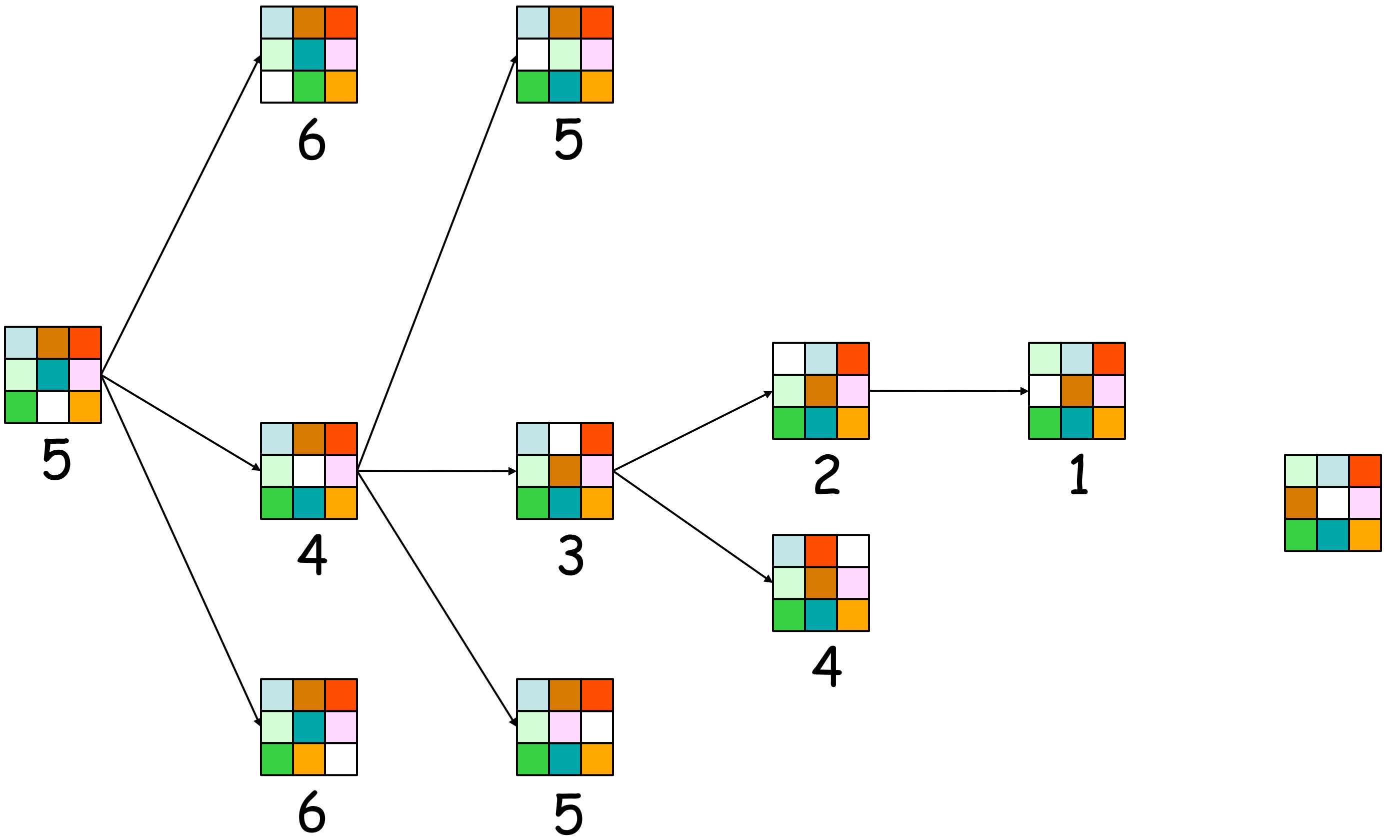

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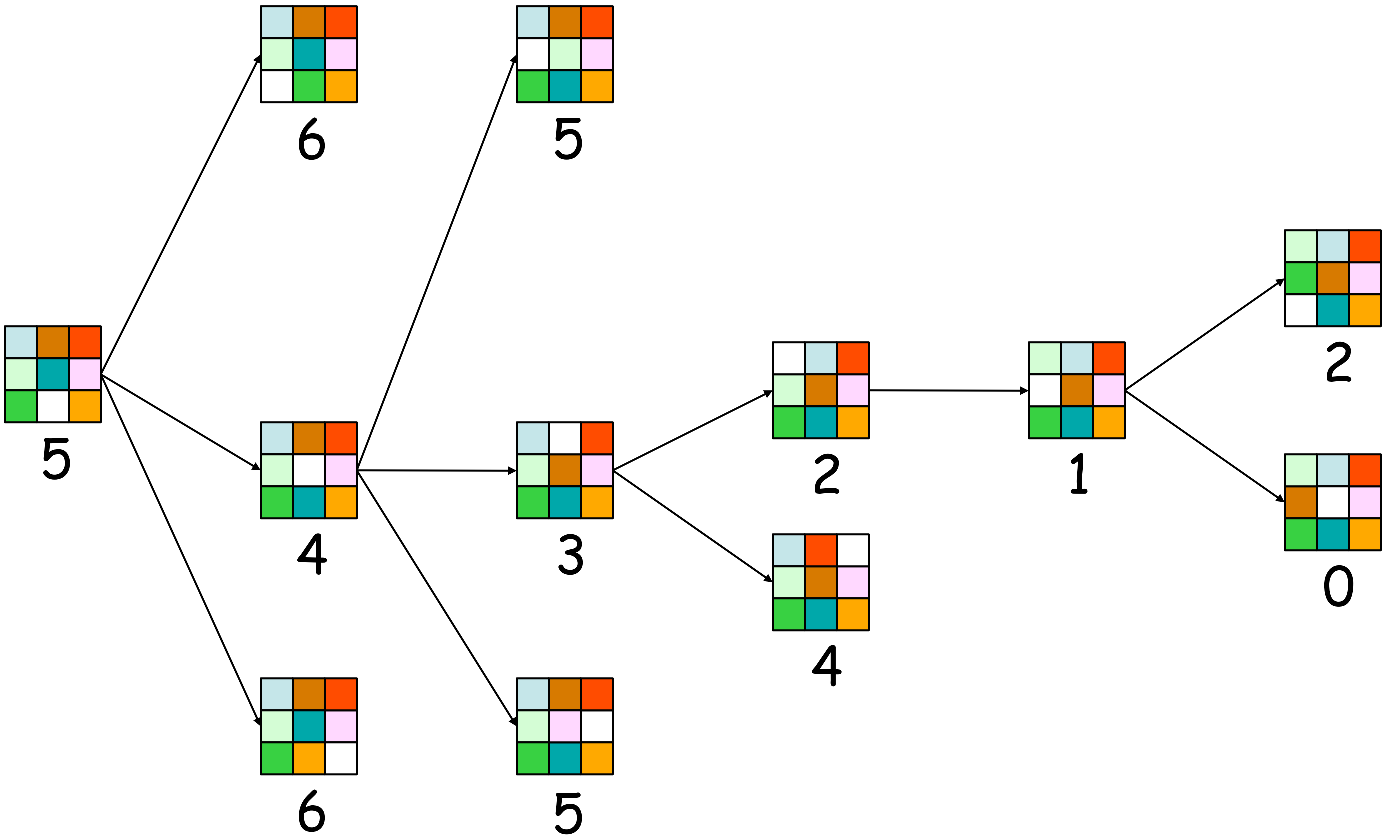

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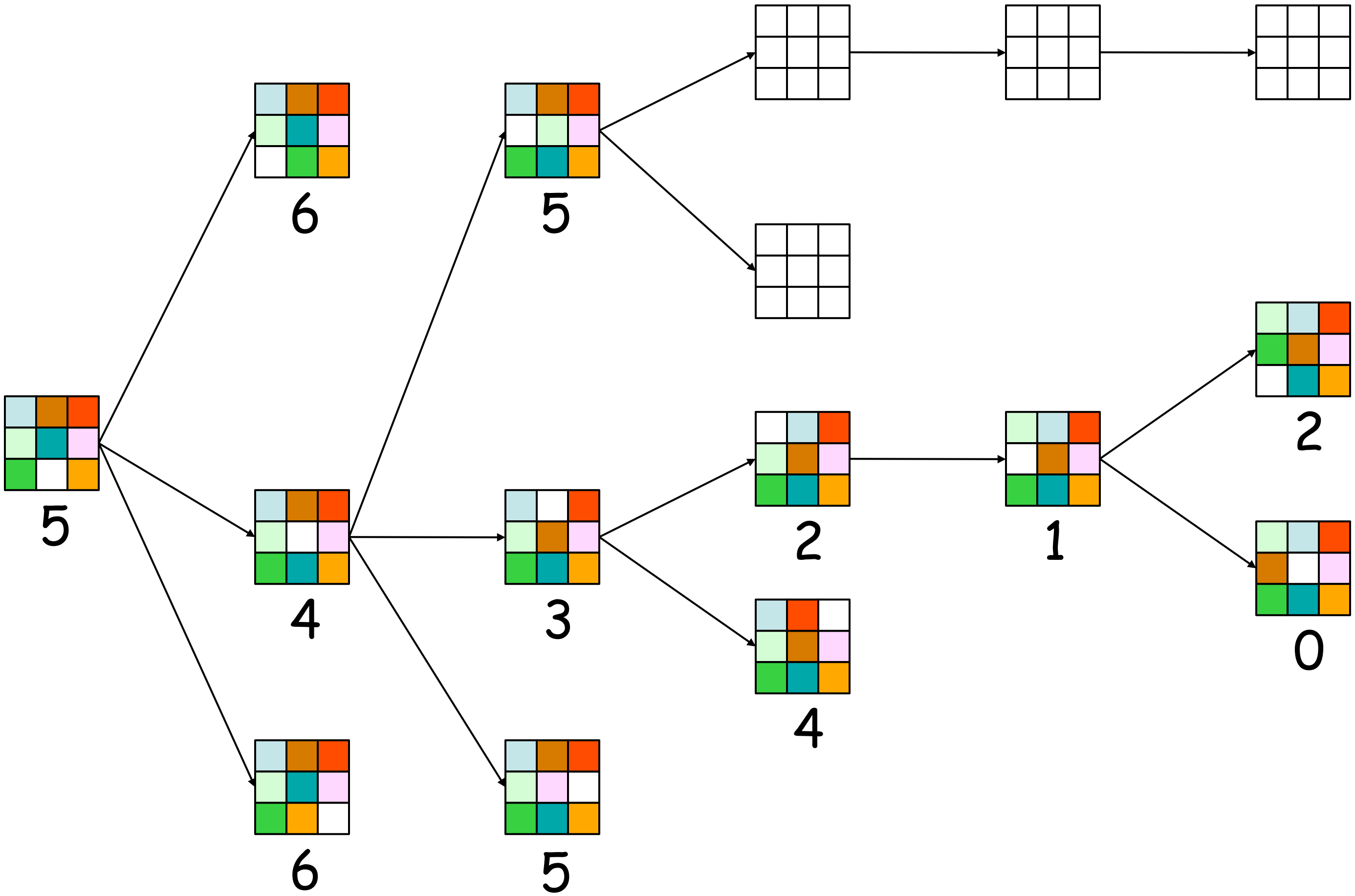
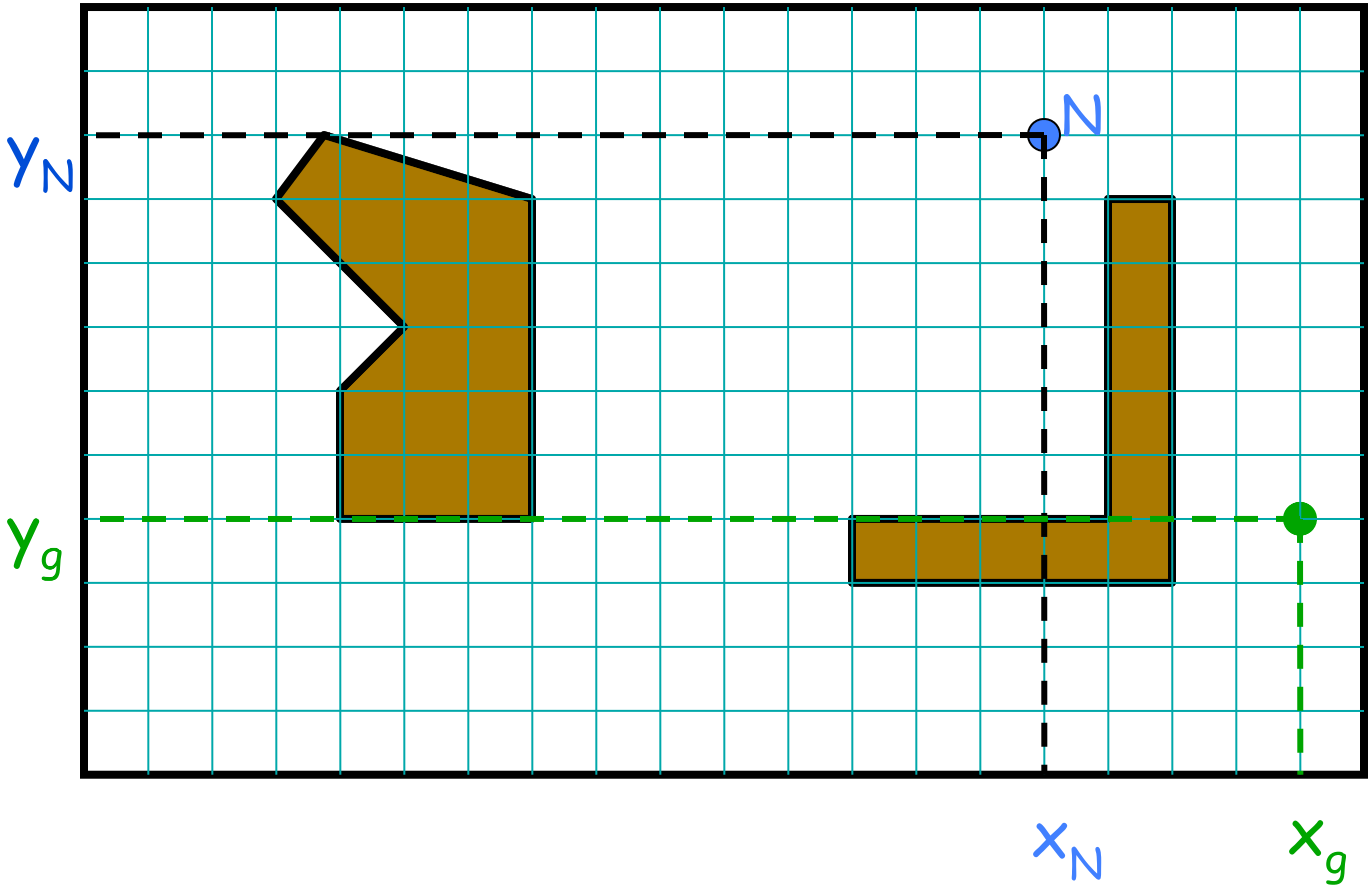


Robot Navigation



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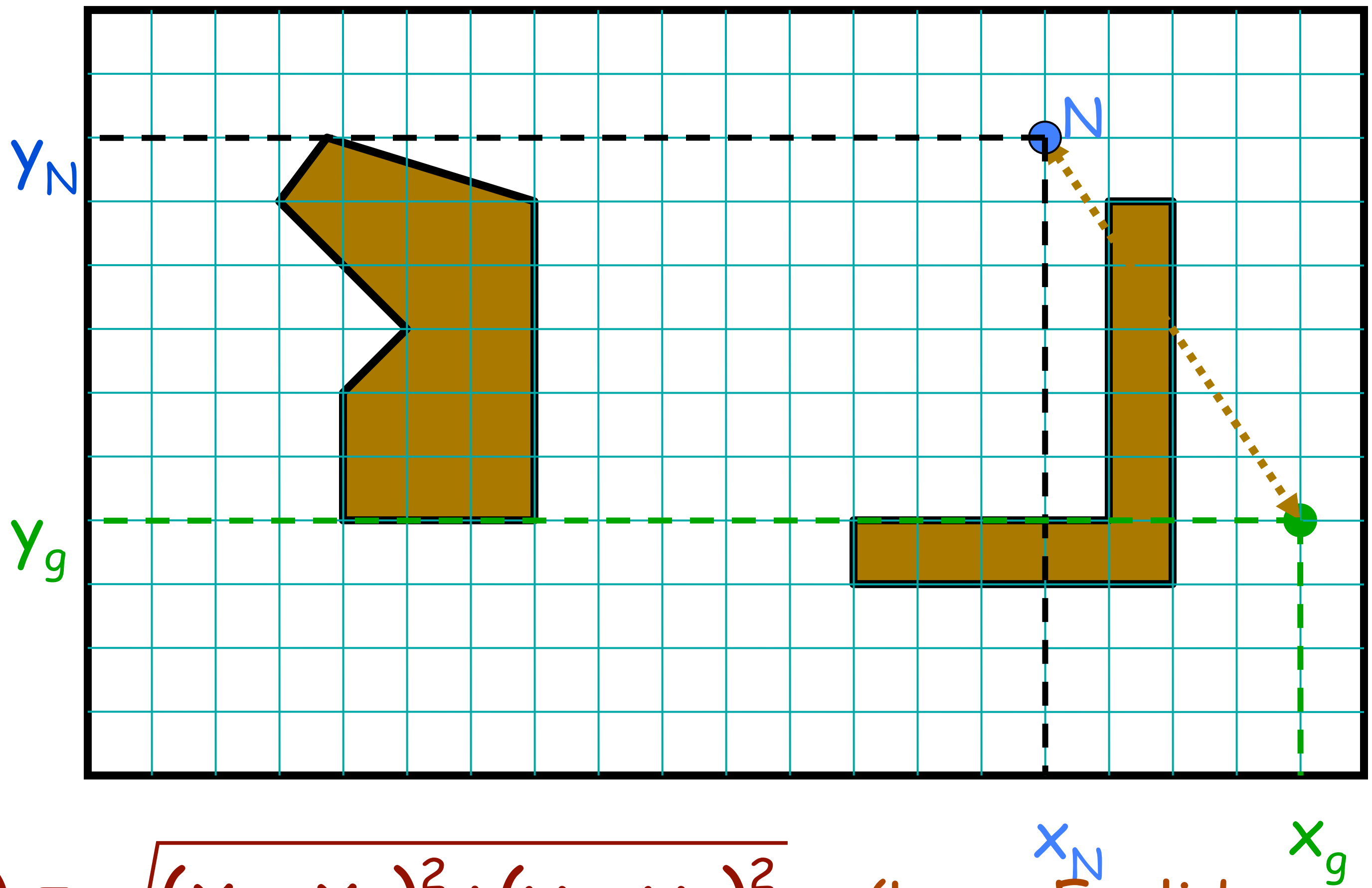

$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$ (L_2 or Euclidean distance)

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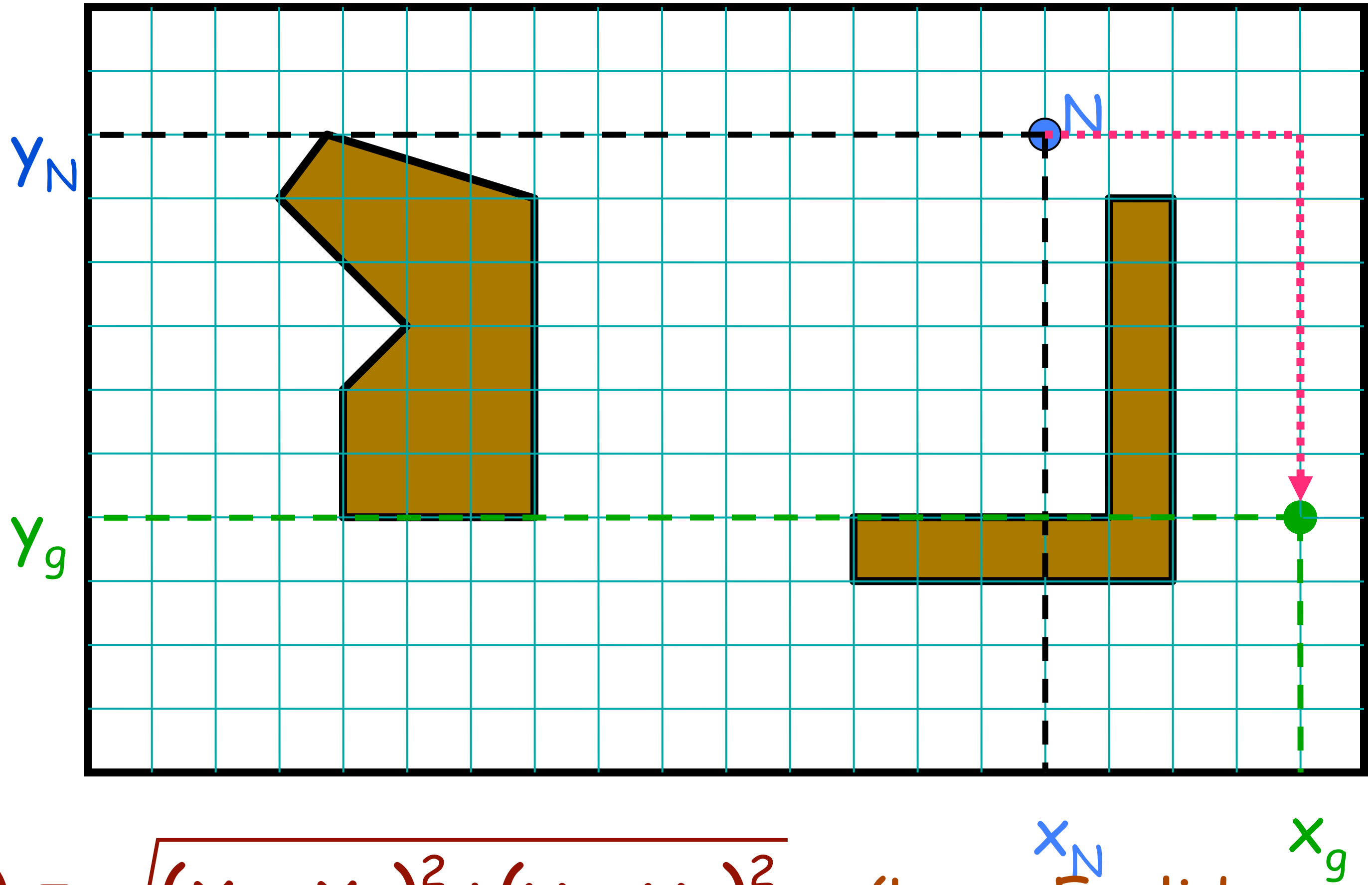

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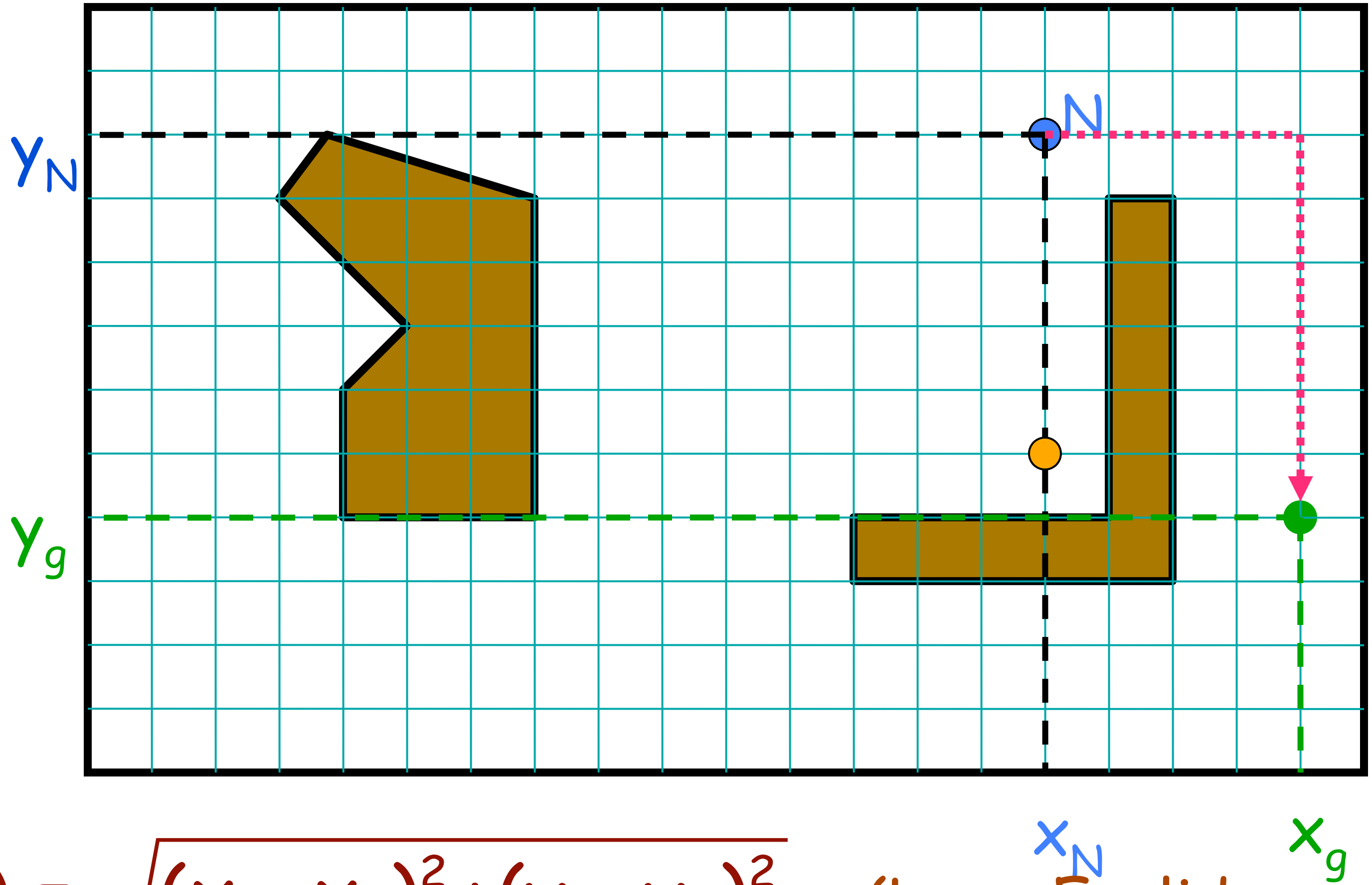

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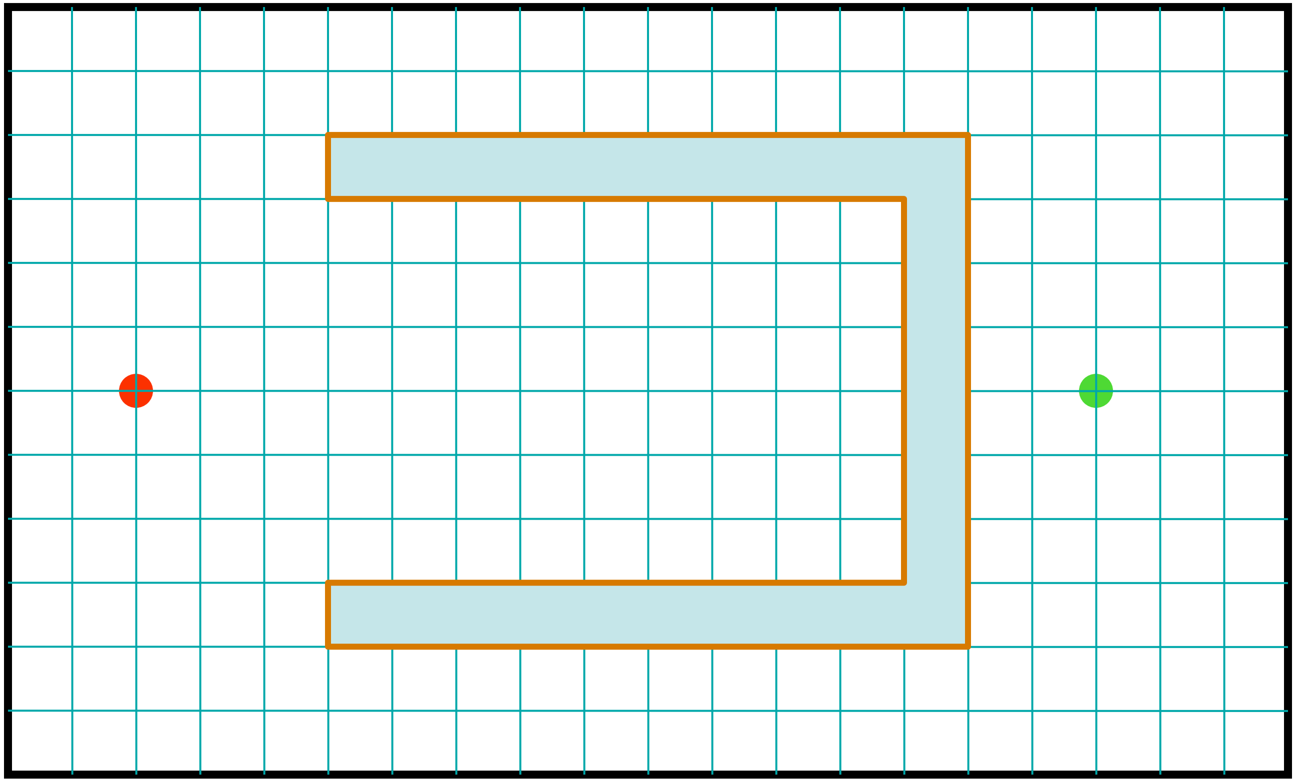
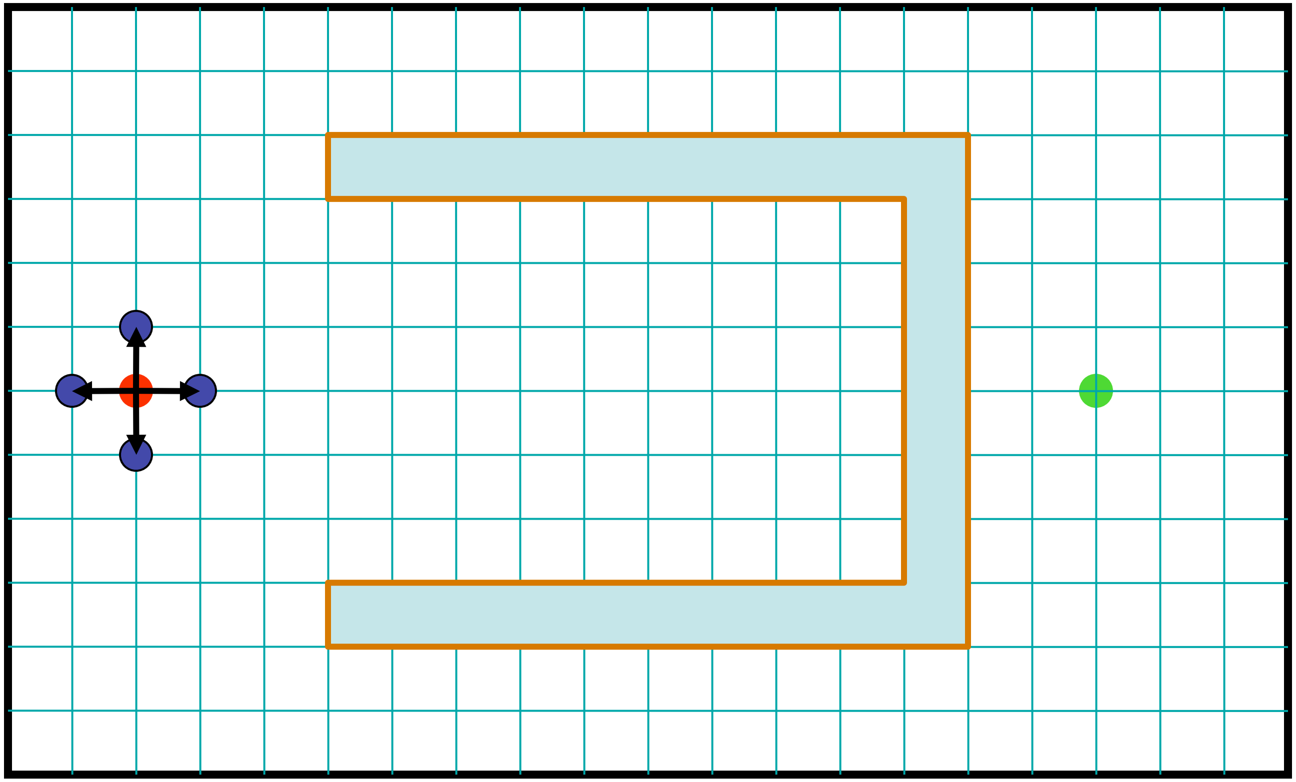
Best-First ↛ Efficiency



f(N) = h(N) = straight distance to the goal

Best-First $\not\rightarrow$ Efficiency



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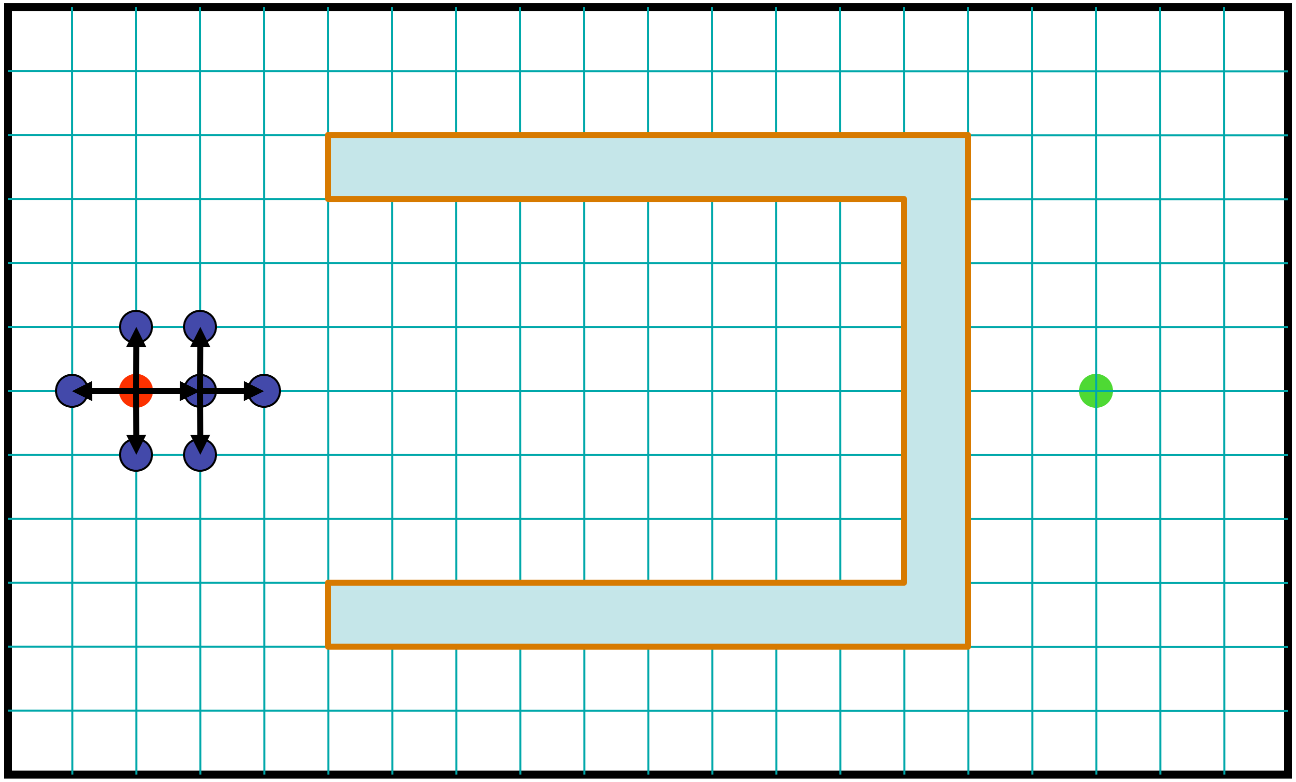

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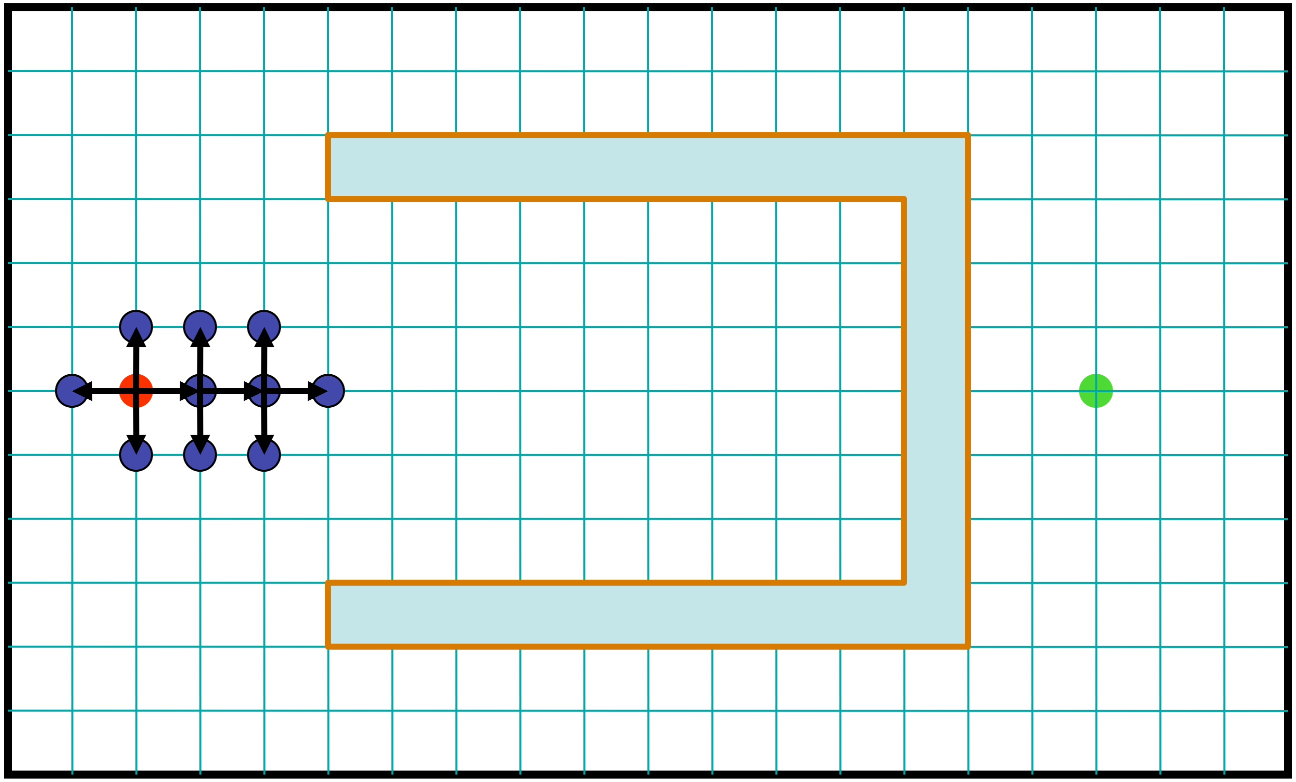

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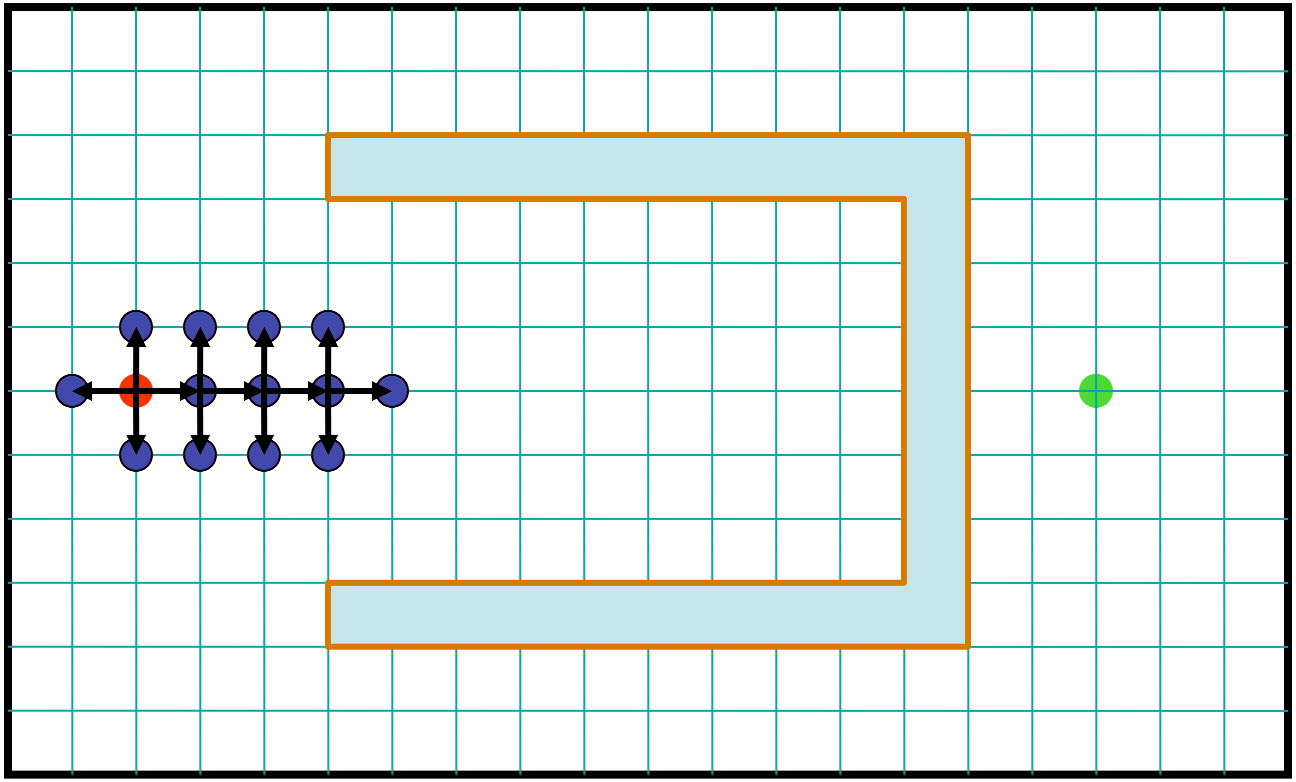

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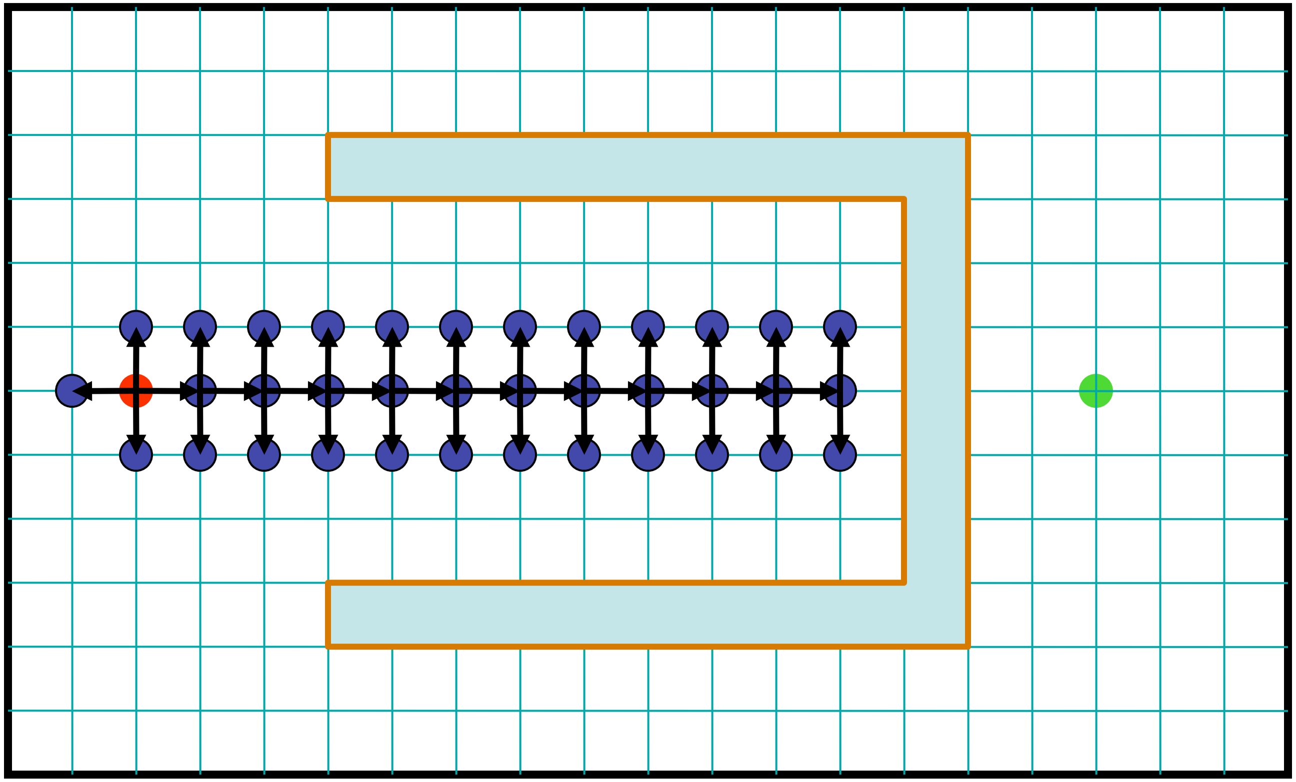

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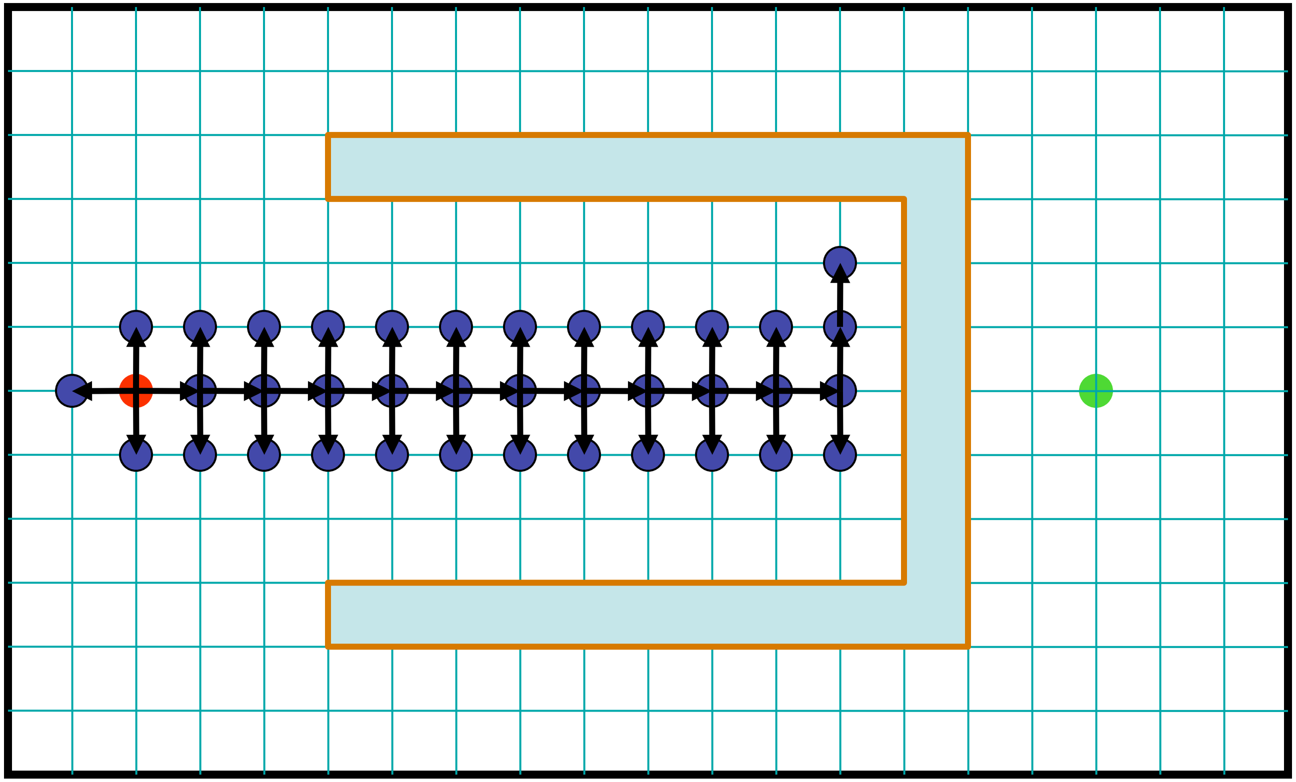

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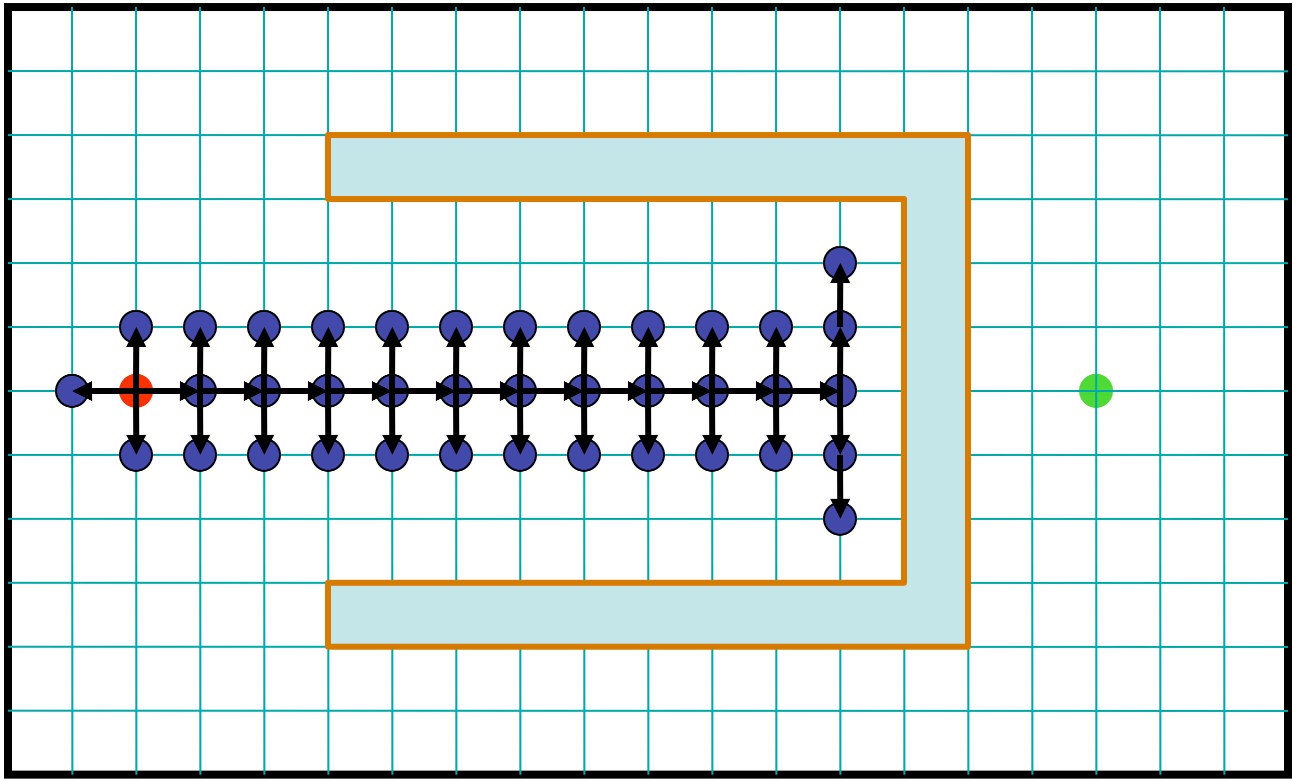

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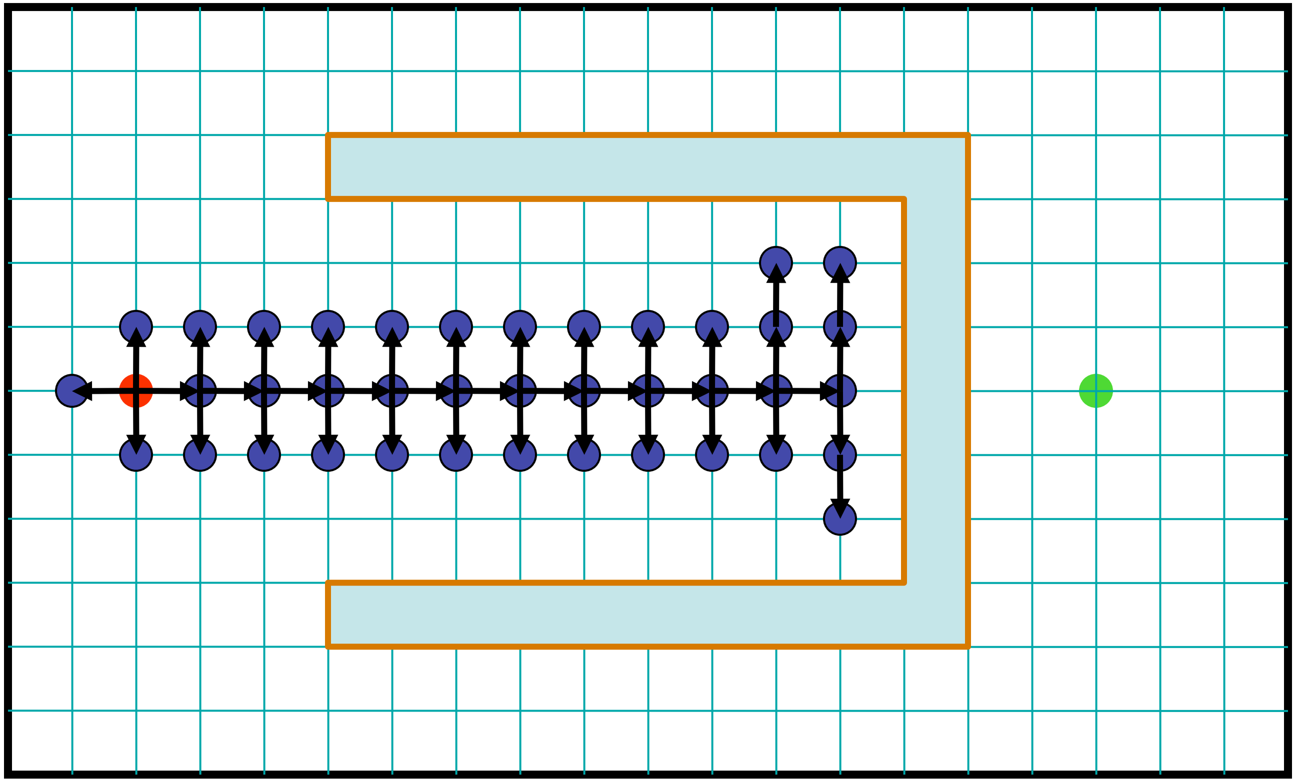

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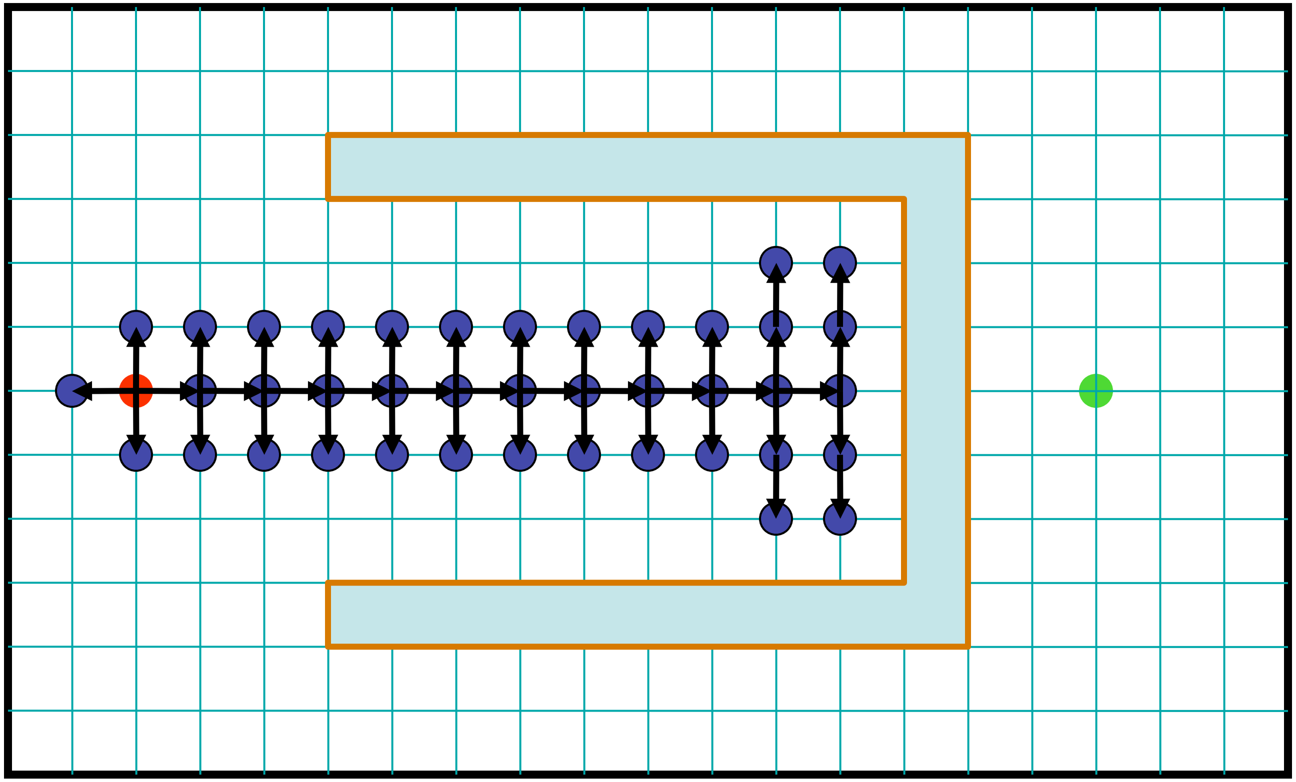

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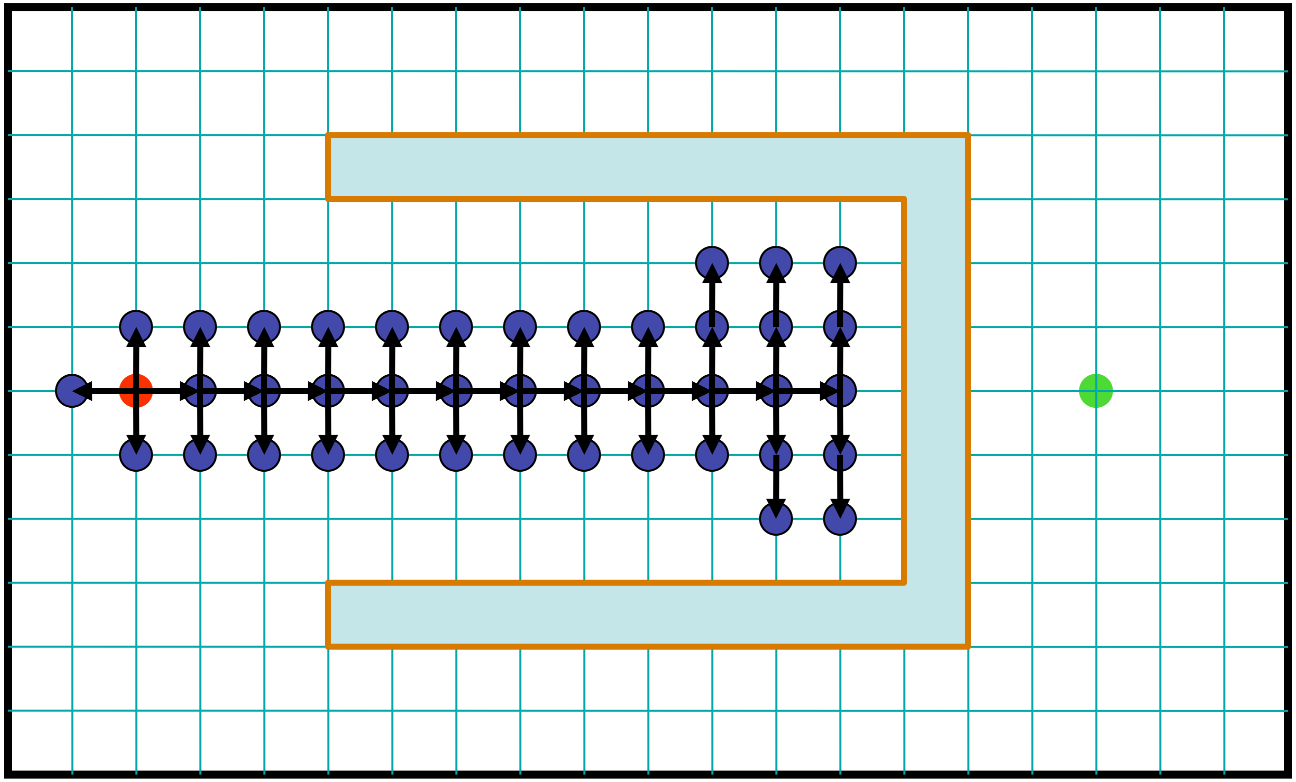

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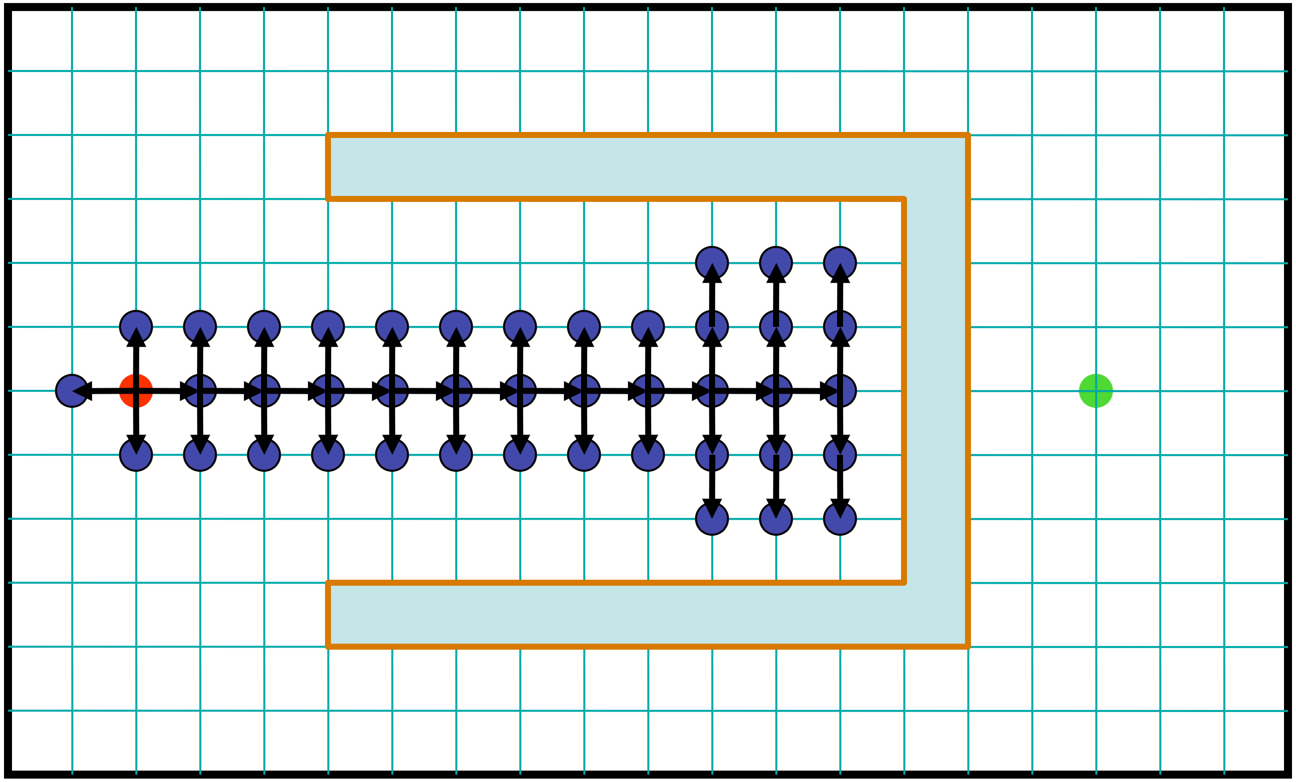

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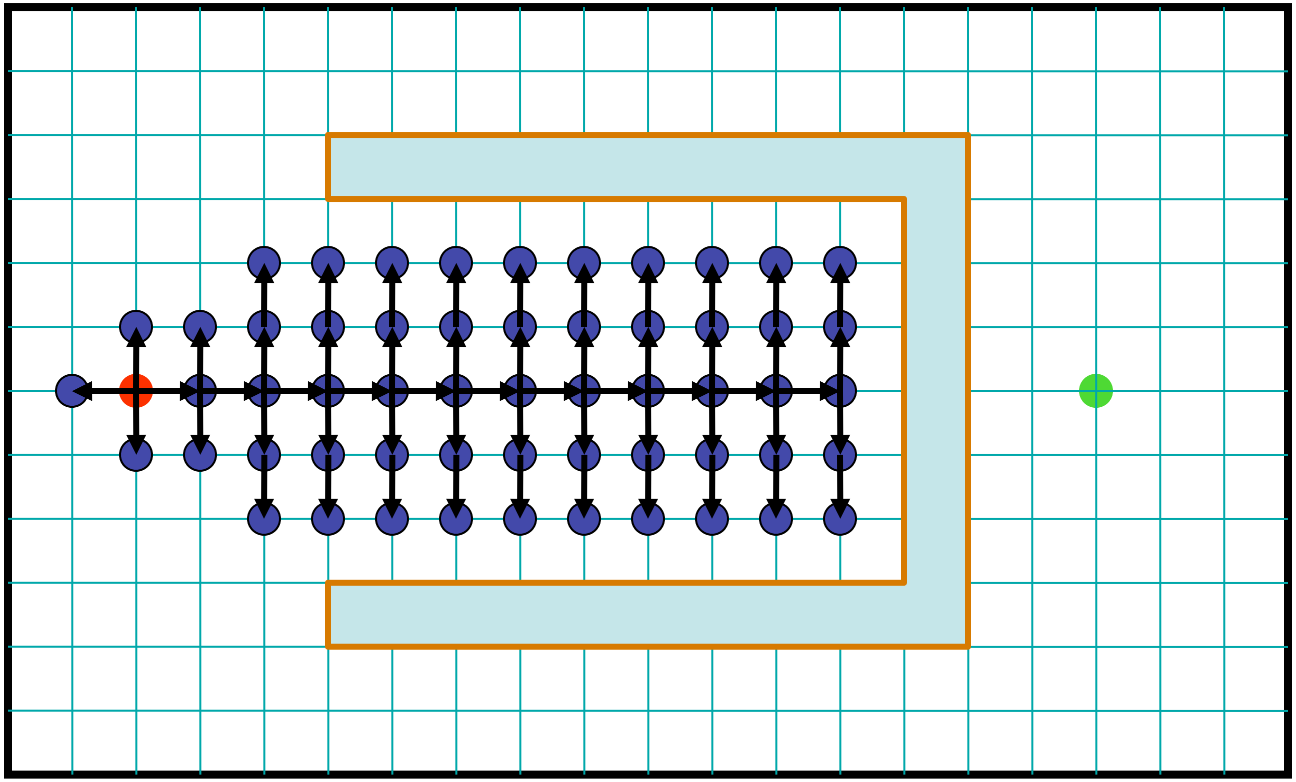

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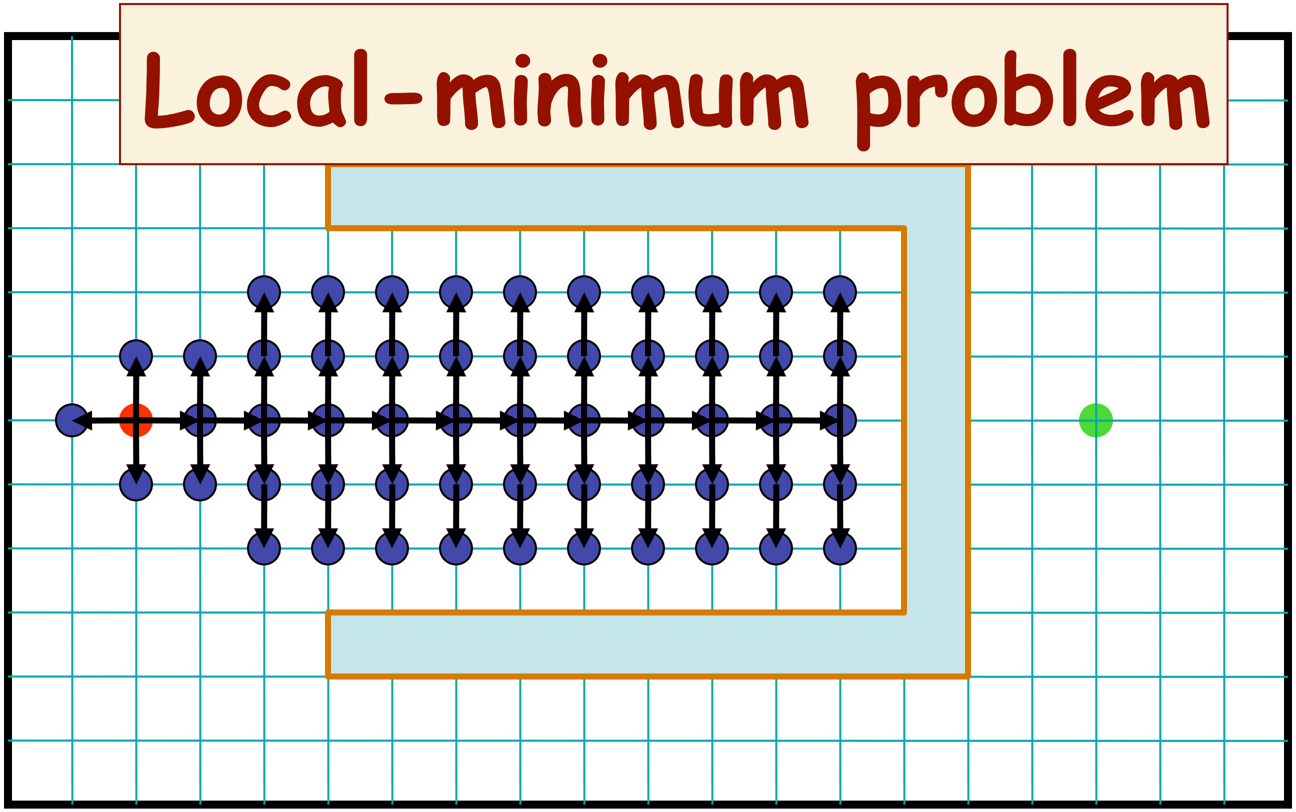

f(N) = h(N) = straight distance to the goal

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Can we prove anything?

- If the state space is infinite, in general the search is not complete
- If the state space is finite and we do not discard nodes that revisit states, in general the search is not complete
- If the state space is finite and we discard nodes that revisit states, the search is complete, but in general is not optimal

Admissible Heuristic

- Let $h^*(N)$ be the cost of the optimal path from N to a goal node
- The heuristic function $h(N)$ is admissible if:

$$0 \leq h(N) \leq h^*(N)$$

- An admissible heuristic function is always optimistic !

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G is a goal node → $h(G) = 0$

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is ???

8-Puzzle Heuristics

5		8
4	2	1
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STATE(N)

1	2	3
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Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is admissible
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
  is ???

8-Puzzle Heuristics

5		8
4	2	1
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Goal state

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  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
  is admissible
- $h_3(N)$ = sum of permutation inversions
  = 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16

8-Puzzle Heuristics

5		8
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Goal state

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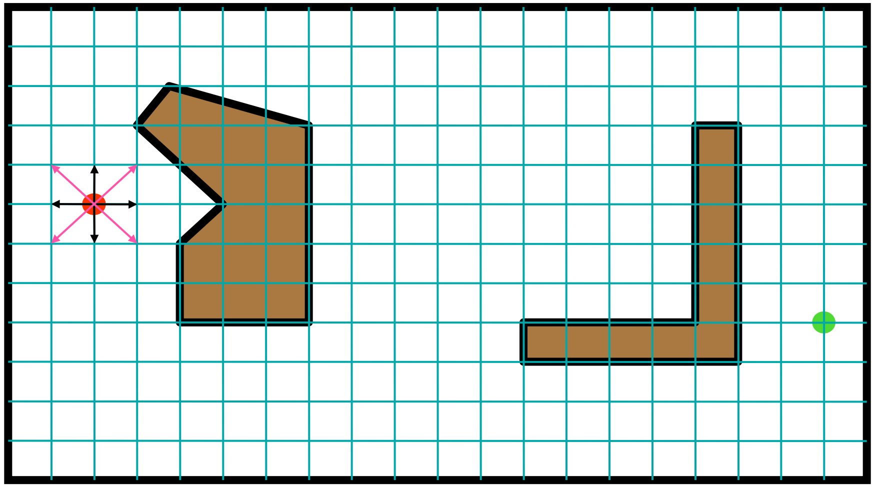
Robot Navigation Heuristics

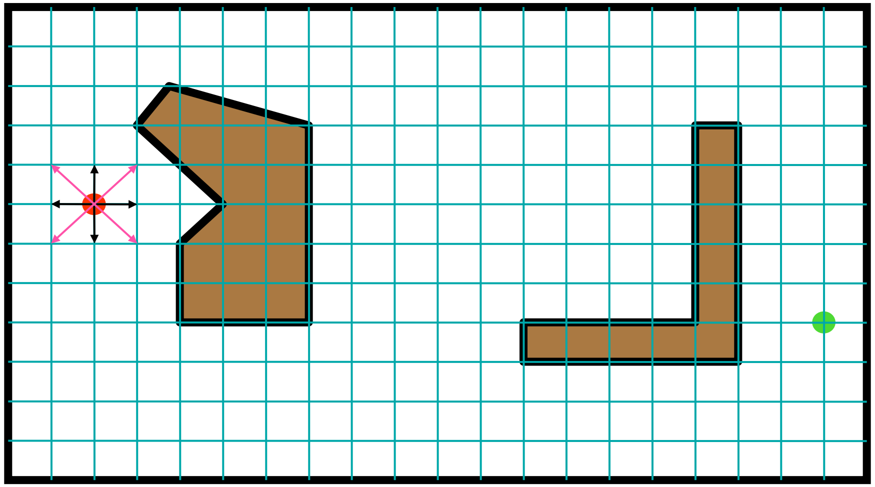


Cost of one horizontal/vertical step = 1

Cost of one diagonal step = $\sqrt{2}$

$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$ is admissible

Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1

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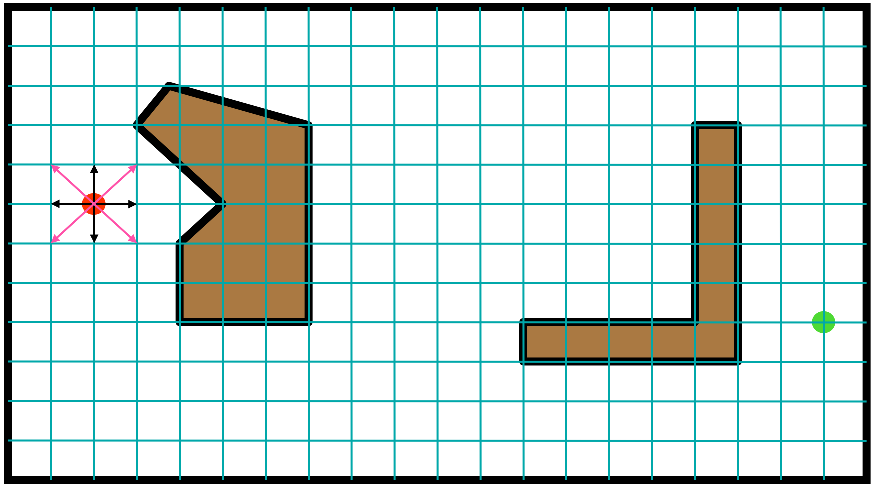
$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is ???

Robot Navigation Heuristics



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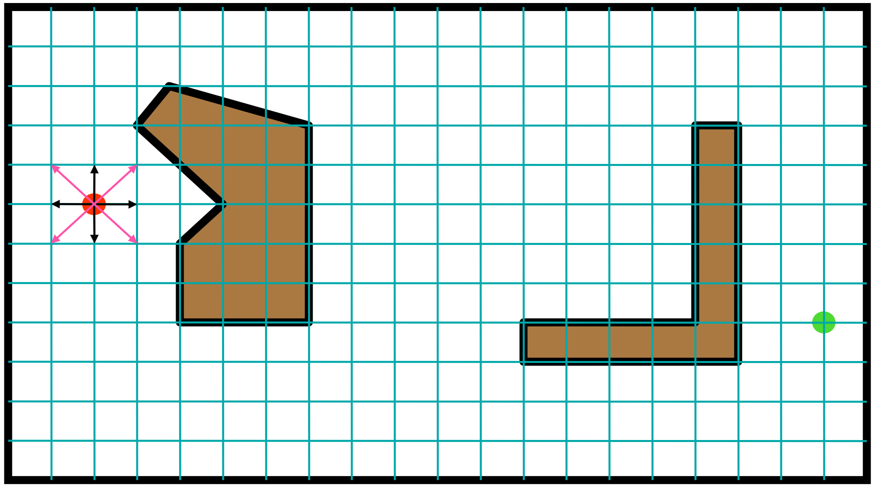
$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is admissible if moving along diagonals is not allowed, and not admissible otherwise

Robot Navigation Heuristics



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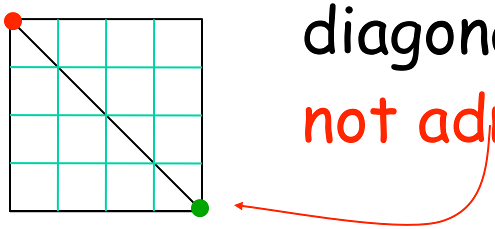
$h^*(I) = 4\sqrt{2}$

$h_2(I) = 8$



How to create an admissible h?

- An admissible heuristic can usually be seen as the cost of an optimal solution to a relaxed problem (one obtained by removing constraints)
- In robot navigation:
  - The Manhattan distance corresponds to removing the obstacles
  - The Euclidean distance corresponds to removing both the obstacles and the constraint that the robot moves on a grid
- More on this topic later

A* Search
(most popular algorithm in AI)

1) $f(N) = g(N) + h(N)$, where:
   - $g(N)$ = cost of best path found so far to N
   - $h(N)$ = **admissible** heuristic function

2) for all arcs: $c(N,N') \geq \varepsilon > 0$
3) SEARCH#2 algorithm is used

→ Best-first search is then called A* search

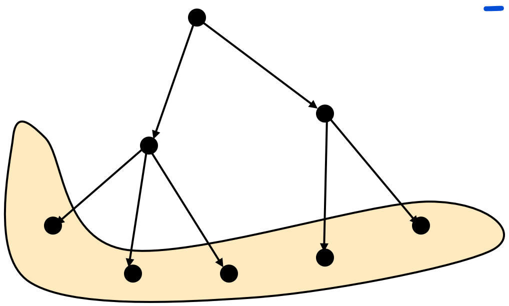
Result #1

A* is complete and optimal

[This result holds if nodes revisiting states are not discarded]

Proof (1/2)

1) If a solution exists, A* terminates and returns a solution



- For each node N on the OL,
  $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\varepsilon$,
  where d(N) is the depth of N in the tree

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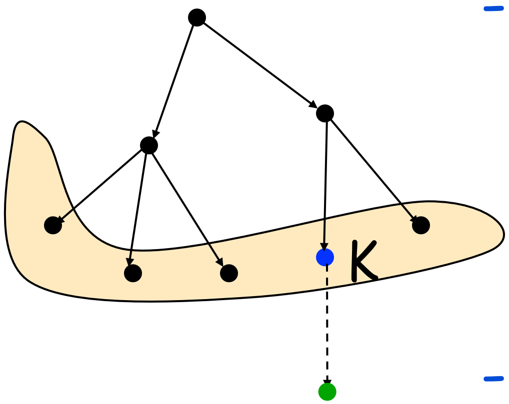

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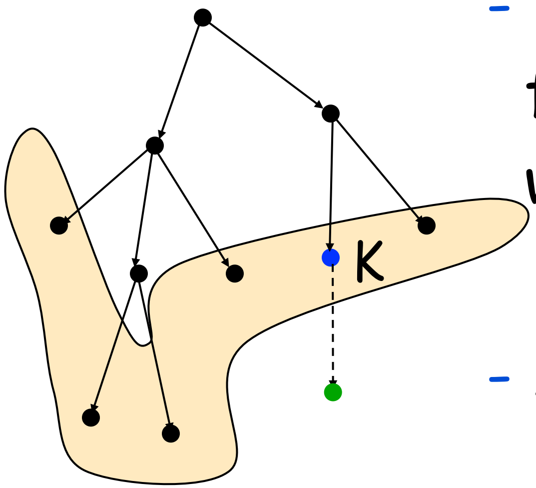

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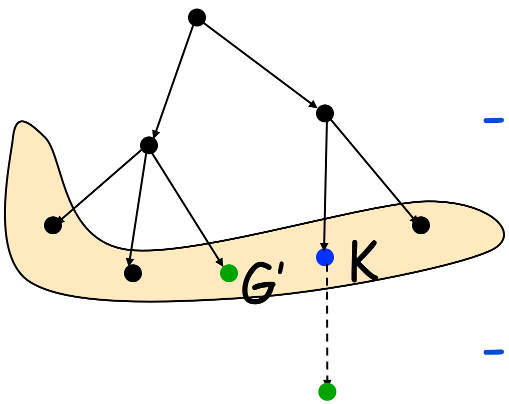
- Since each node expansion increases the length of one path, K will eventually be selected for expansion, unless a solution is

Proof (2/2)

2) Whenever A* chooses to expand a goal node, the path to this node is optimal



- C^*= cost of the optimal solution path
- G': non-optimal goal node in the OL
  $$f(G') = g(G') + h(G') = g(G') > C^*$$
- A node K in the OL lies on an optimal path:
  $$f(K) = g(K) + h(K) \leq C^*$$
- So, G' will not be selected for expansion

Time Limit Issue

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- When a problem has no solution, A* runs for ever if the state space is infinite. In other cases, it may take a huge amount of time to terminate

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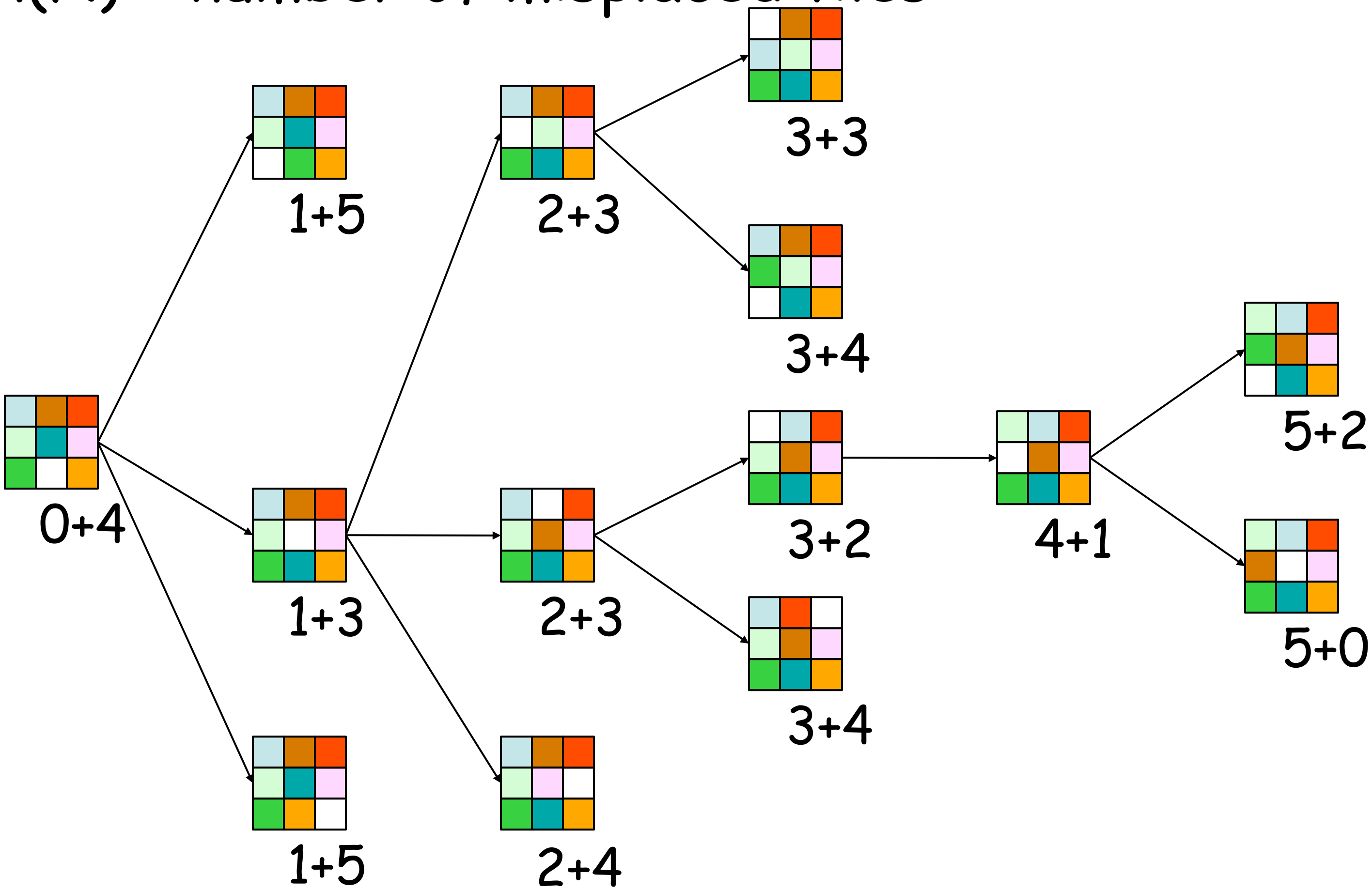
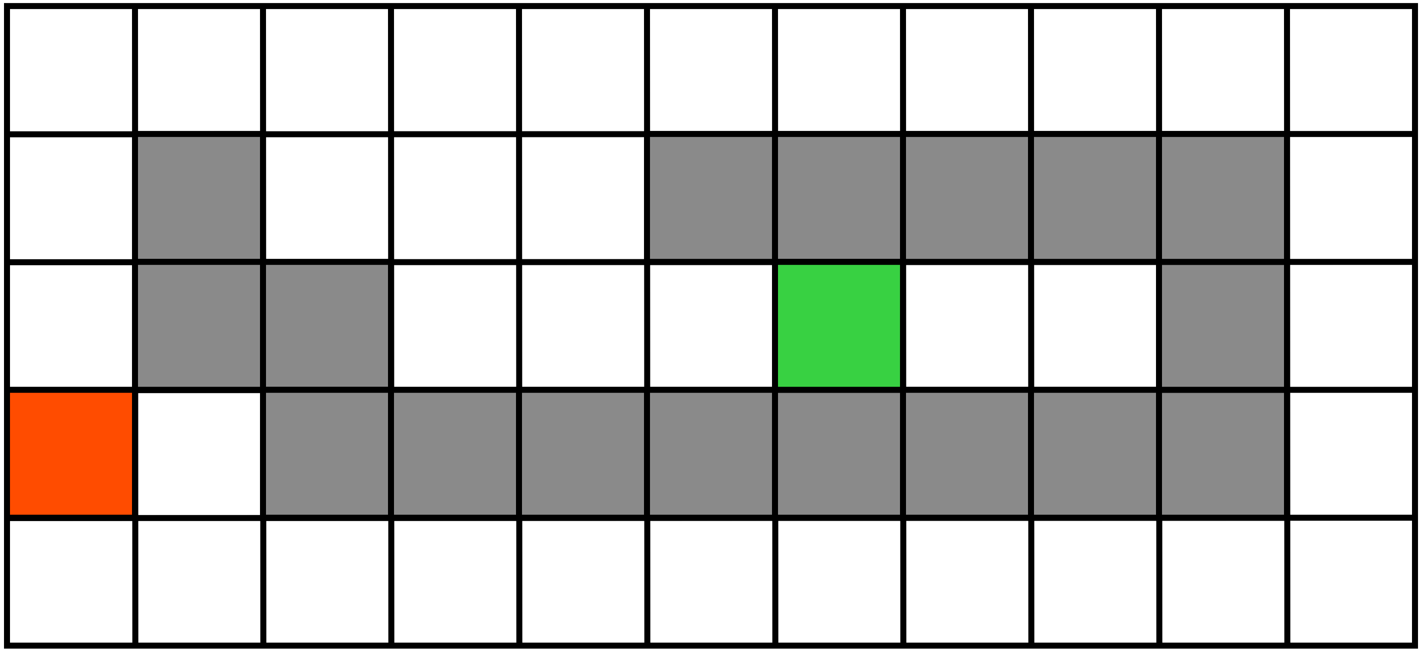
- When AI systems are "small" and solving a single search problem at a time, this is not too much of a concern.
- When AI systems become larger, they solve many search problems concurrently, some with no solution. What should be the time limit for each of them? More on this in the lecture on Motion Planning ...

8-Puzzle

f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles



Robot Navigation



Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

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Robot Navigation

f(N) = g(N)+h(N), with h(N) = Manhattan distance to goal
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7		5	4	3						5
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6+1			3	2	1	0	1	2		4
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Best-First Search

- An evaluation function f maps each node N of the search tree to a real number $f(N) \geq 0$
- Best-first search sorts the OL in increasing f

A* Search

1) $f(N) = g(N) + h(N)$, where:
   - $g(N)$ = cost of best path found so far to N
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2) for all arcs: $c(N,N') \geq \varepsilon > 0$
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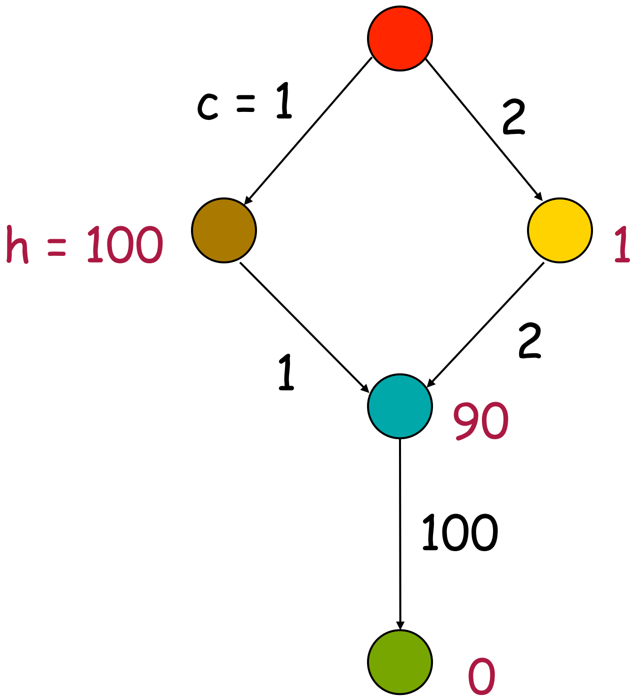
→| Best-first search is then called A* search

Result #1

A* is complete and optimal

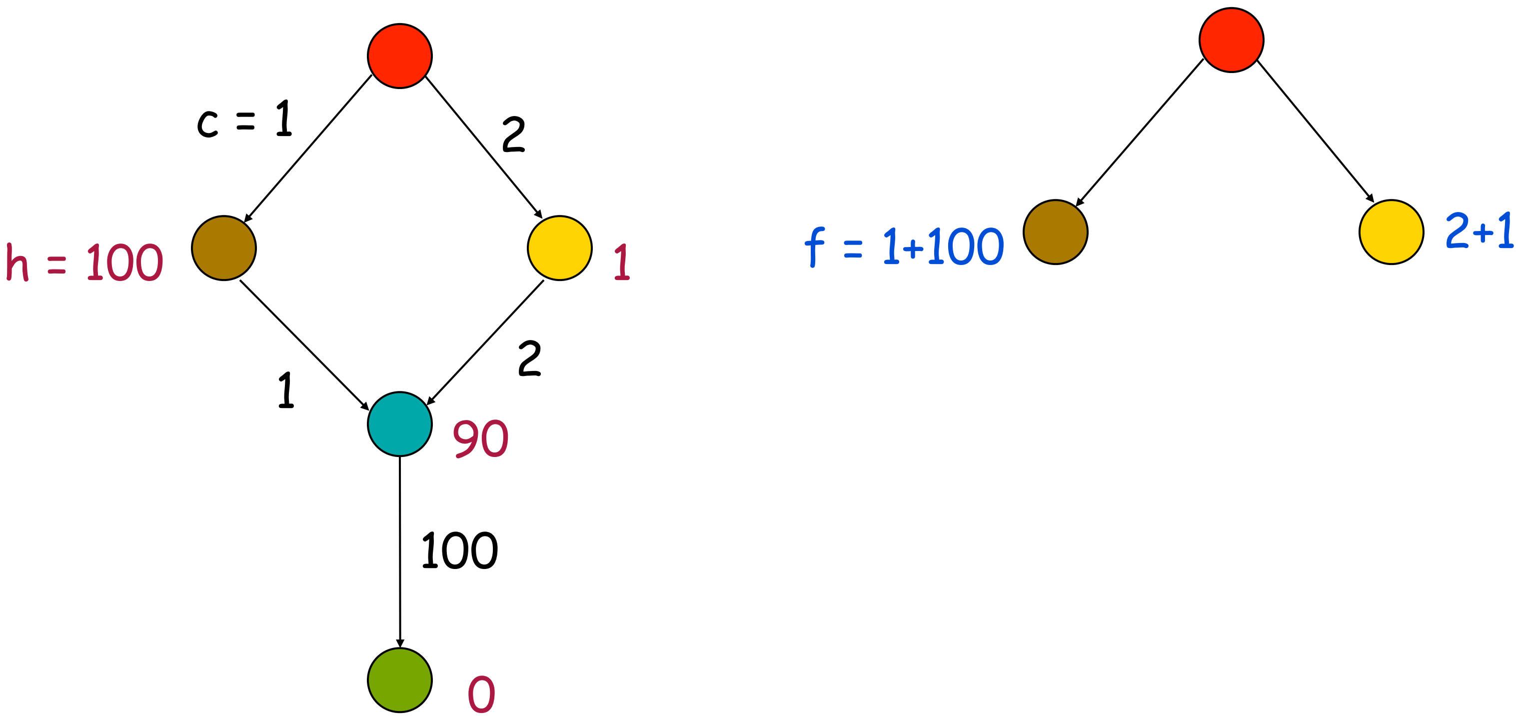
[This result holds if nodes revisiting states are not discarded]

What to do with revisited states?



The heuristic h is clearly admissible

What to do with revisited states?



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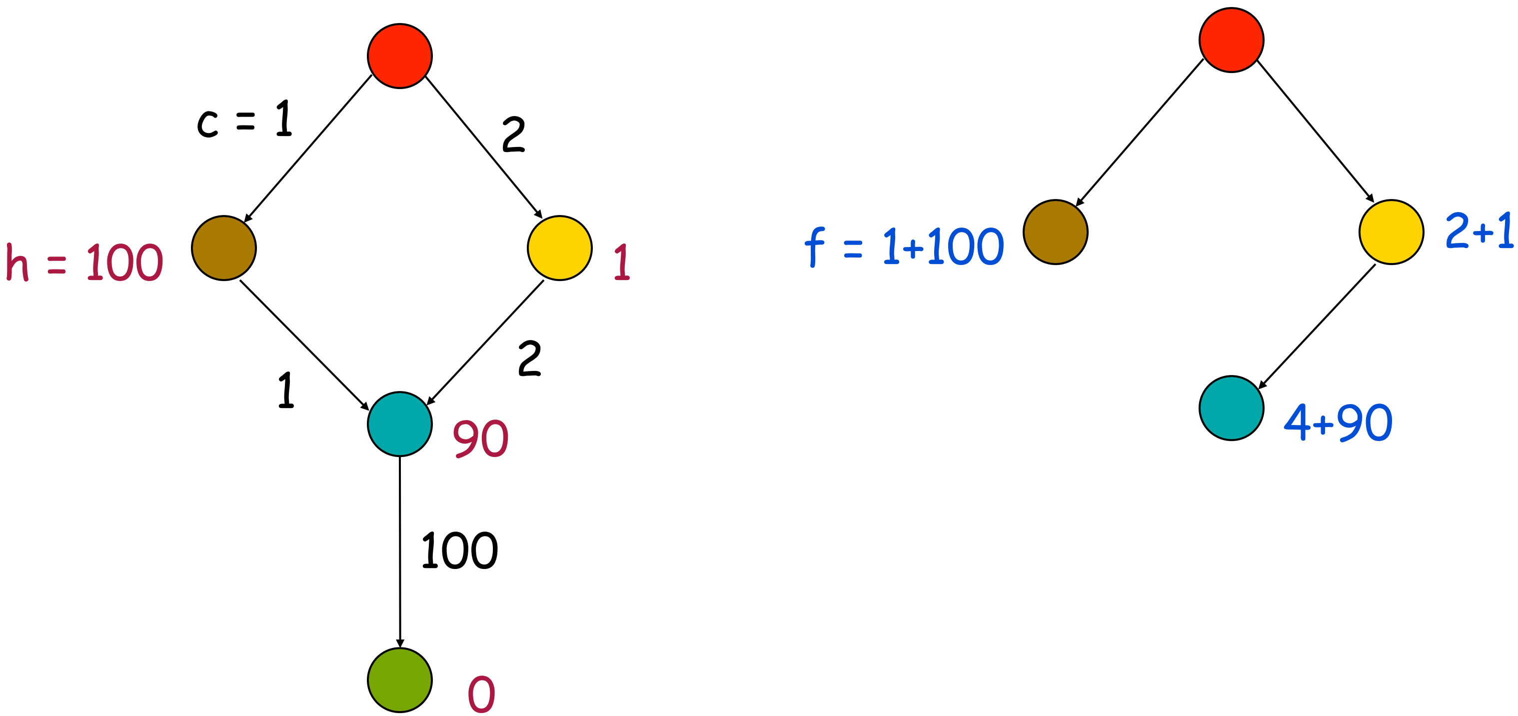

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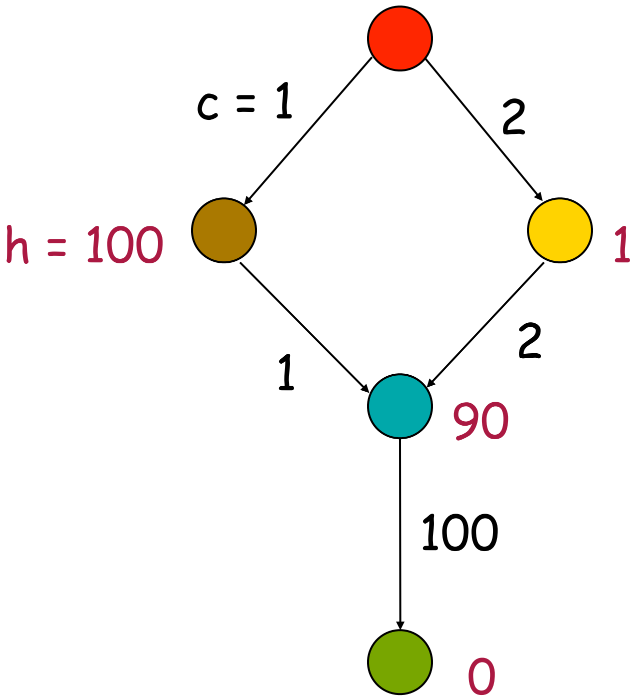

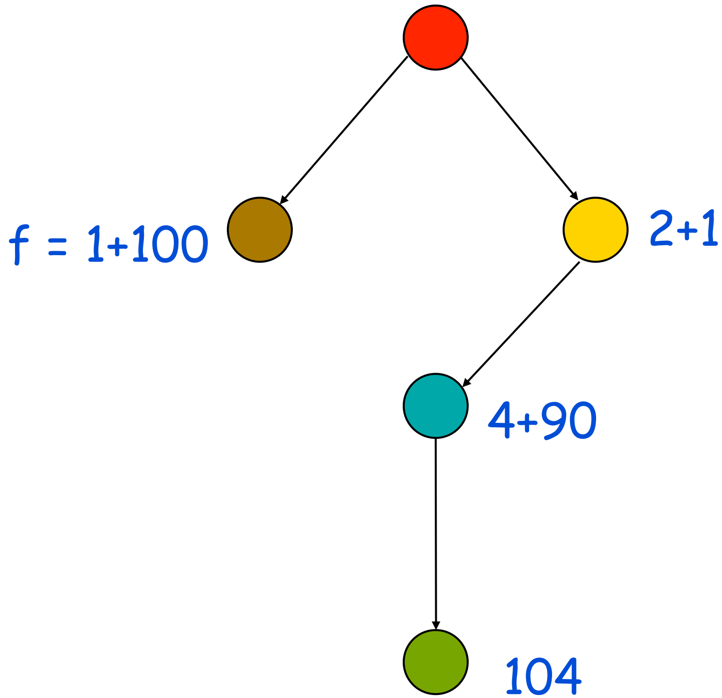

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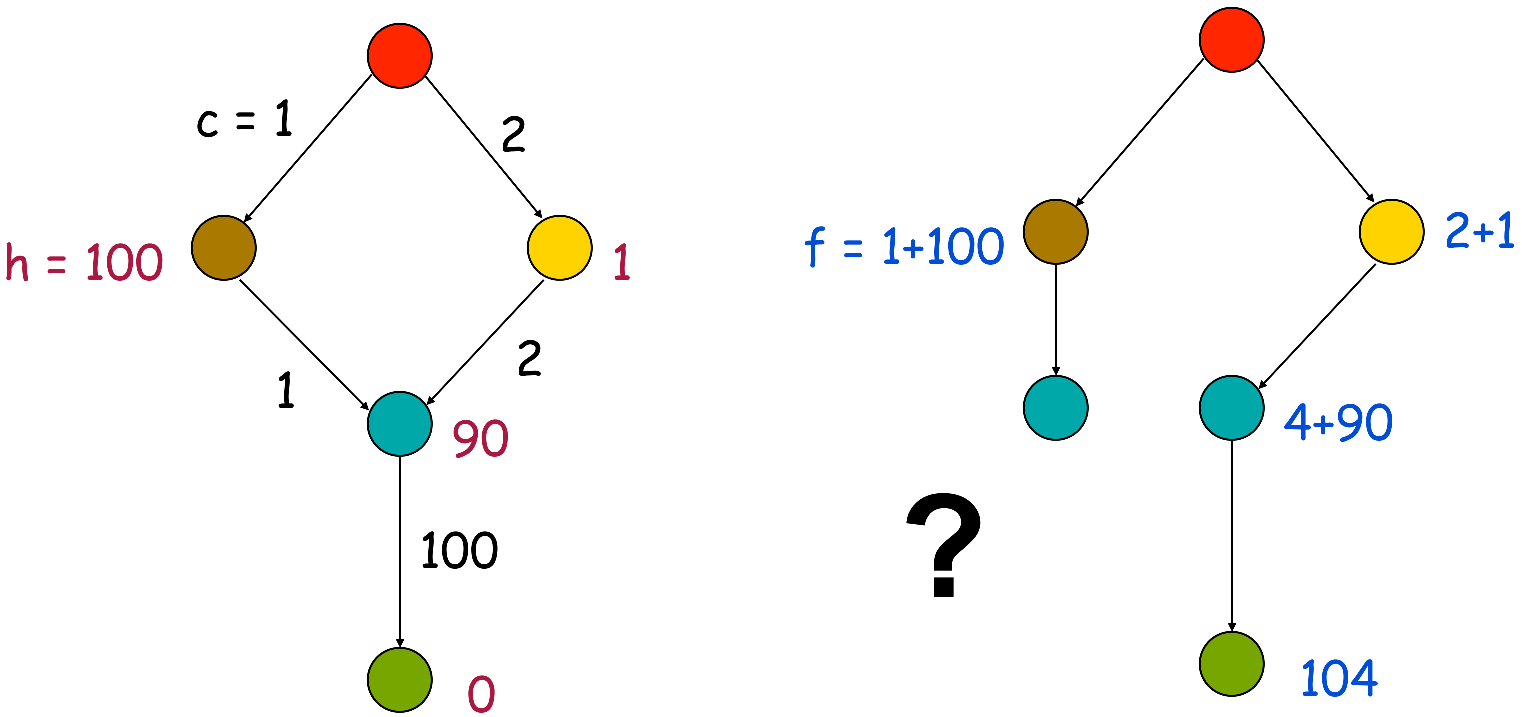

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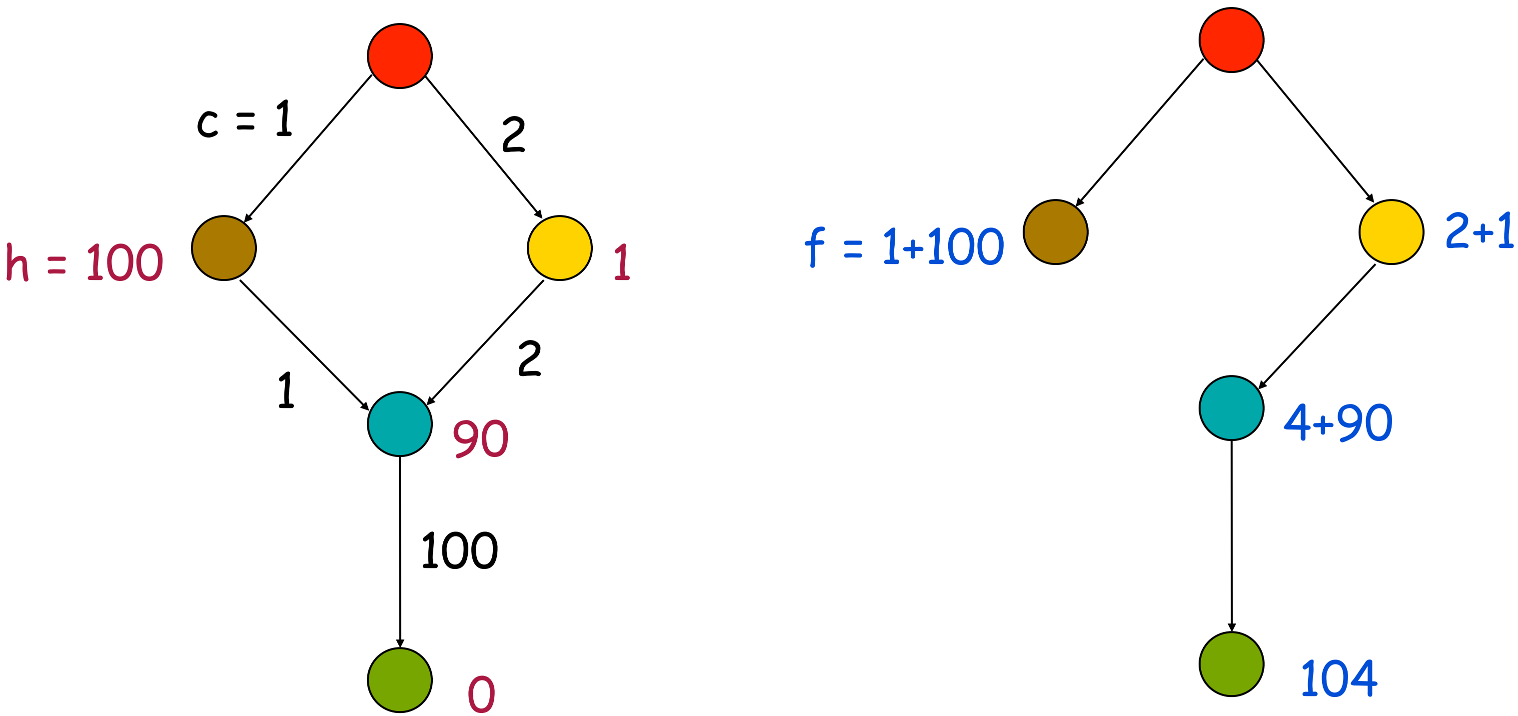
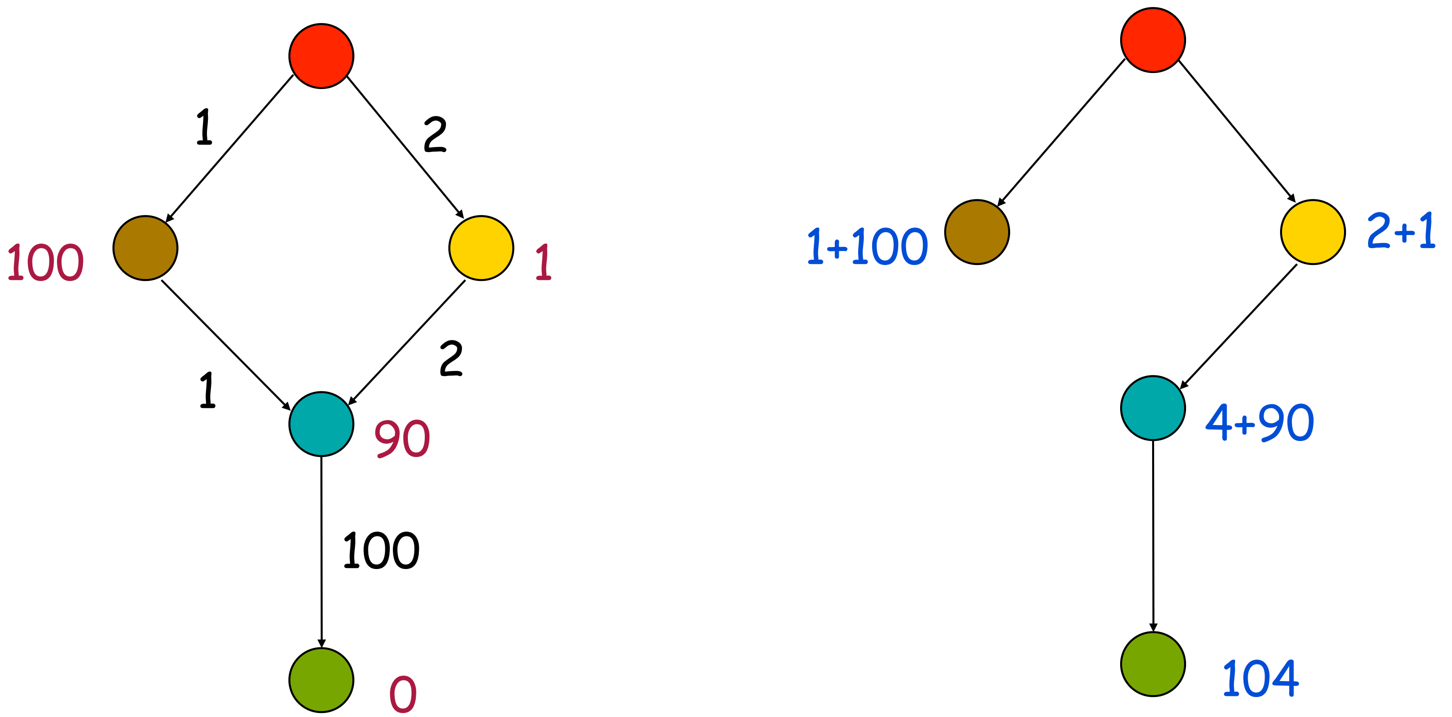


If we discard this new node, then the search algorithm expands the goal node next and returns a non-optimal solution

What to do with revisited states?



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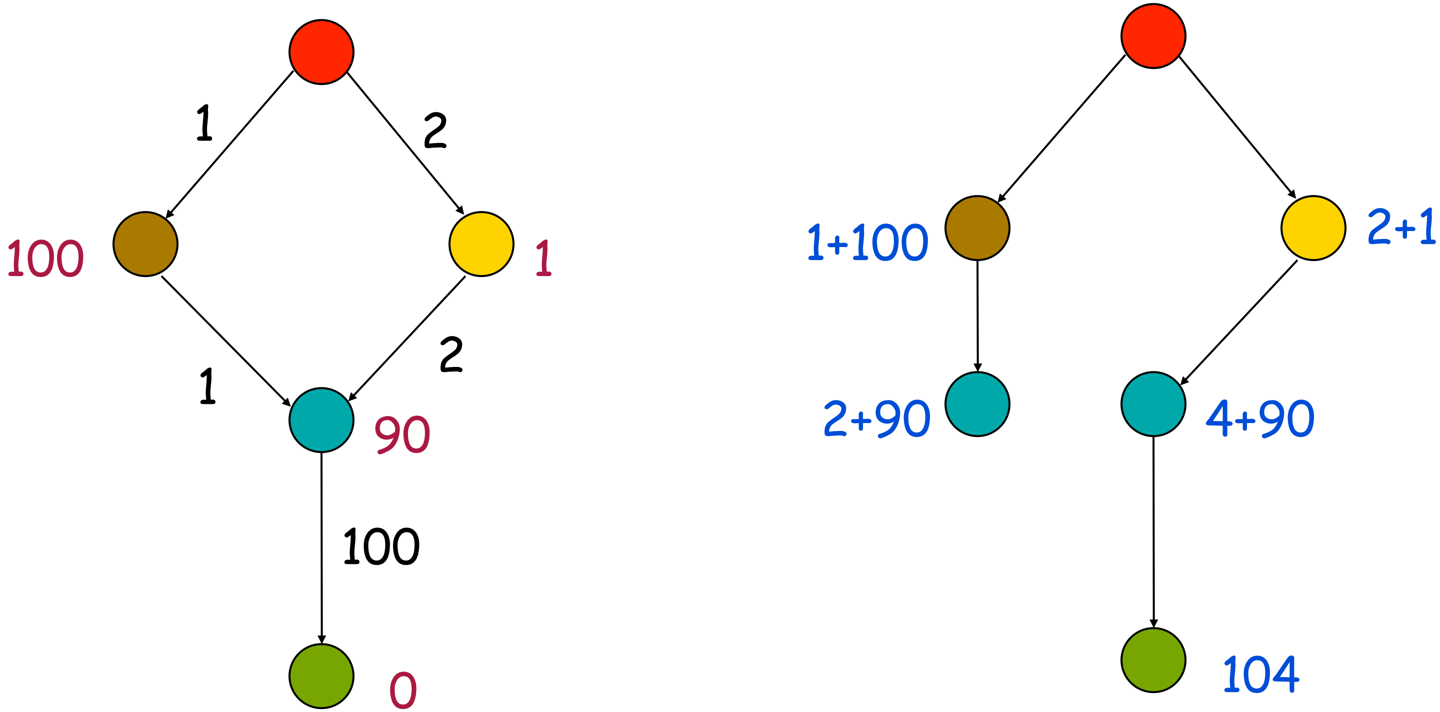

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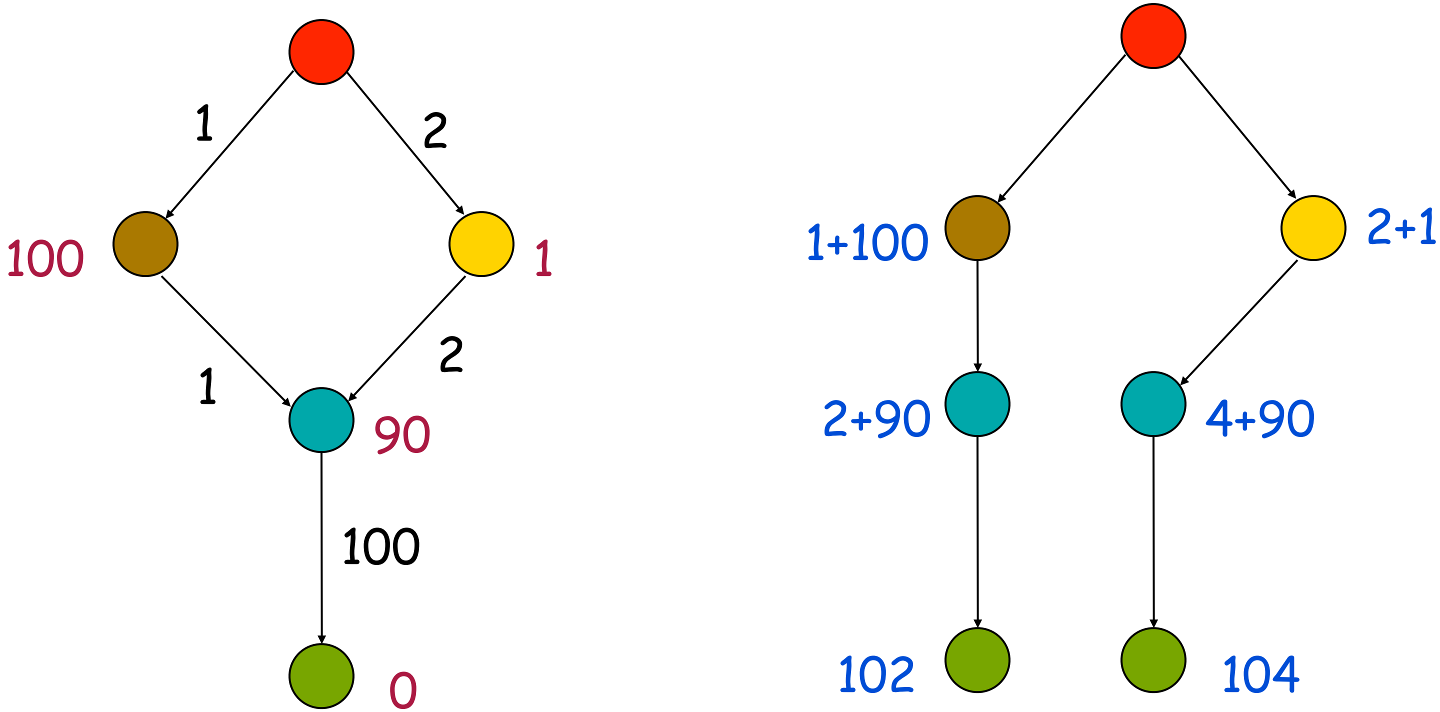

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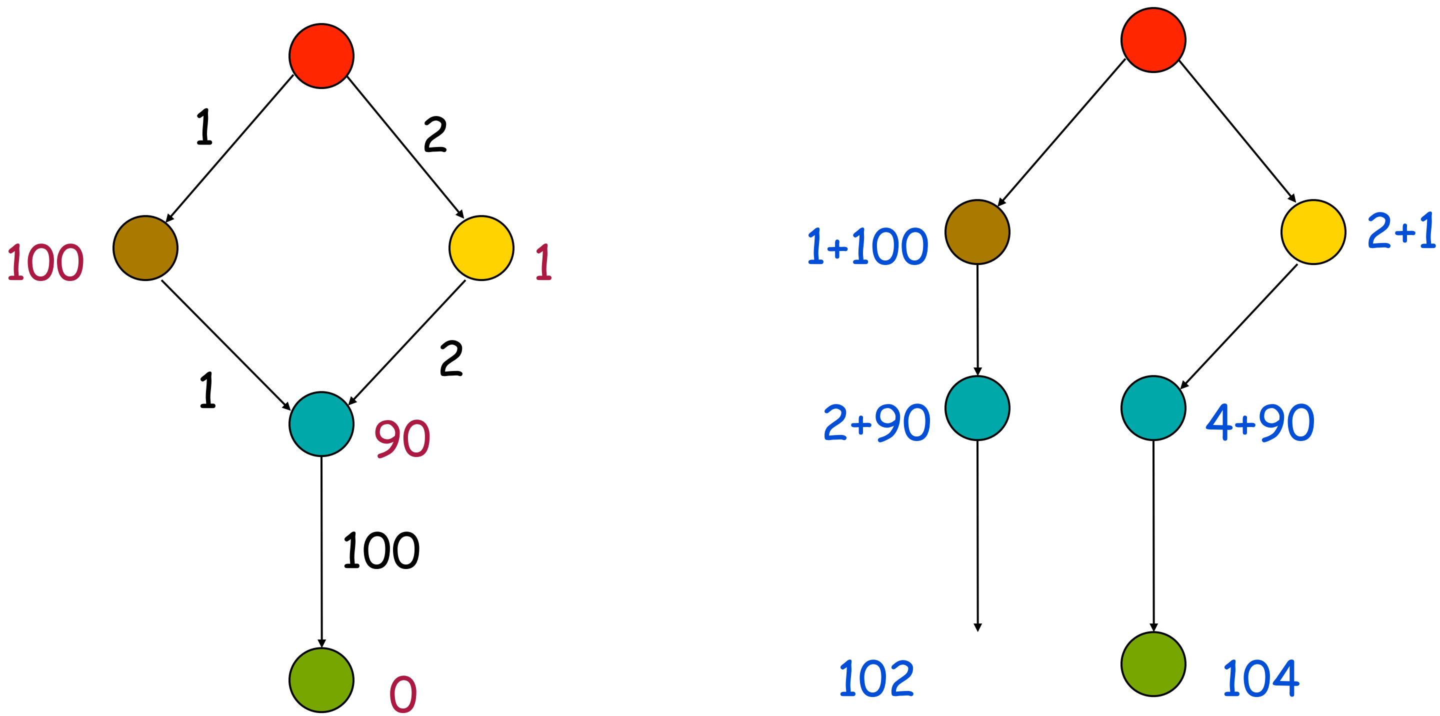

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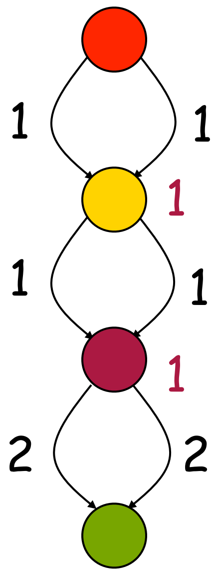
If we do not discard nodes revisiting states, the size of the search tree can be exponential in the number of visited states



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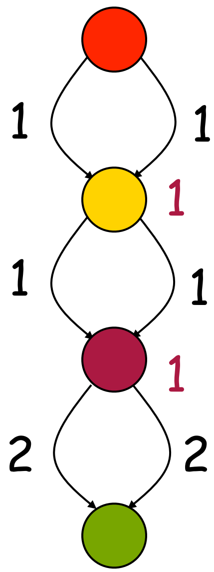

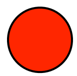

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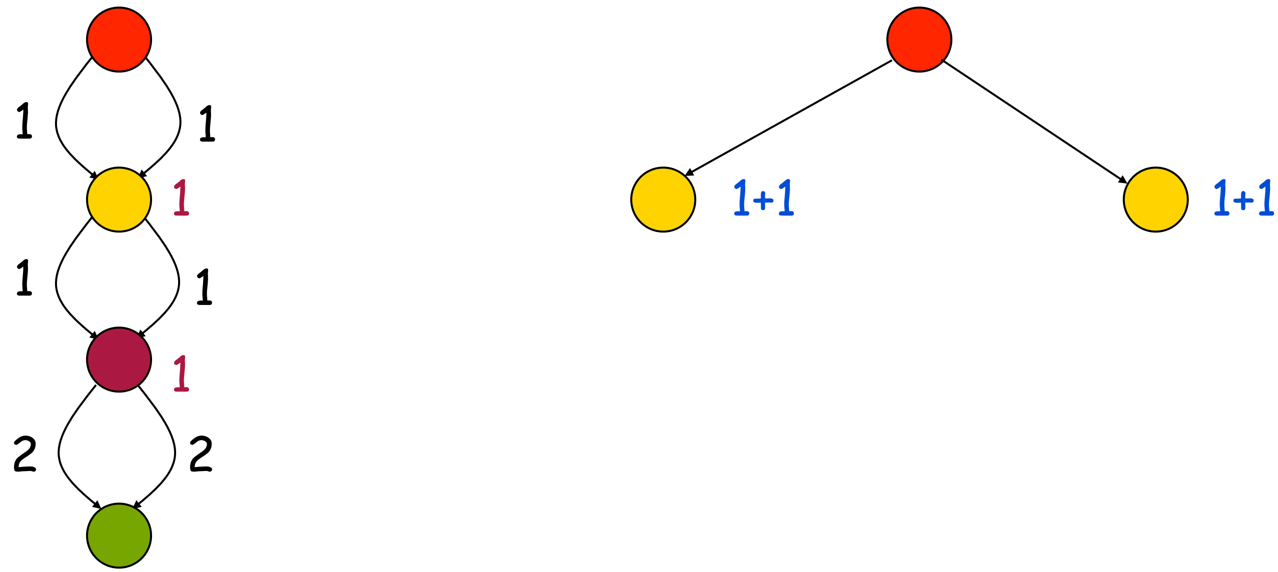

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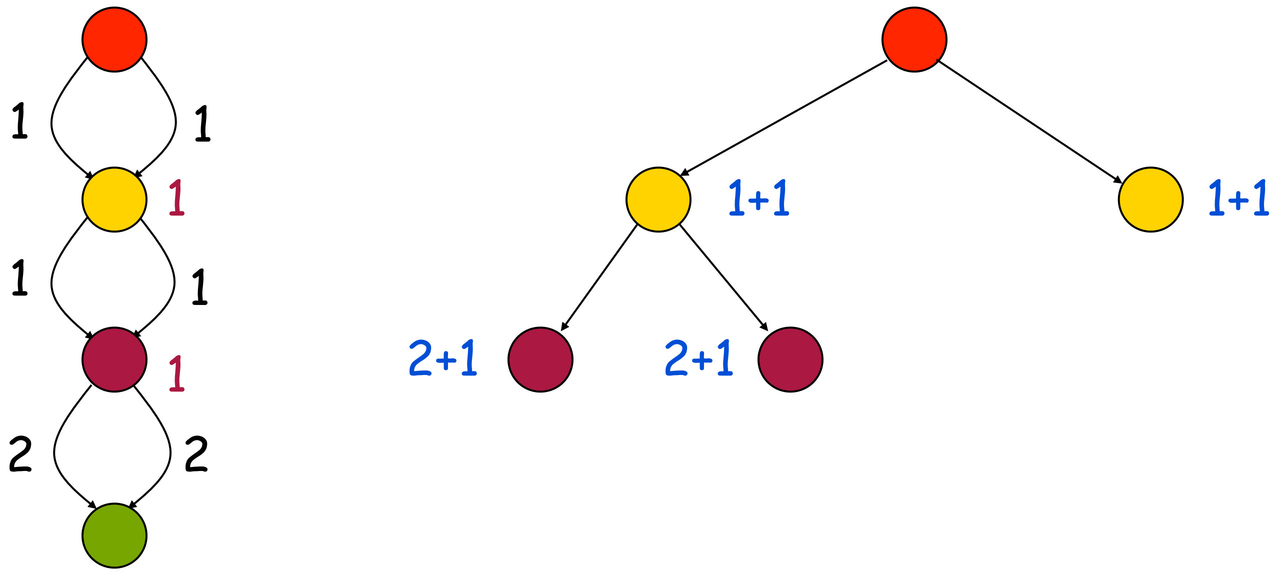

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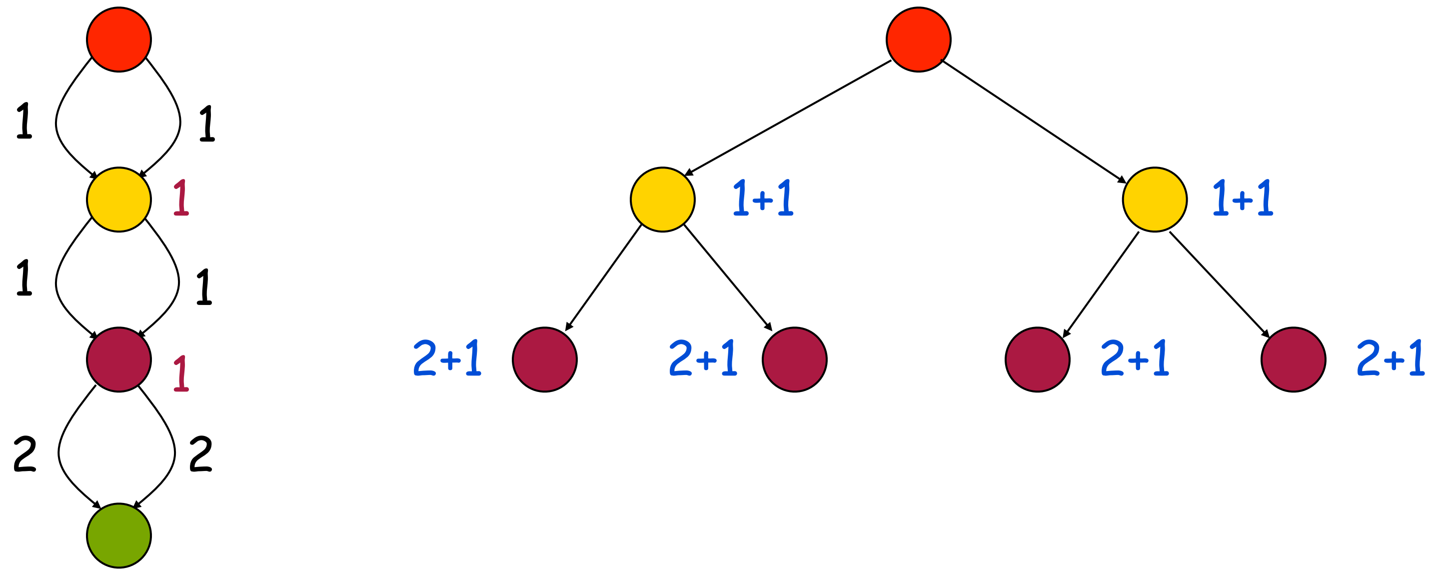

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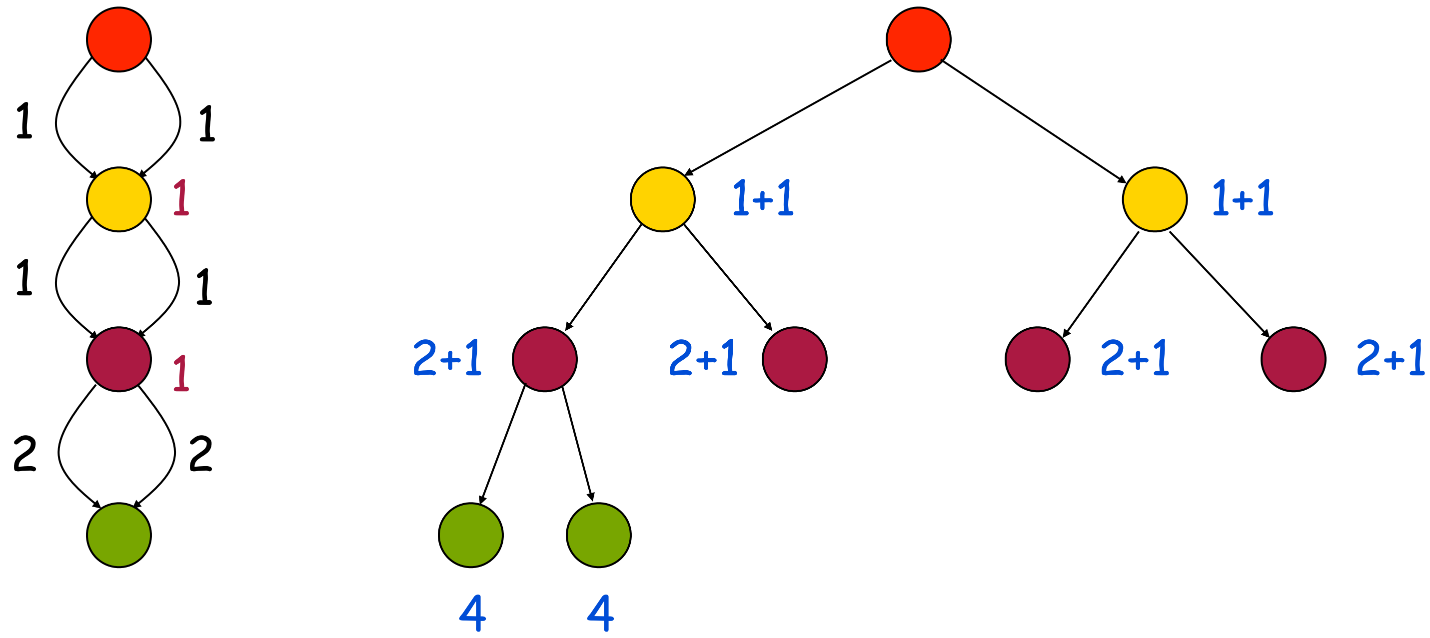

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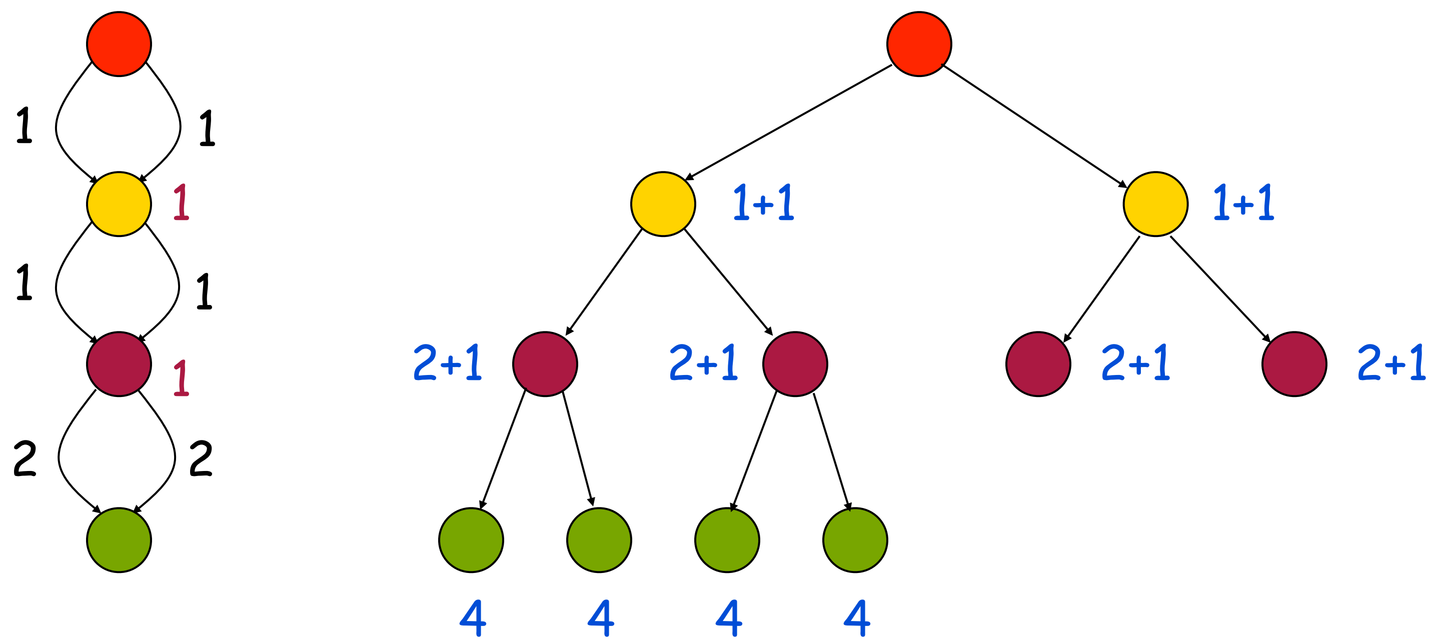

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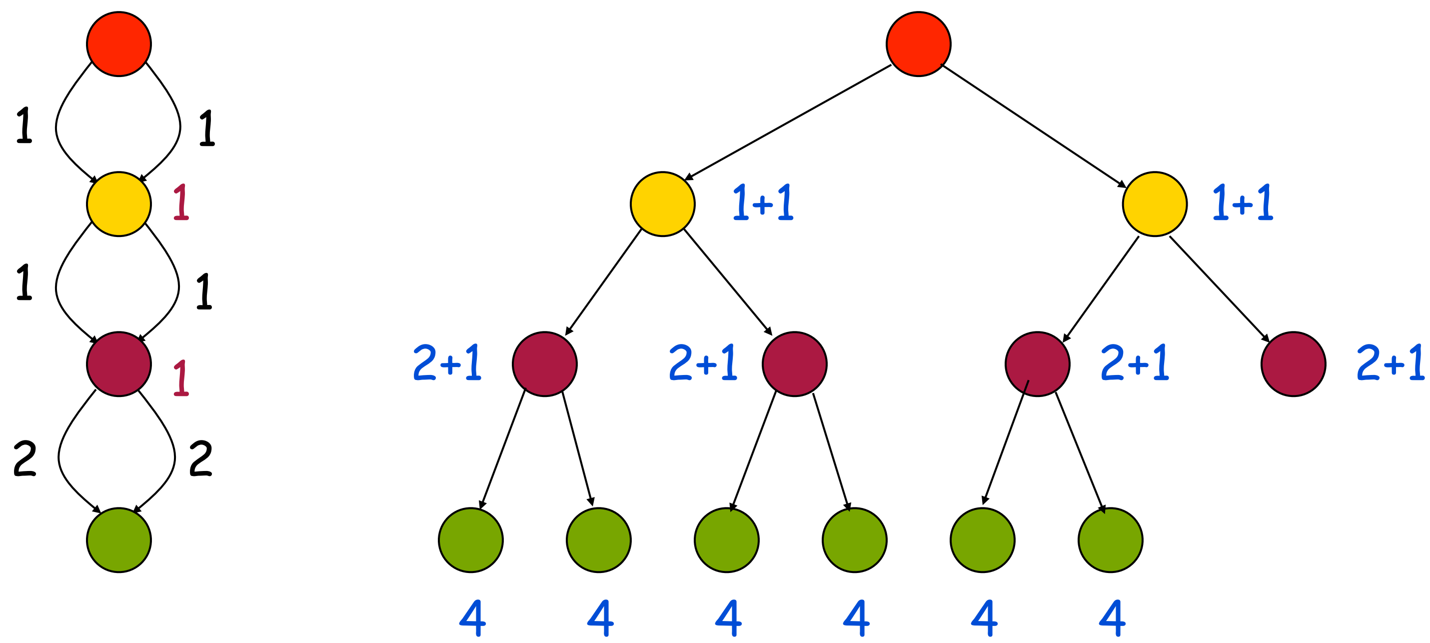

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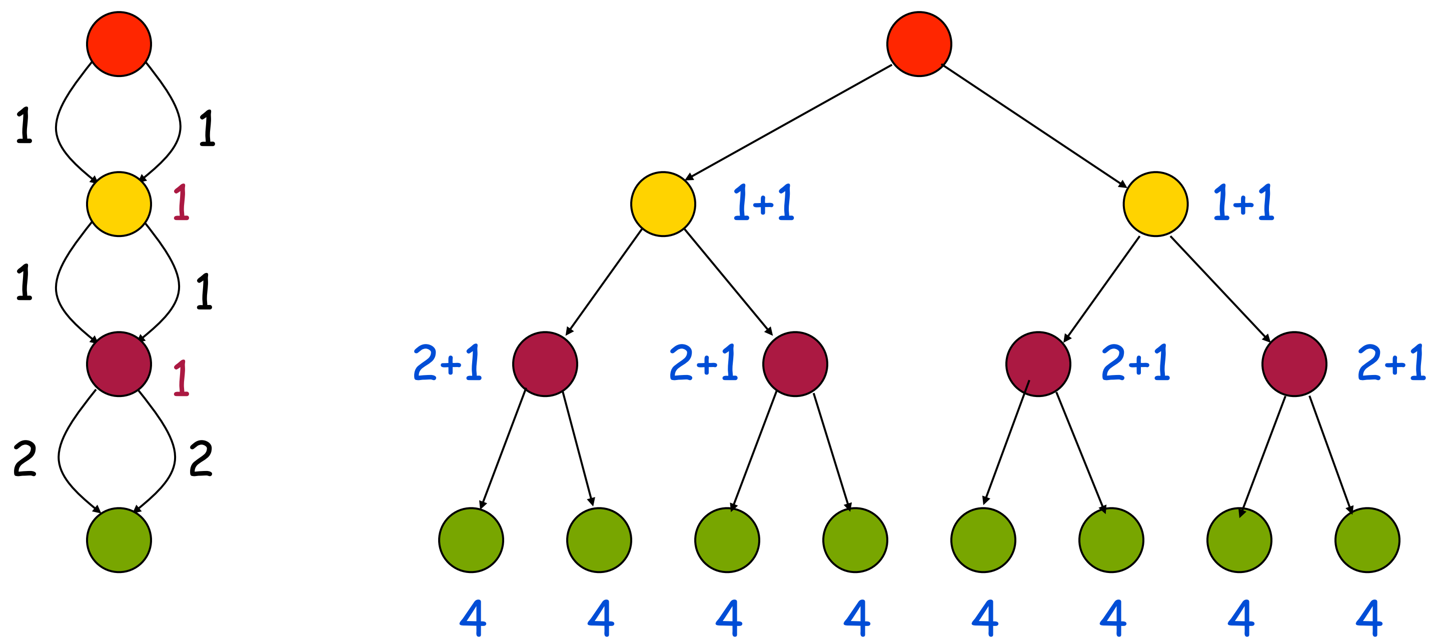

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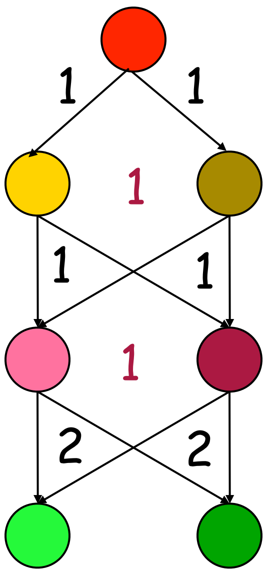
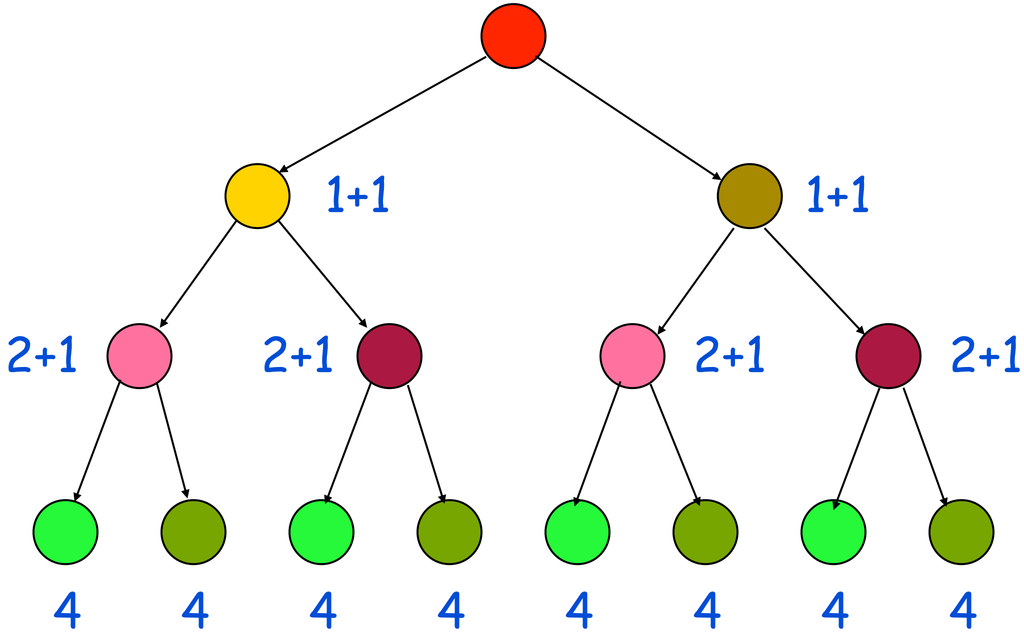


2n+1 states



$O(2^n)$ nodes

- It is not harmful to discard a node revisiting a state if the cost of the new path to this state is ≥ cost of the previous path
  [so, in particular, one can discard a node if it re-visits a state already visited by one of its ancestors]

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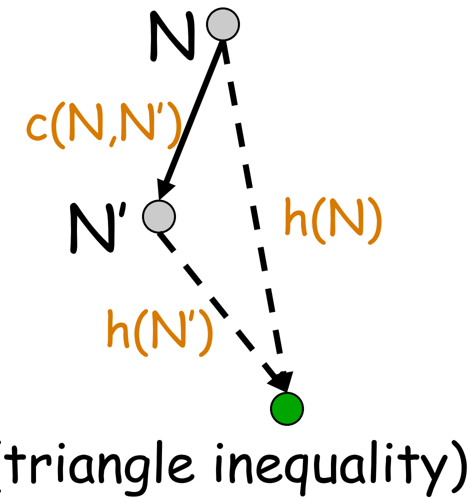
- Fortunately, for a large family of admissible heuristics – consistent heuristics – there is a much more efficient way to handle revisited states

Consistent Heuristic

An admissible heuristic h is consistent (or monotone) if for each node N and each child N' of N:

$$h(N) \leq c(N,N') + h(N')$$



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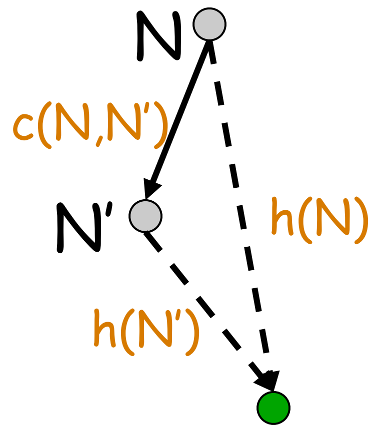

(triangle inequality)

→ Intuition: a consistent heuristics becomes more precise as we get deeper in the search tree

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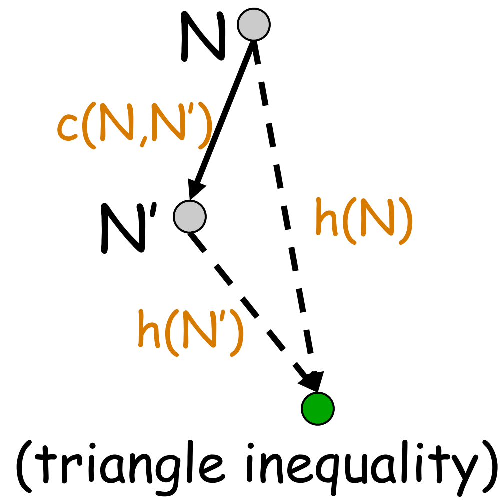
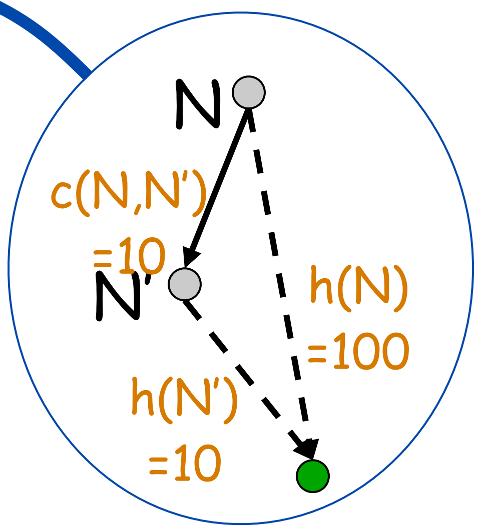


$$h(N) \leq C^*(N) \leq c(N,N') + h^*(N')$$
$$h(N) - c(N,N') \leq h^*(N')$$
$$h(N) - c(N,N') \leq h(N') \leq h^*(N')$$

→ Intuition: a consistent heuristics becomes more precise as we get deeper in the search tree

Consistency Violation

If h tells that N is 100 units from the goal, then moving from N along an arc costing 10 units should **not** lead to a node N' that h estimates to be 10 units away from the goal



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Consistent Heuristic
(alternative definition)

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1) for each node N and each child N' of N:

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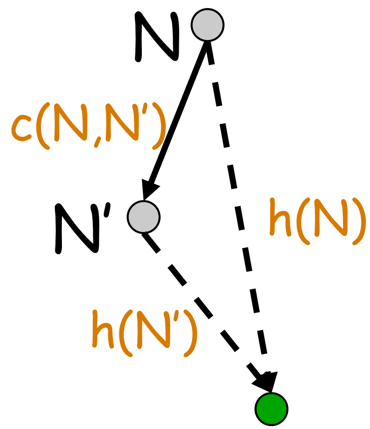
   $h(G) = 0$



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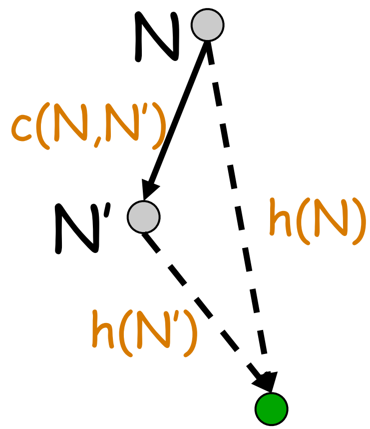


(triangle inequality)

A consistent heuristic is also admissible

Admissibility and Consistency

- A consistent heuristic is also admissible
- An admissible heuristic may not be consistent, but many admissible heuristics are consistent

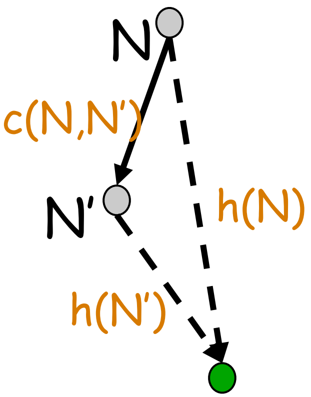
8-Puzzle

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

goal

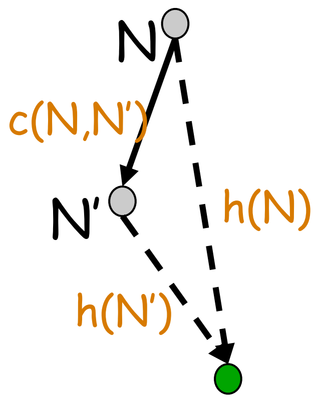


$h(N) \leq c(N,N') + h(N')$

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position

are both consistent (why?)

Robot Navigation



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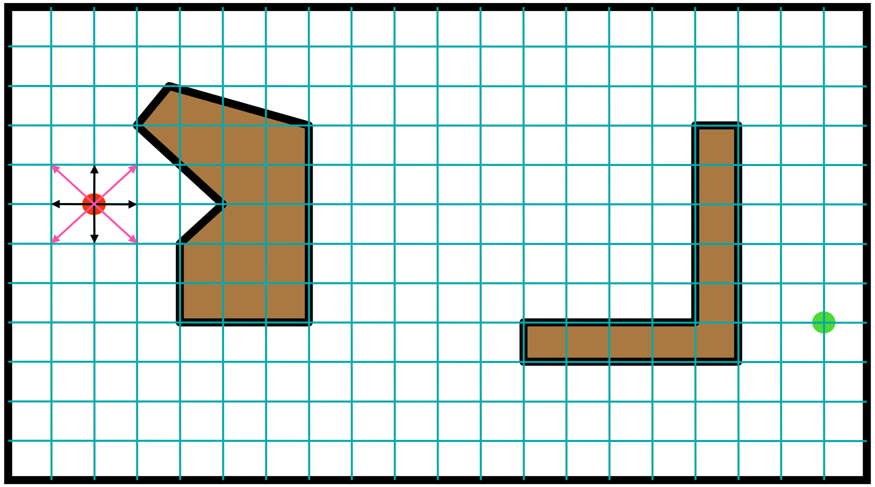


Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

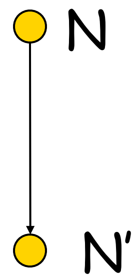
$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$ is consistent

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is consistent if moving along diagonals is not allowed, and not consistent otherwise

Result #2

If h is consistent, then whenever A* expands a node, it has already found an optimal path to this node's state

Proof (1/2)

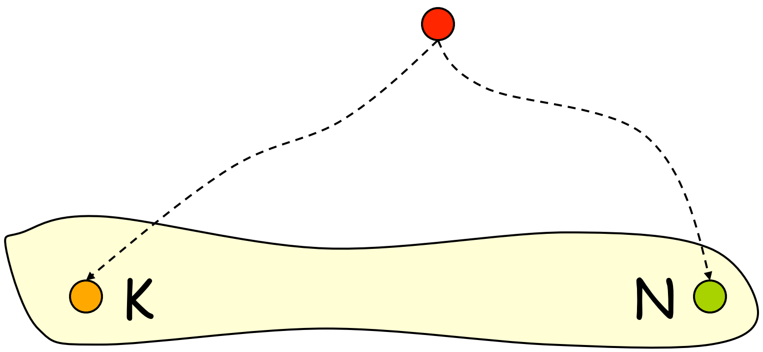


1) Consider a node N and its child N'
Since h is consistent: $h(N) \le c(N,N')+h(N')$

$$f(N) = g(N)+h(N) \le g(N)+c(N,N')+h(N') = f(N')$$

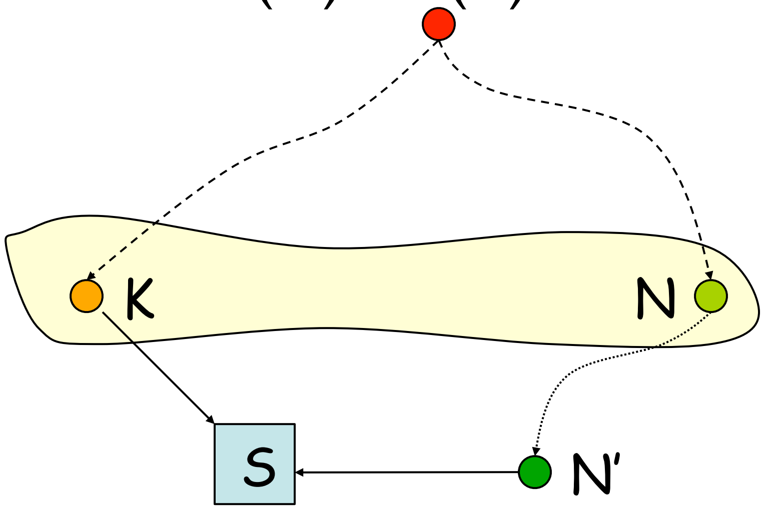
So, f is non-decreasing along any path

Proof (2/2)



Proof (2/2)

2) If a node K is selected for expansion, then any other node N in the fringe verifies f(N) ≥ f(K)



If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

f(N') ≥ f(N) ≥ f(K)   and   h(N') = h(K)

So, g(N') ≥ g(K)

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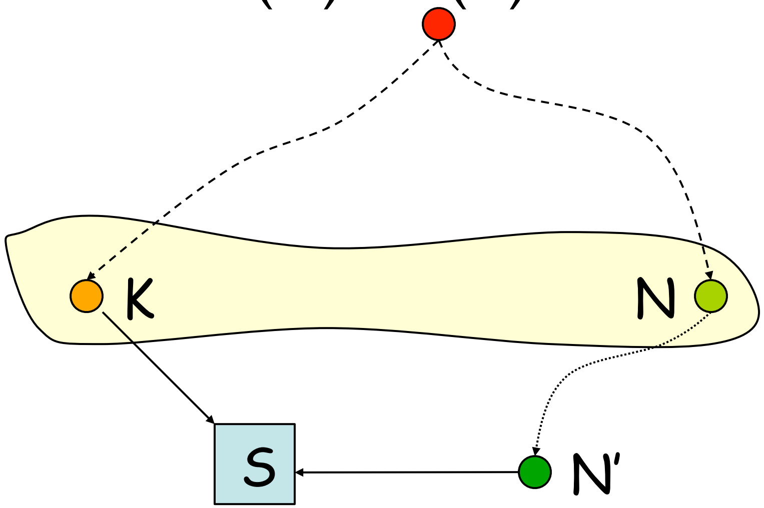

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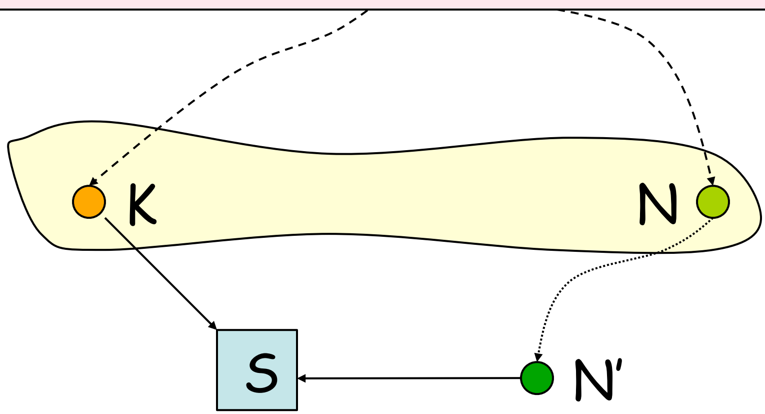

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Implication of Result #2

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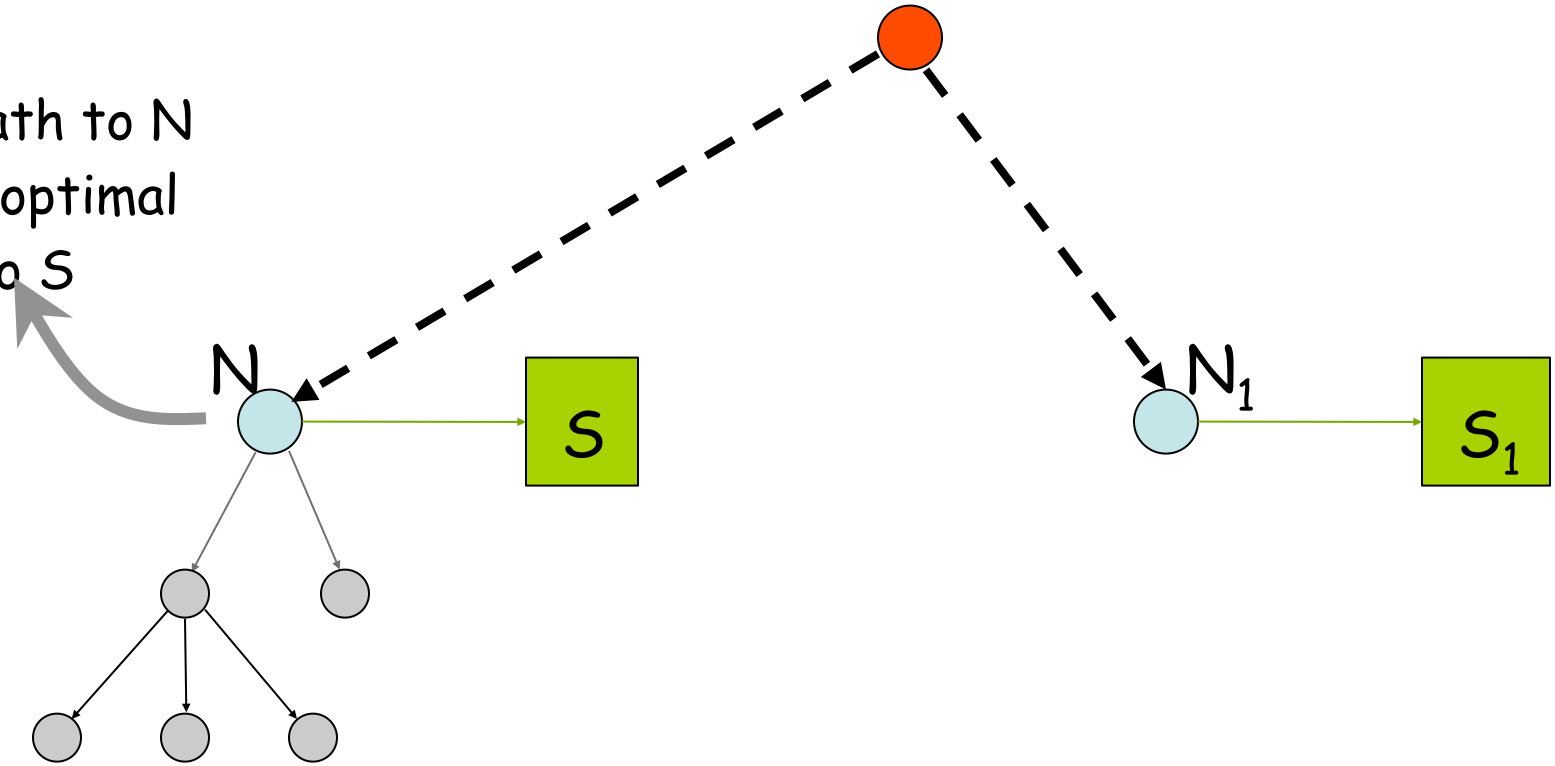
The path to N is the optimal path to S



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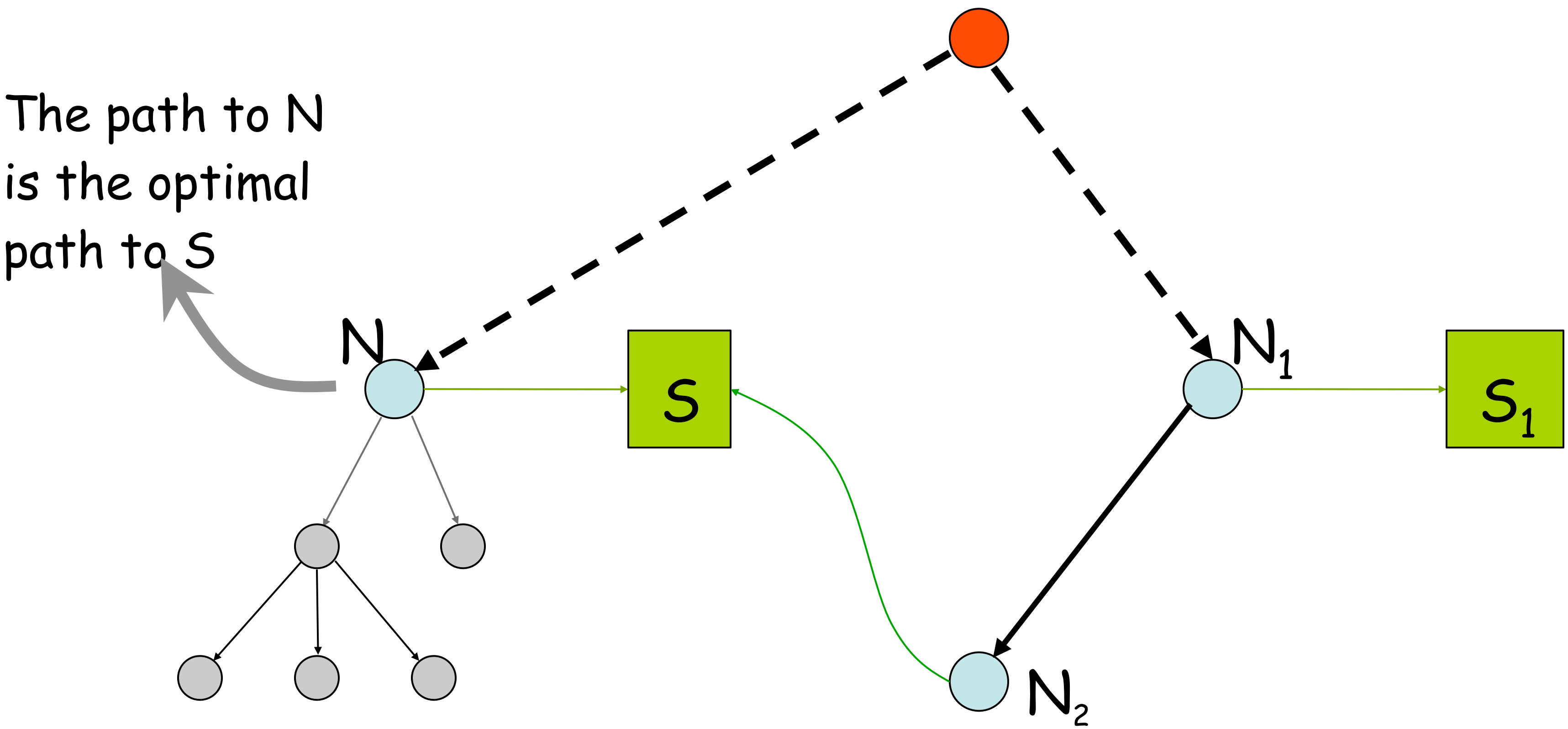

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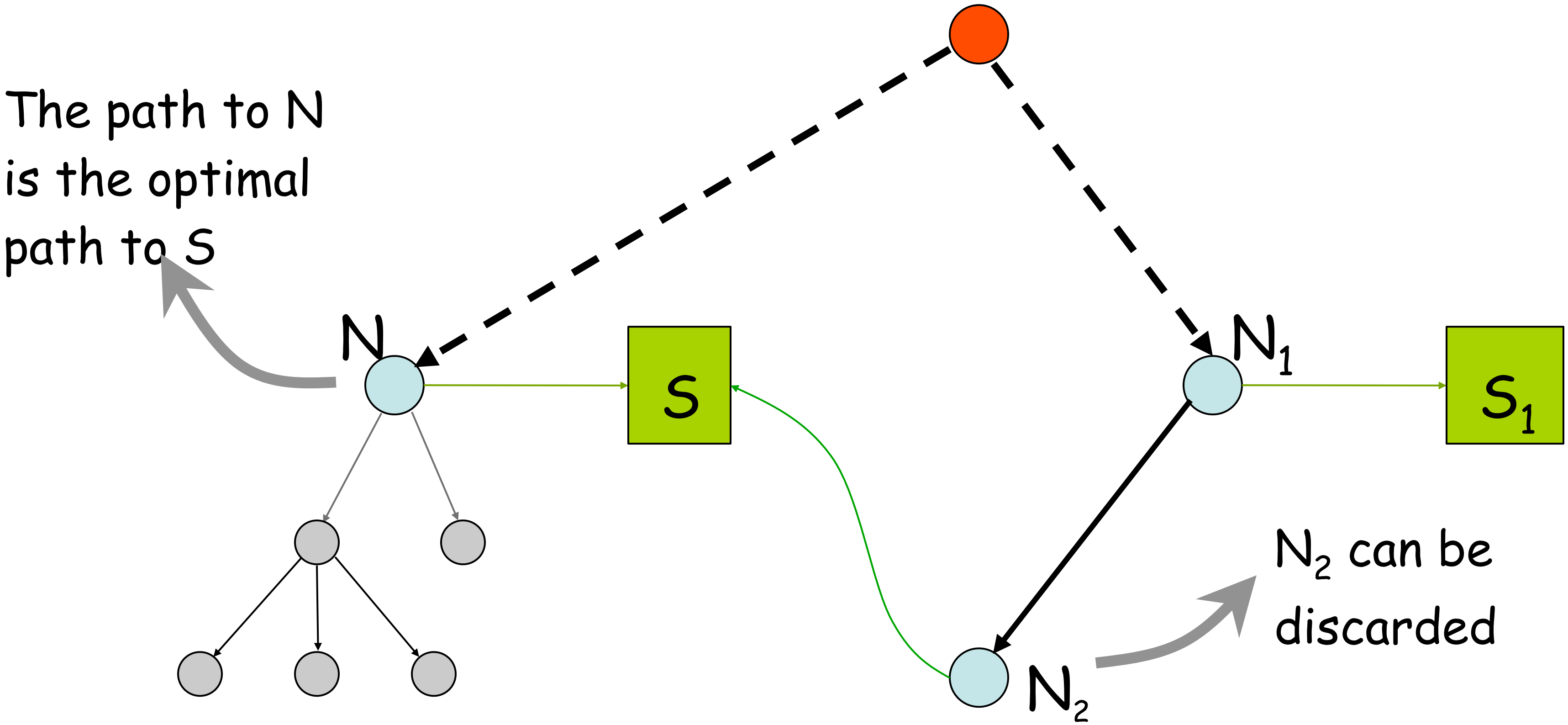

Revisited States with Consistent Heuristic

- When a node is expanded, store its state into CLOSED
- When a new node N is generated:
  - If STATE(N) is in CLOSED, discard N
  - If there exists a node N’ in the fringe such that STATE(N’) = STATE(N), discard the node – N or N’ – with the largest f (or, equivalently, g)

Is A* with some consistent heuristic all that we need?

No !

There are **very dumb** consistent heuristic functions

For example: $h \equiv 0$

- It is consistent (hence, admissible) !
- A* with h≡0 is uniform-cost search
- Breadth-first and uniform-cost are particular cases of A*

Heuristic Accuracy

Let h_1 and h_2 be two consistent heuristics such that for all nodes N:

$$h_1(N) \leq h_2(N)$$

h_2 is said to be more accurate (or more informed) than h_1

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Goal state

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of distances of every tile to its goal position
- h_2 is more accurate than h_1

Result #3

- Let h_2 be more accurate than h_1
- Let A_1^* be A^* using h_1 and A_2^* be A^* using h_2
- Whenever a solution exists, all the nodes expanded by A_2^*, except possibly for some nodes such that
  $f_1(N) = f_2(N) = C^*$ (cost of optimal solution)
  are also expanded by A_1^*

Proof

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- C* = h*(initial-node) [cost of optimal solution]

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Proof

- $C^* = h^*$(initial-node) [cost of optimal solution]
- Every node N such that $f(N) < C^*$ is eventually expanded. No node N such that $f(N) > C^*$ is ever expanded
- Every node N such that $h(N) < C^* - g(N)$ is eventually expanded. So, every node N such that $h_2(N) < C^* - g(N)$ is expanded by A_2^*. Since $h_1(N) \leq h_2(N)$, N is also expanded by A_1^*

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- If there are several nodes N such that $f_1(N) = f_2(N) = C^*$ (such nodes include the optimal goal nodes, if there exists a solution), A_1^* and A_2^* may or may not expand them in the same order (until one goal node is expanded)

Effective Branching Factor

- It is used as a measure the effectiveness of a heuristic
- Let n be the total number of nodes expanded by A* for a particular problem and d the depth of the solution
- The effective branching factor b* is defined by $n = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$

Experimental Results

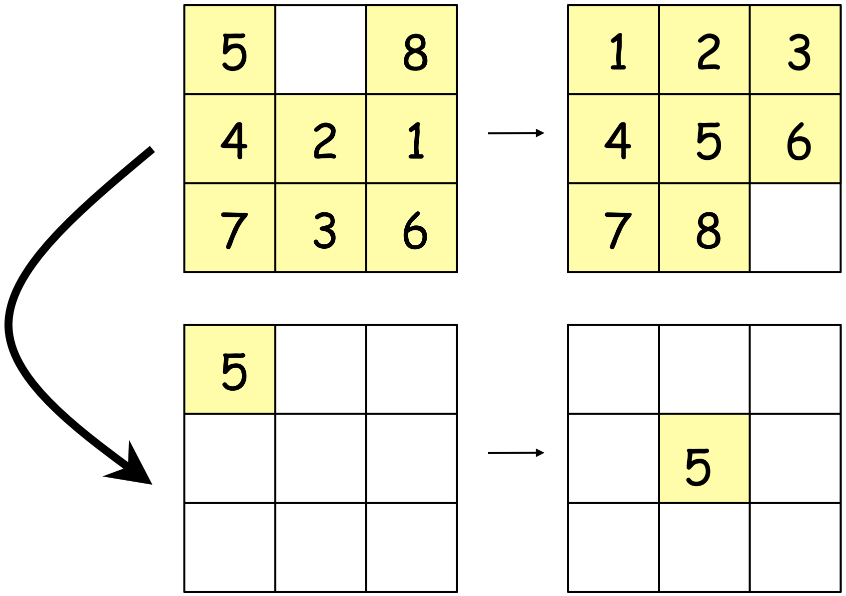
(see R&N for details)

- 8-puzzle with:
  - h_1 = number of misplaced tiles
  - h_2 = sum of distances of tiles to their goal positions
- Random generation of many problem instances
- Average effective branching factors (number of expanded nodes):

d	IDS	A_1*	A_2*
2	2.45	1.79	1.79
6	2.73	1.34	1.30
12	2.78 (3,644,035)	1.42 (227)	1.24 (73)
16	--	1.45	1.25
20	--	1.47	1.27
24	--	1.48 (39,135)	1.26 (1,641)

How to create good heuristics?

- By solving relaxed problems at each node
- In the 8-puzzle, the sum of the distances of each tile to its goal position (h_2) corresponds to solving 8 simple problems:

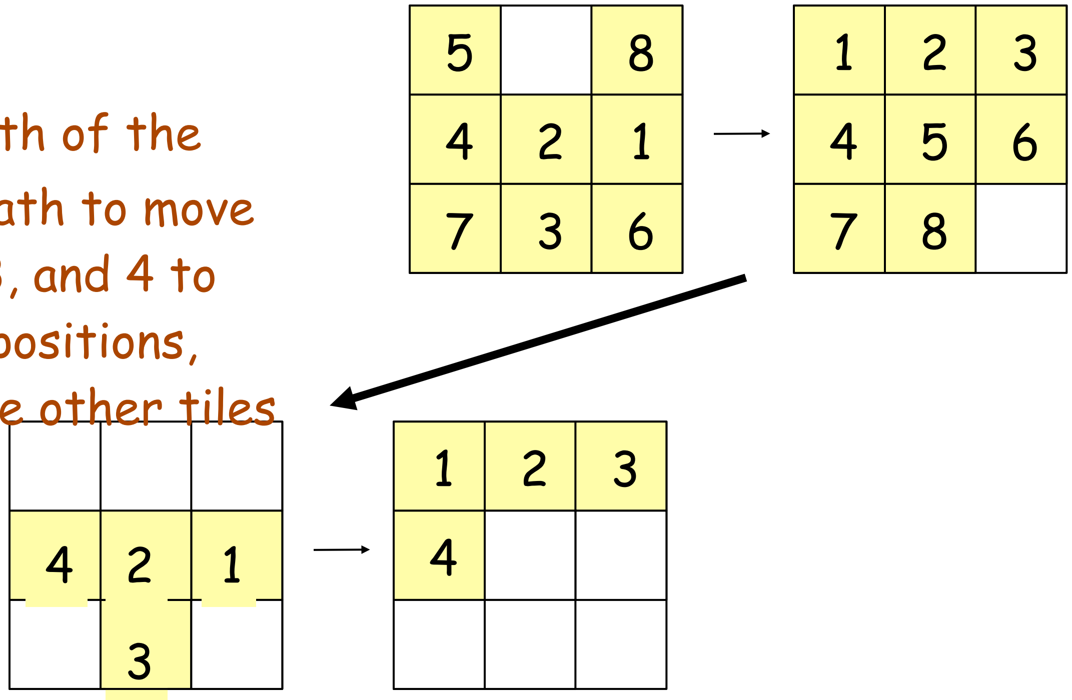


d_i is the length of the shortest path to move tile i to its goal position, ignoring the other tiles, e.g., $d_5 = 2$

$h_2 = \Sigma_{i=1,...8}\ d_i$

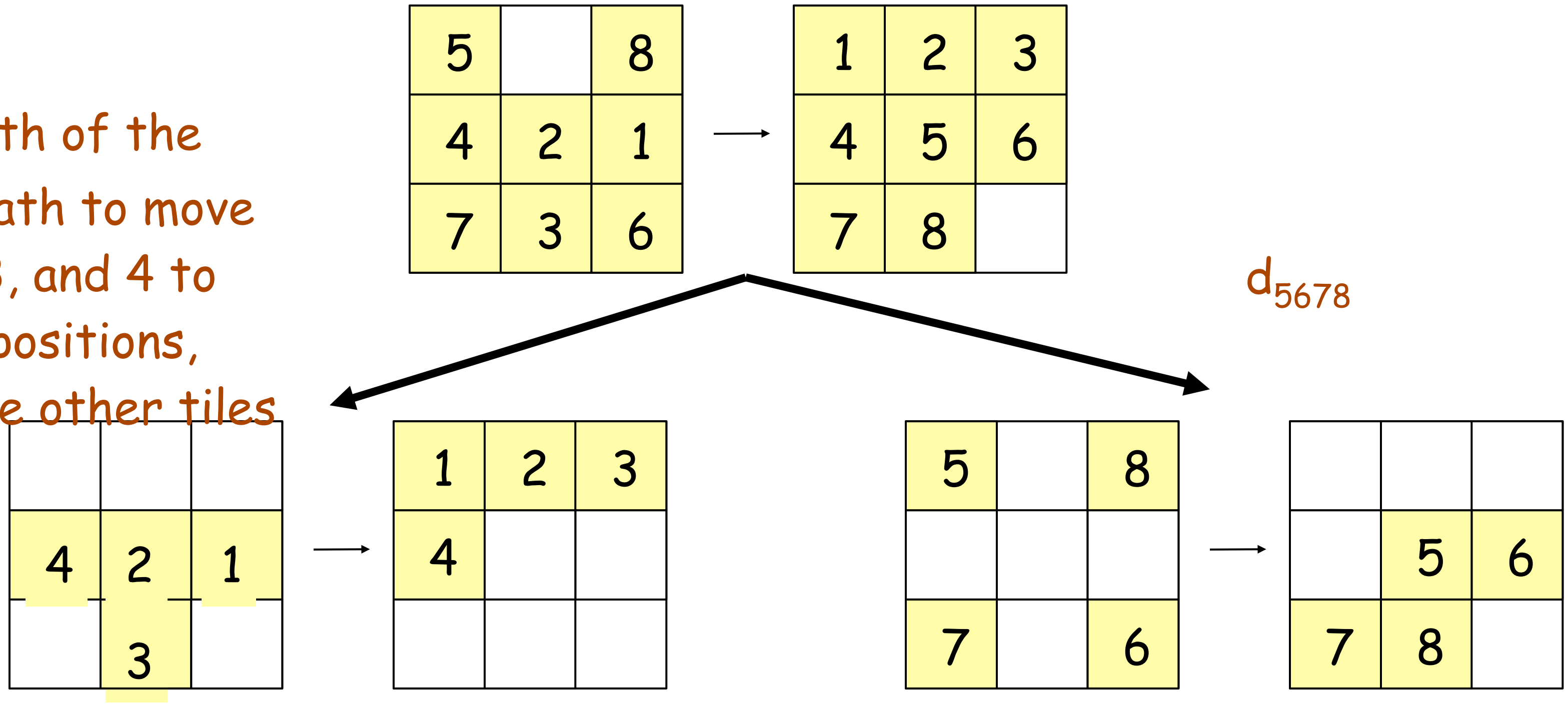
Can we do better?

d_{1234} = length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles



Can we do better?

d_{1234} = length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles



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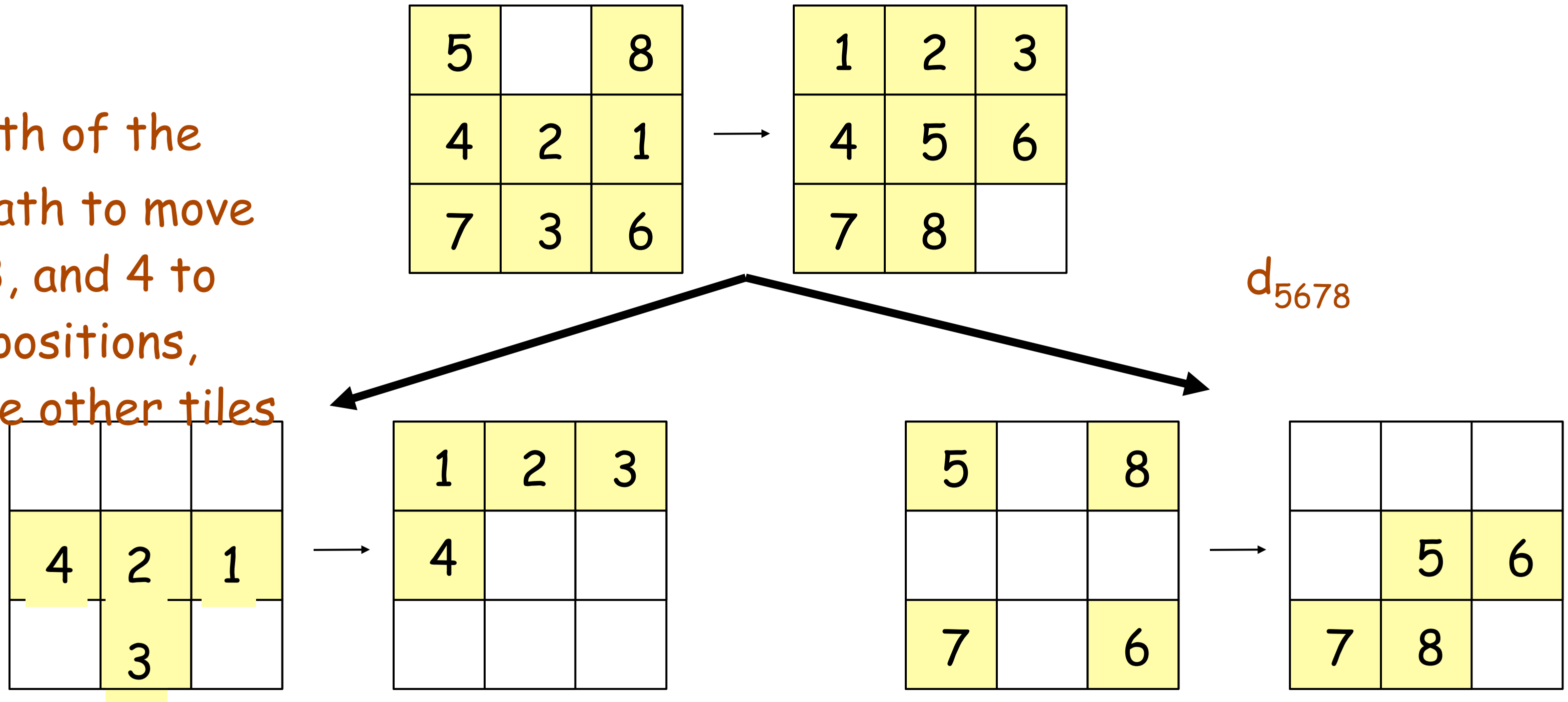

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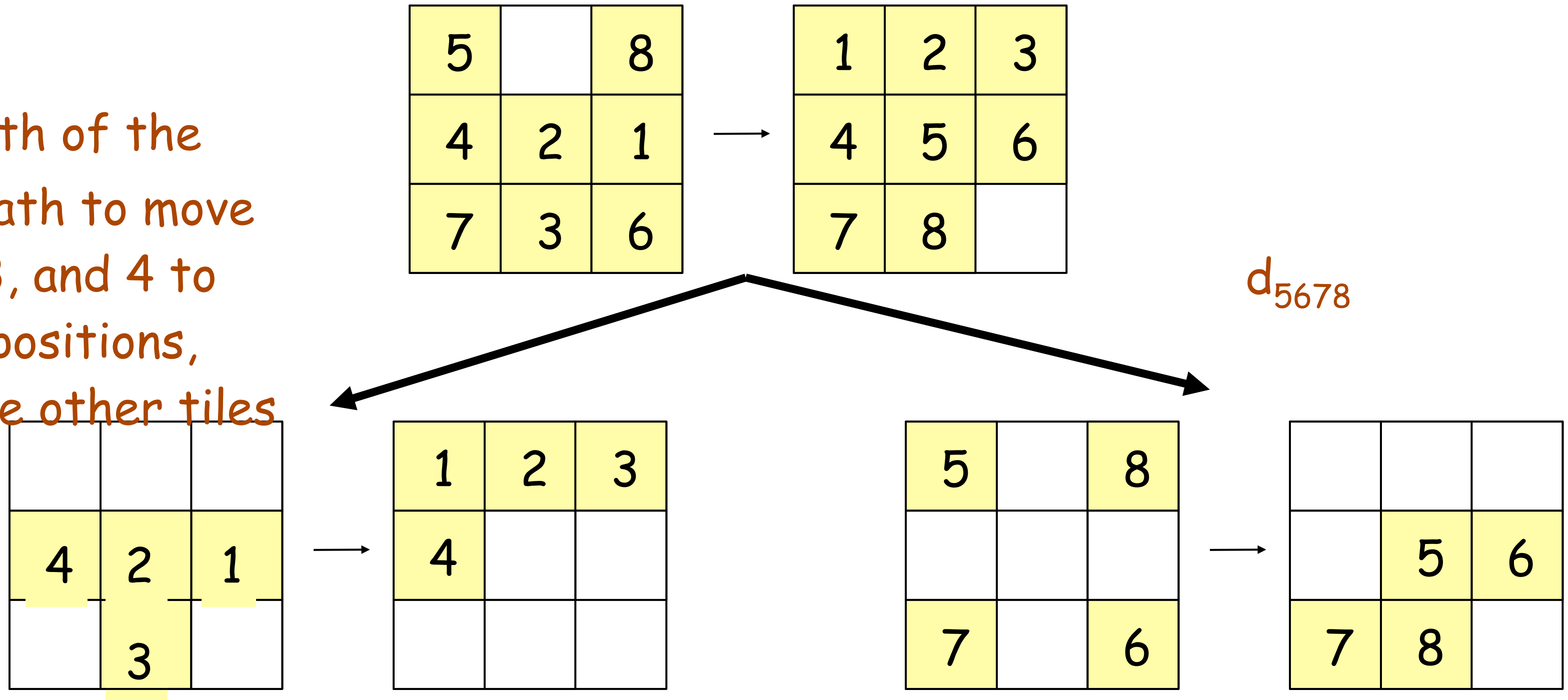

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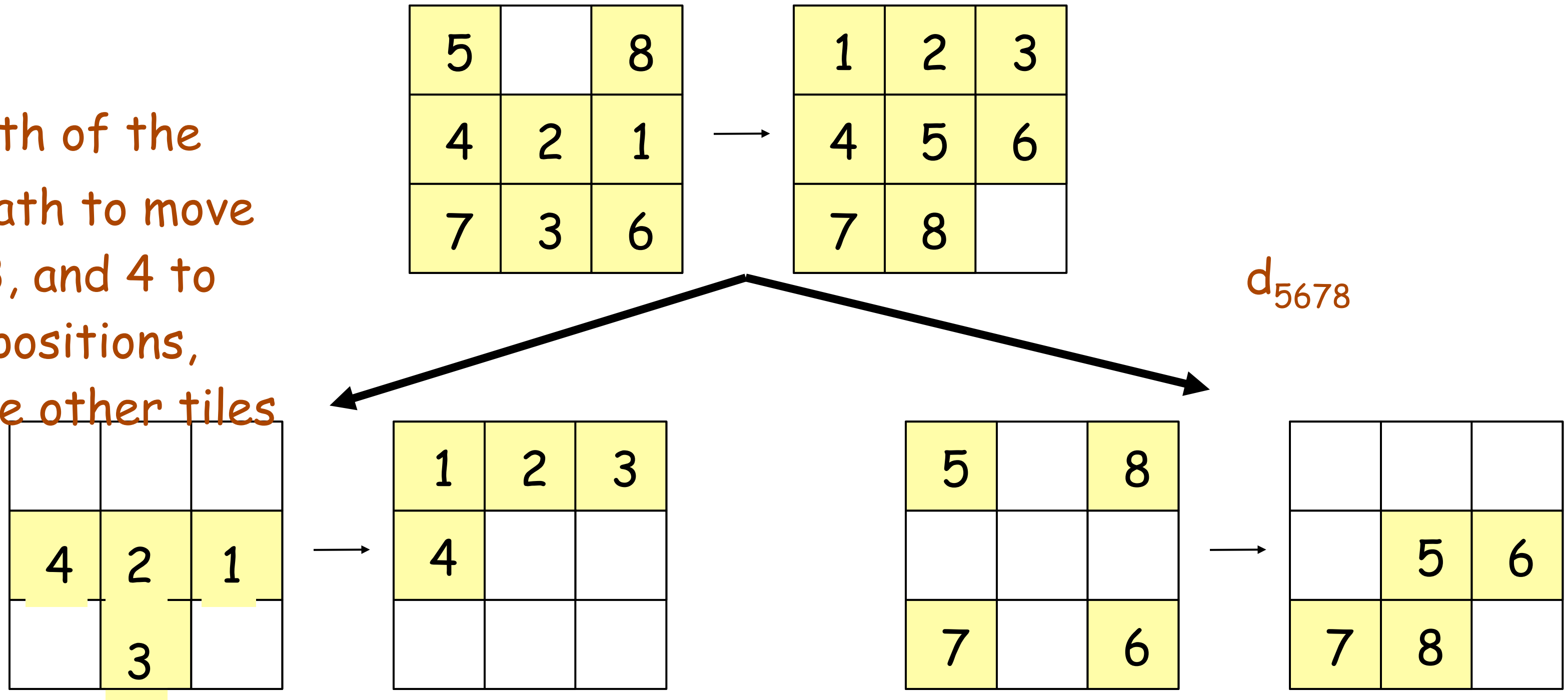

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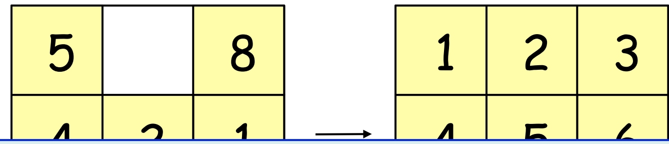


d = length of the

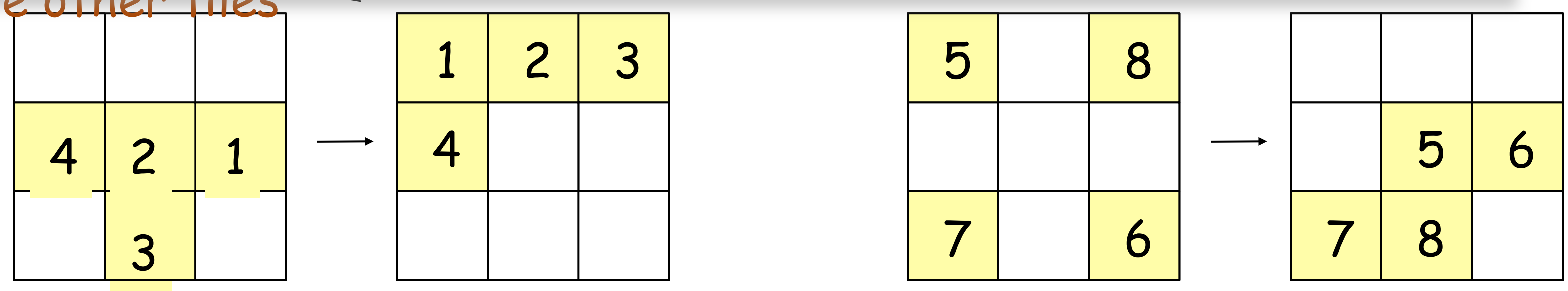
s

t

t

→ Several order-of-magnitude speedups for the 15- and 24-puzzle (see R&N)

ignoring the other tiles



On Completeness and

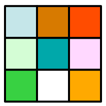
- A* with a consistent heuristic function has nice properties: completeness, optimality, no need to revisit states
- Theoretical completeness does not mean "practical" completeness if you must wait too long to get a solution (remember the time limit issue)
- So, if one can't design an accurate consistent heuristic, it may be better to settle for a non-admissible heuristic that "works well in practice", even through completeness and optimality are no longer guaranteed

Iterative Deepening A* (IDA*)

- Idea: Reduce memory requirement of A* by applying cutoff on values of f
- Consistent heuristic function h
- Algorithm IDA*:
  1. Initialize cutoff to f(initial-node)
  2. Repeat:
     a. Perform depth-first search by expanding all nodes N such that $f(N) \leq$ cutoff
     b. Reset cutoff to smallest value f of non-expanded (leaf) nodes

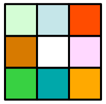
8-Puzzle

f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles



4

Cutoff=4



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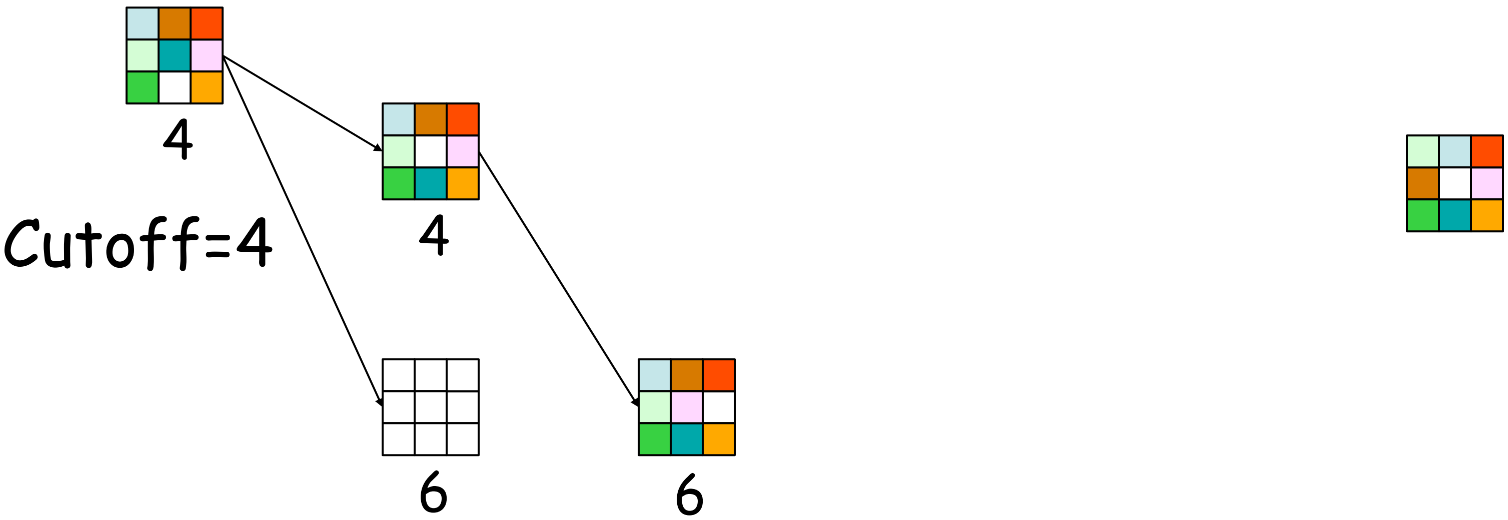

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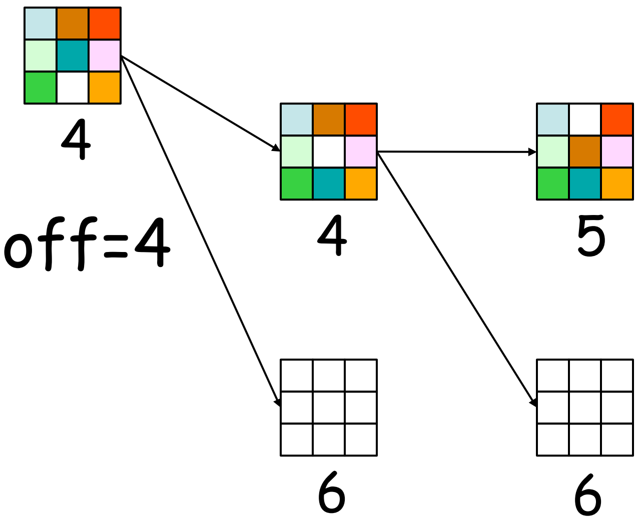

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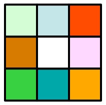

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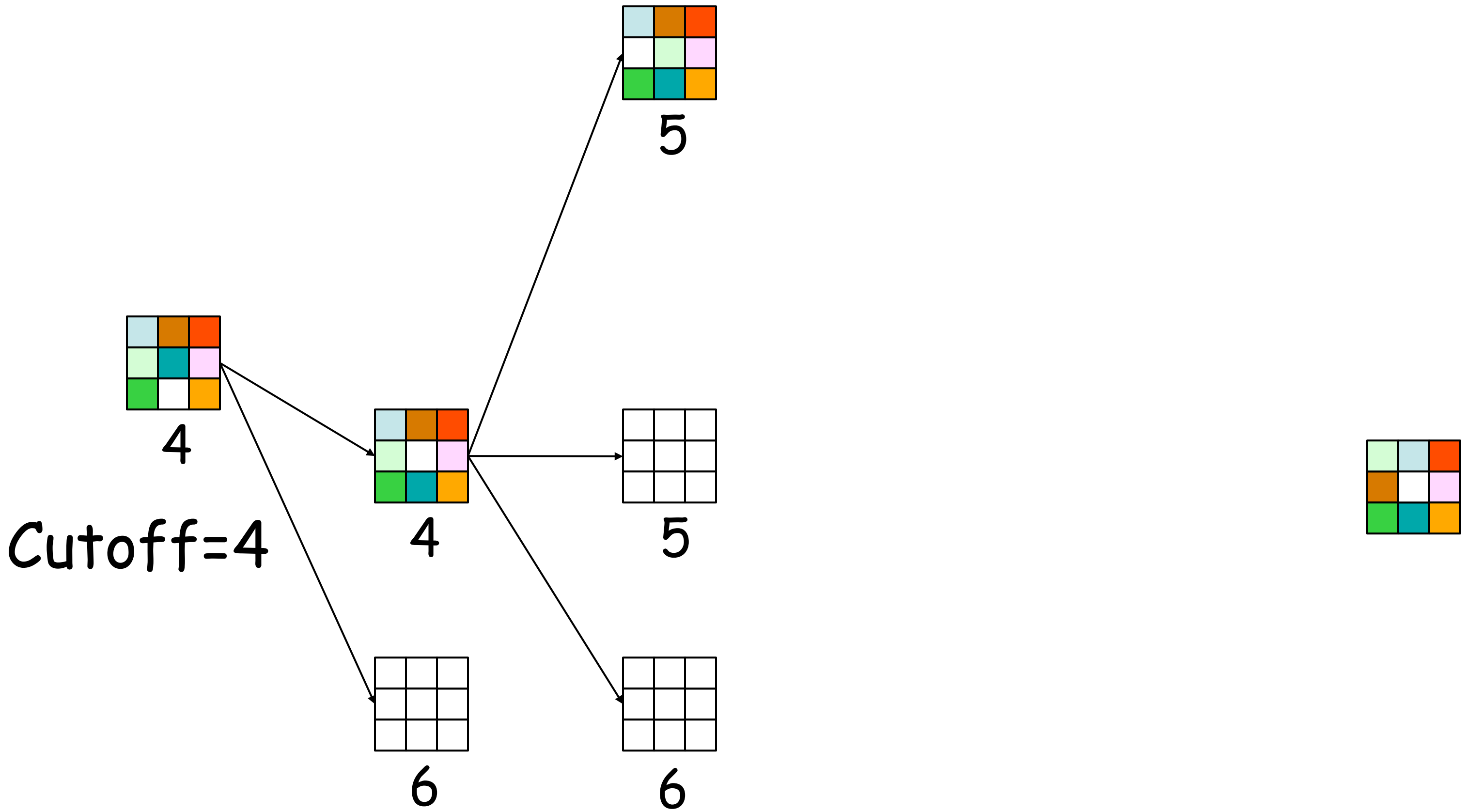

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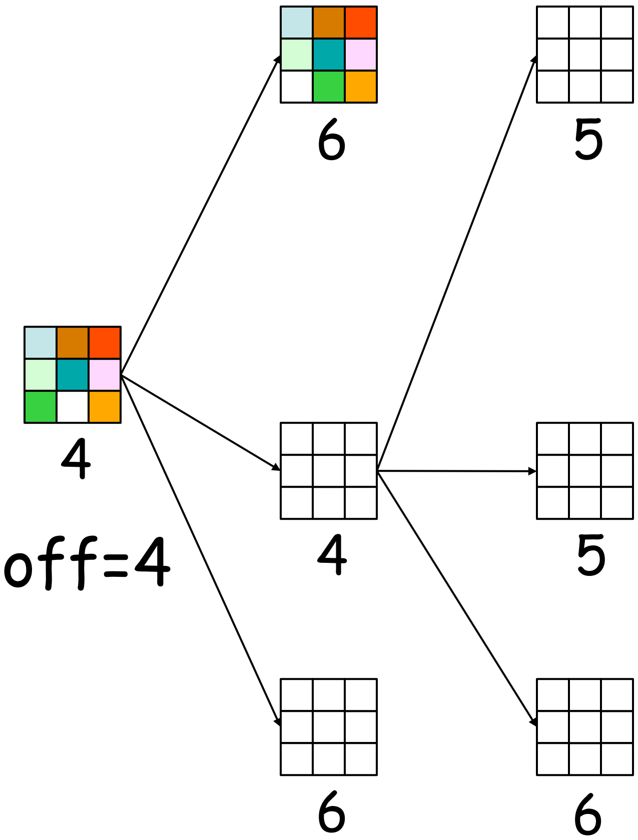

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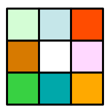

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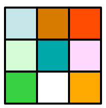
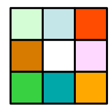


4

Cutoff=5



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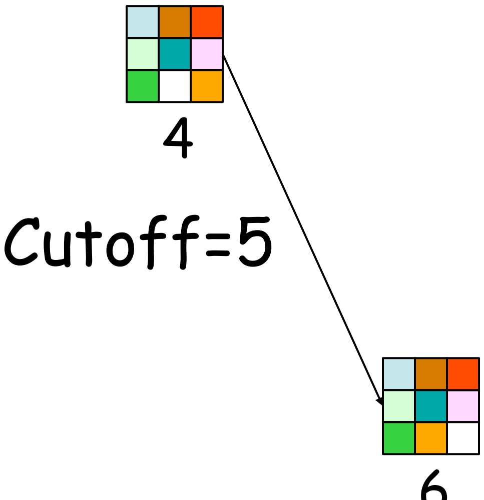

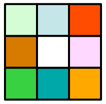

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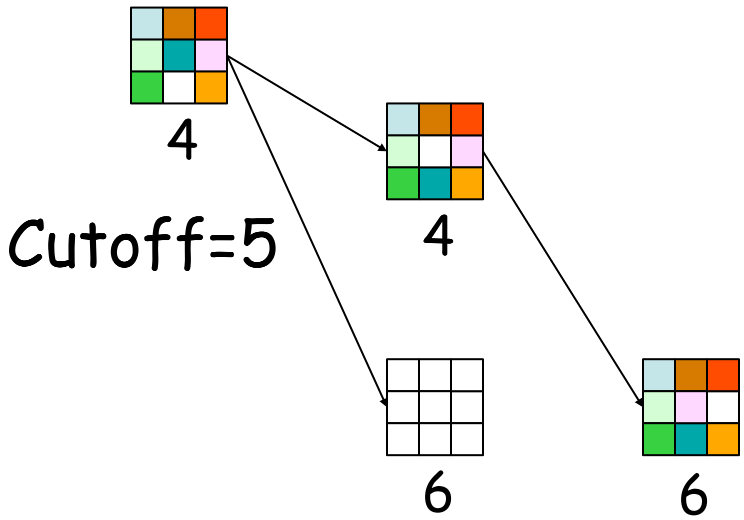

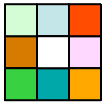

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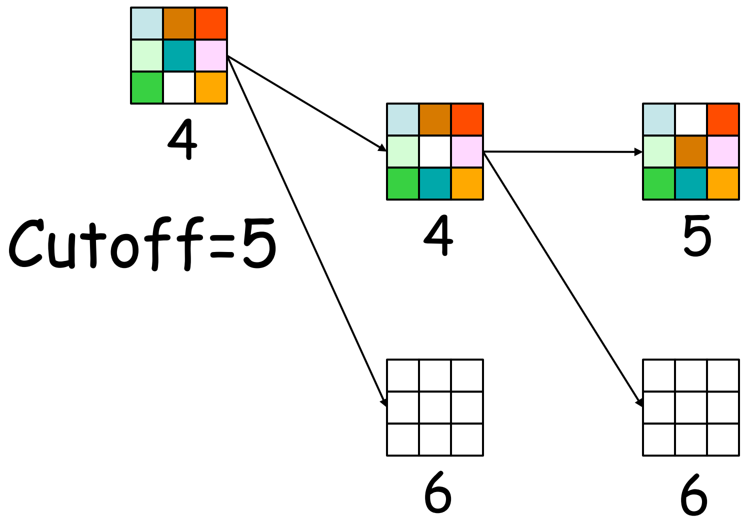

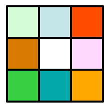

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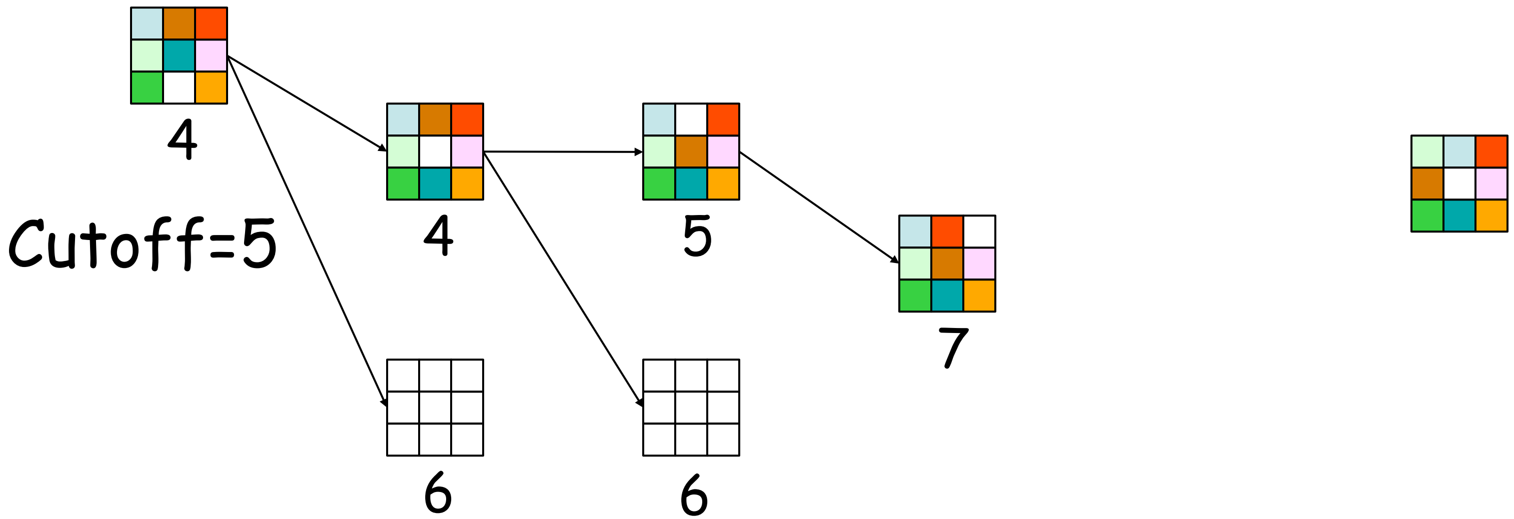

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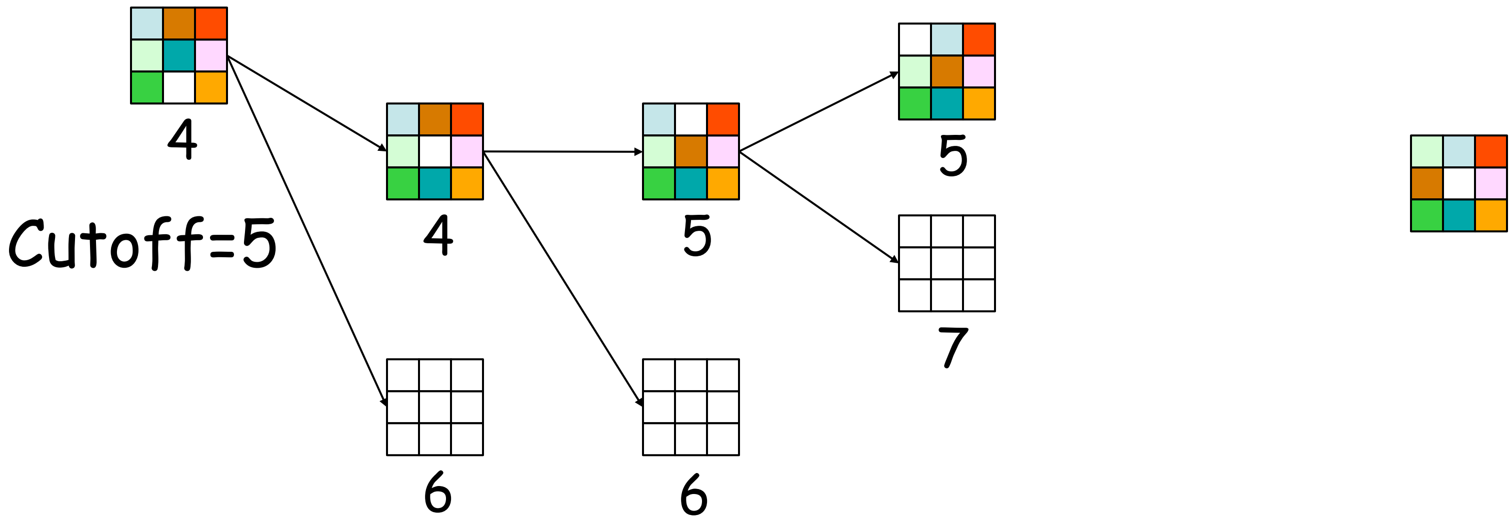

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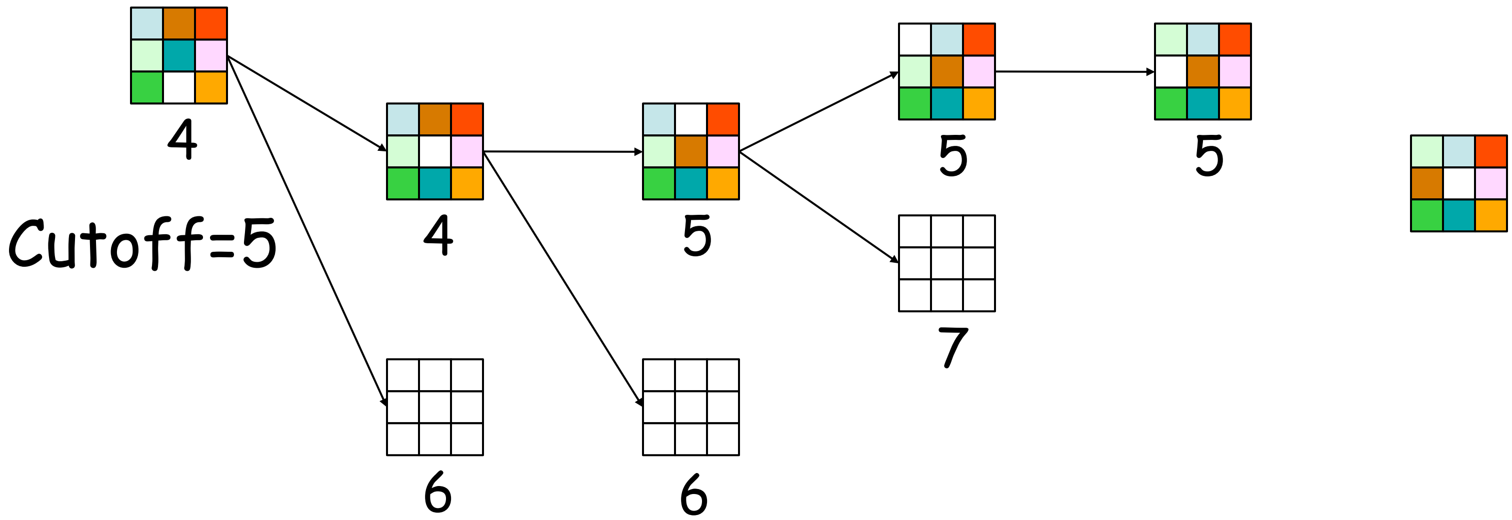

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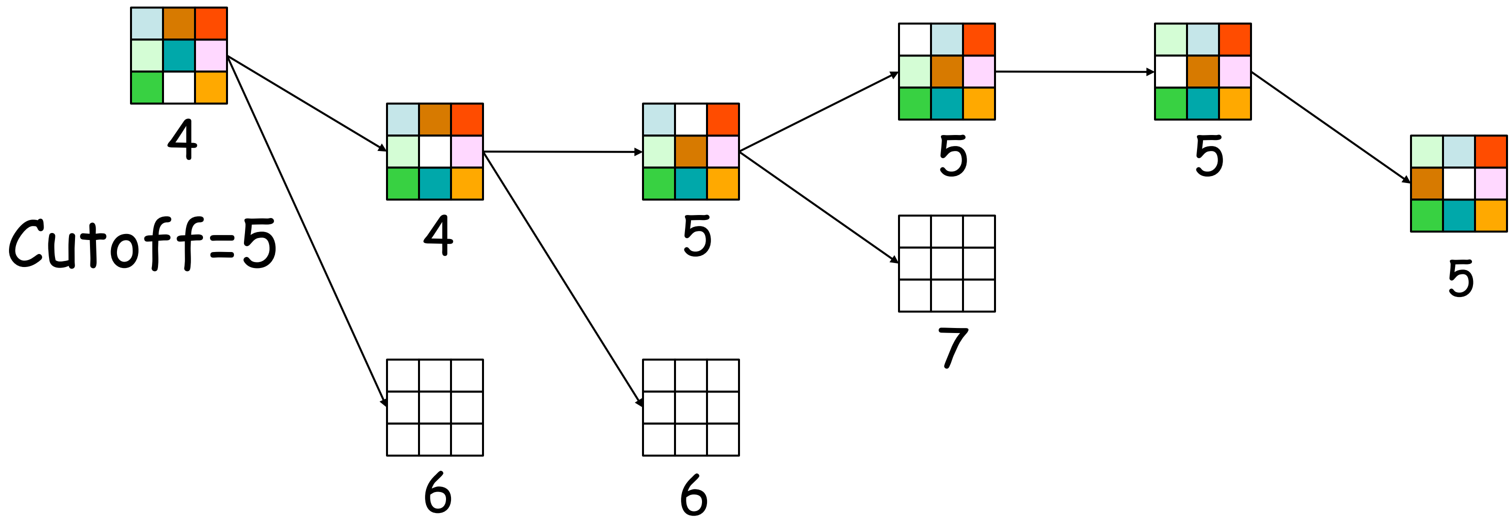


Advantages/Drawbacks of IDA*

- Advantages:
  - Still complete and optimal
  - Requires less memory than A*
  - Avoid the overhead to sort the fringe
- Drawbacks:
  - Can't avoid revisiting states not on the current path
  - Available memory is poorly used
    (memory-bounded search, see R&N p. 101-104)

Local Search

- Light-memory search method
- No search tree; only the current state is represented!
- Only applicable to problems where the path is irrelevant (e.g., 8-queen), unless the path is encoded in the state
- Many similarities with optimization techniques

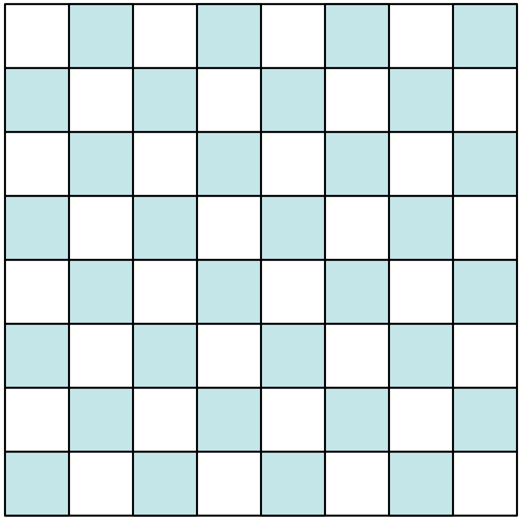
Steepest Descent

1) S $\leftarrow$ initial state
2) Repeat:
   a) S' $\leftarrow$ arg $\min_{S' \in SUCCESSORS(S)}\{h(S')\}$
   b) if GOAL?(S') return S'
   c) if $h(S') < h(S)$ then S $\leftarrow$ S' else return failure

Similar to:

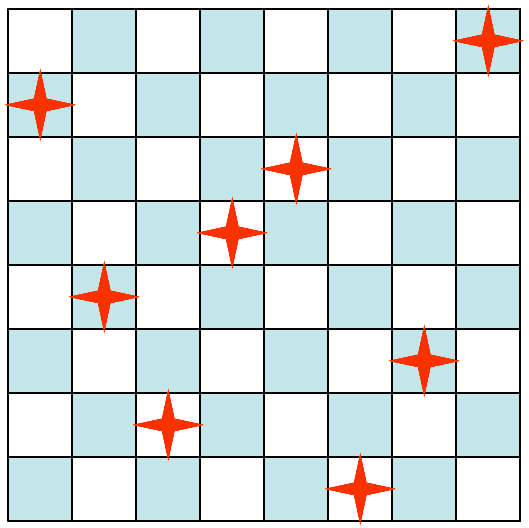
- hill climbing with –h
- gradient descent over continuous space

Application: 8-Queen



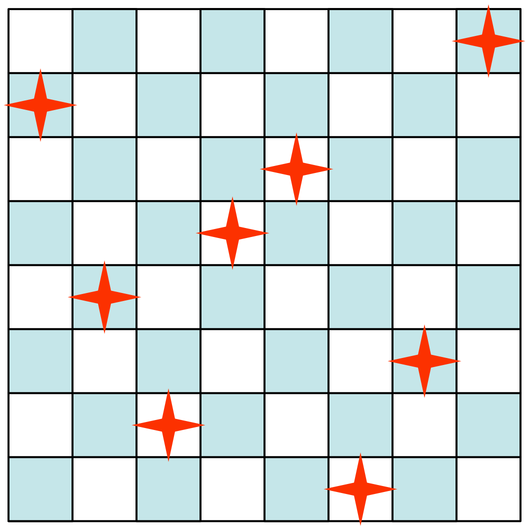
Application: 8-Queen

1) Pick an initial state S at random with one queen in each column



Application: 8-Queen

1) Pick an initial state S at random with one queen in each column
2) Repeat k times:
    a) If GOAL?(S) then return S



Application: 8-Queen

1) Pick an initial state S at random with one queen in each column
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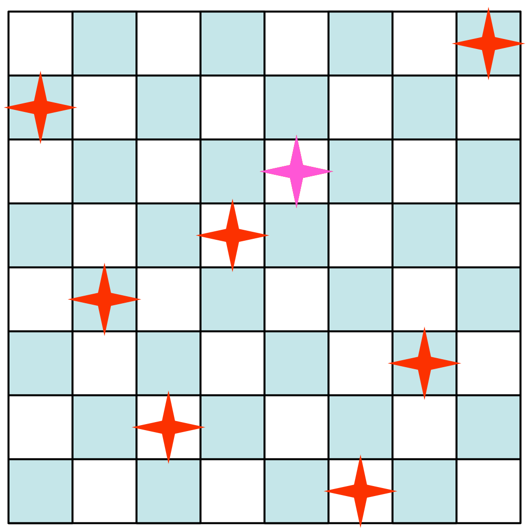
   b) Pick an attacked queen Q at random



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1) Pick an initial state S at random with one queen in each column
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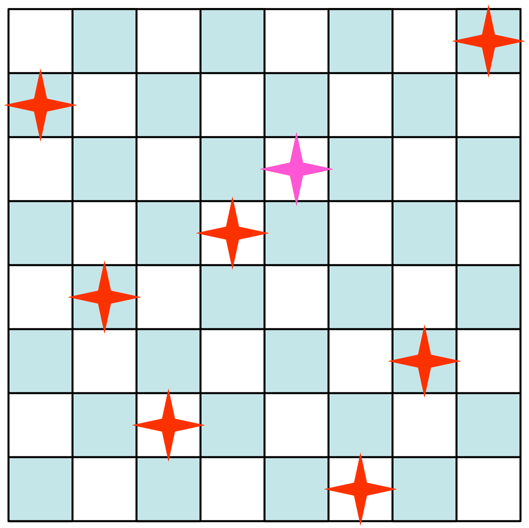
   c) Move Q in its column to minimize the number of attacking queens new S [min-conflicts heuristic]



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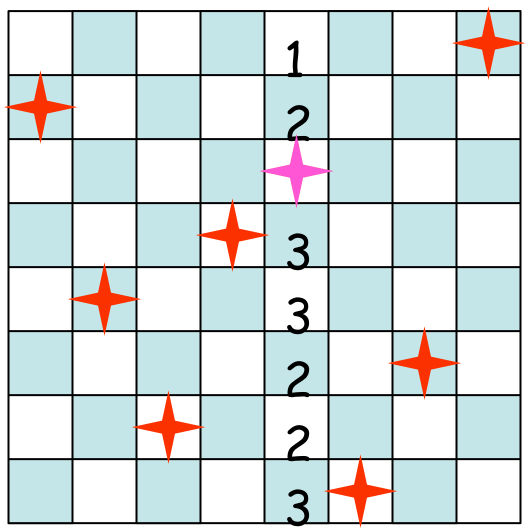

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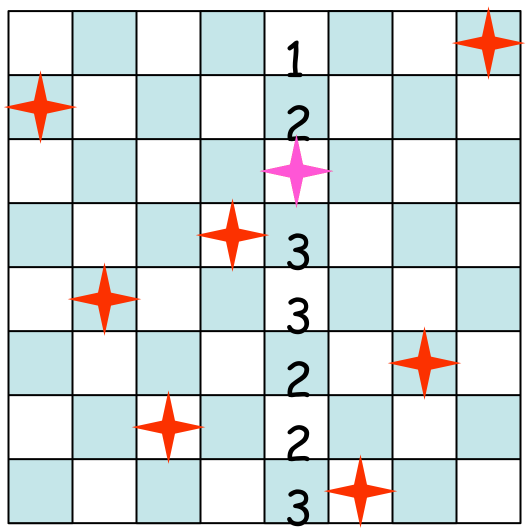

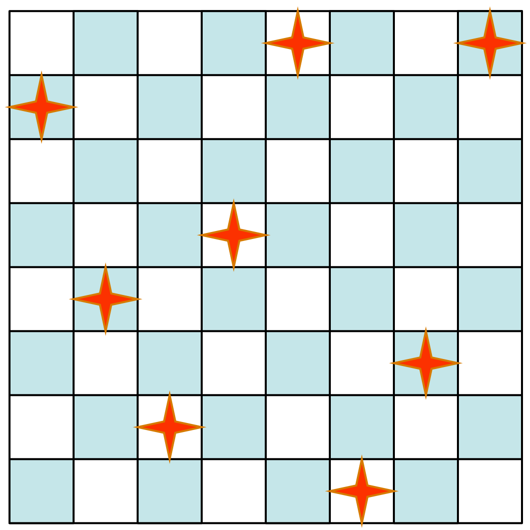

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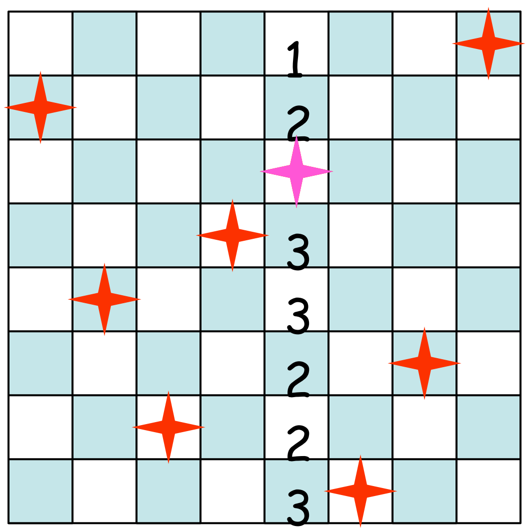

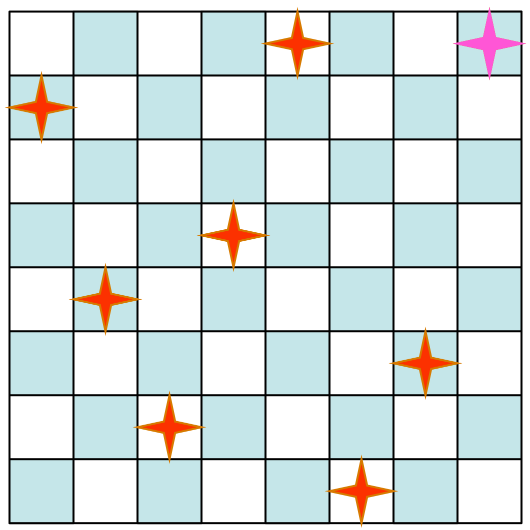

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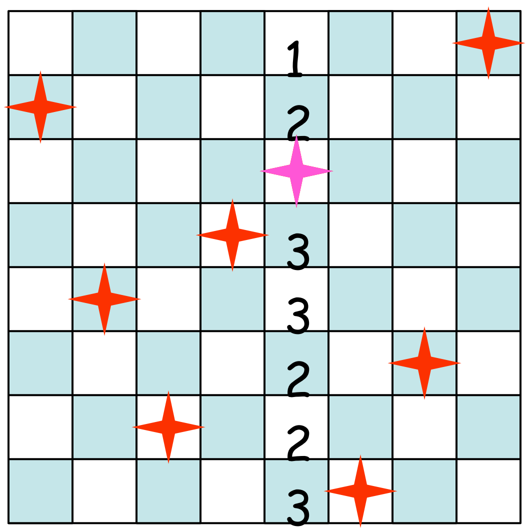

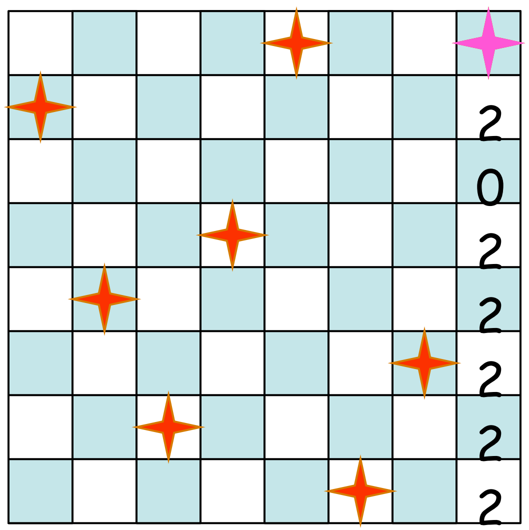

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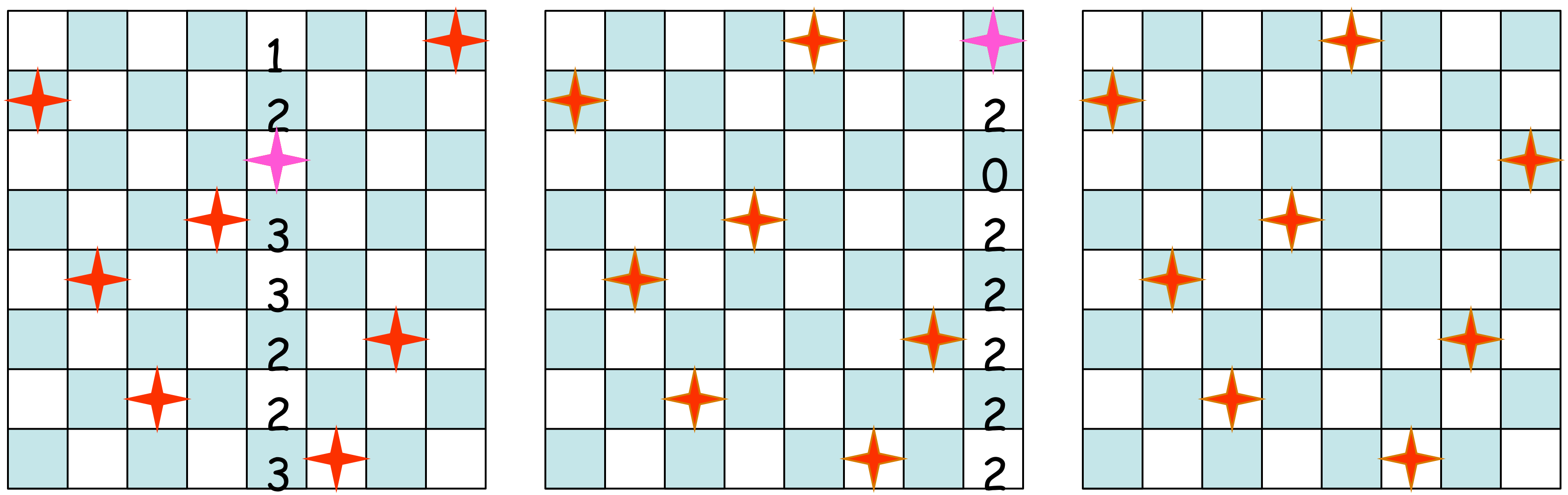

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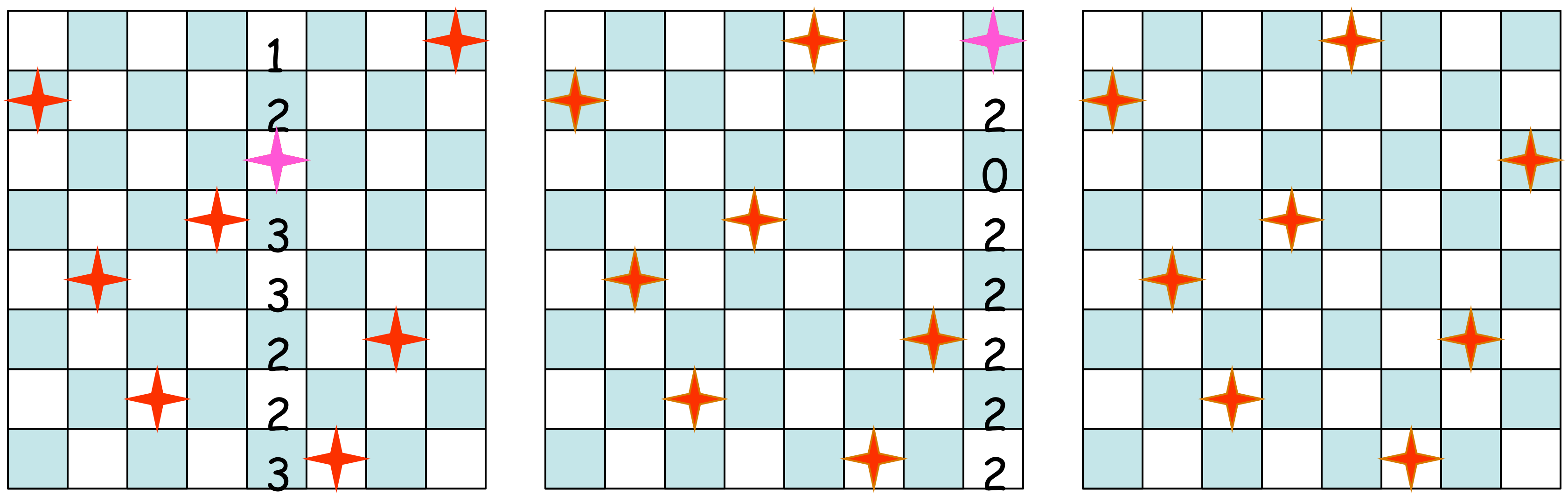
3) Return failure



Application: 8-Queen

Repeat n times:

1) Pick an initial state S at random with one queen in each column
2) Repeat k times:
    a) If GOAL?(S) then return S
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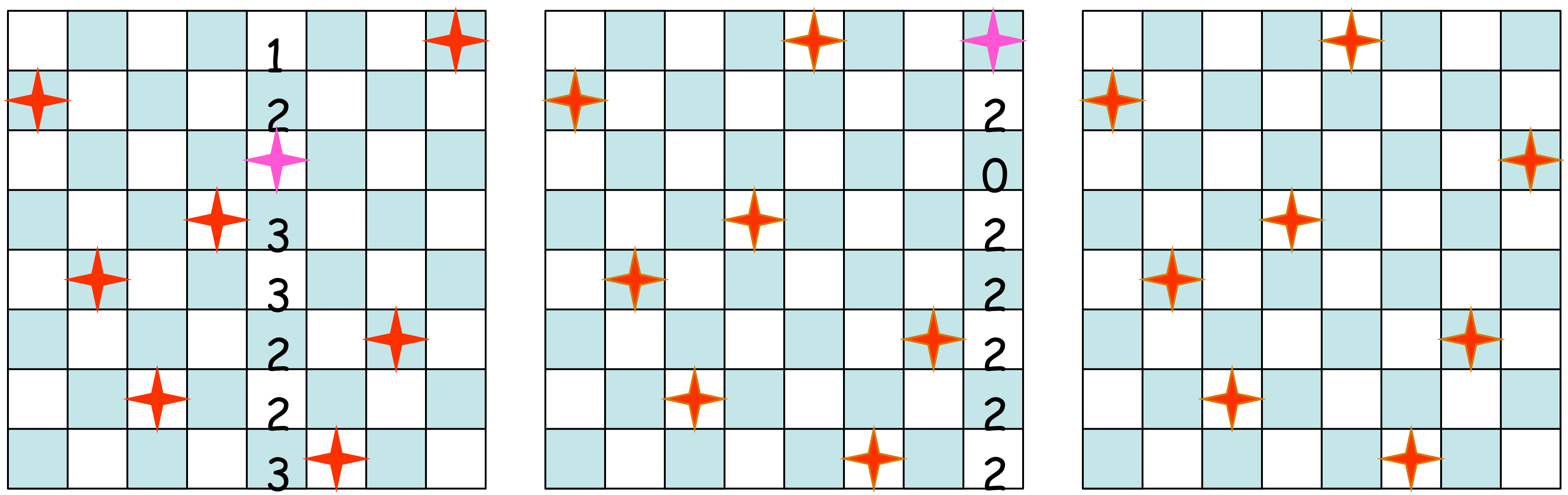
    c) Move Q in its column to minimize the number of attacking queens → new S [min-conflicts heuristic]
3) Return failure



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Repeat n times:

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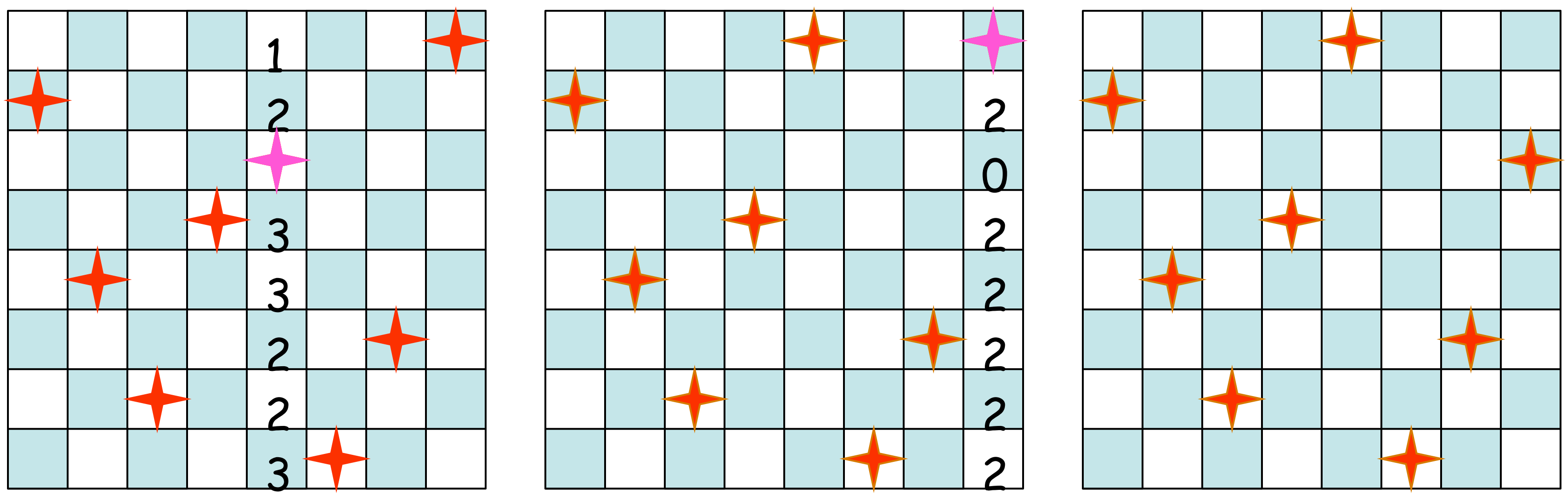
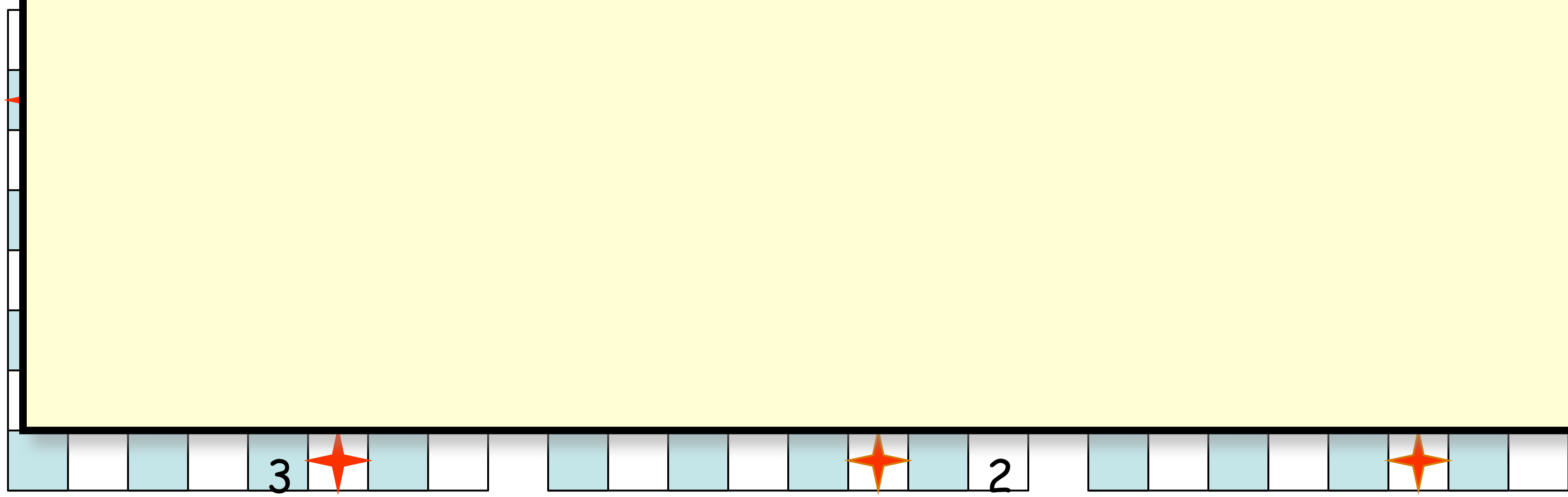


Application: 8-Queen

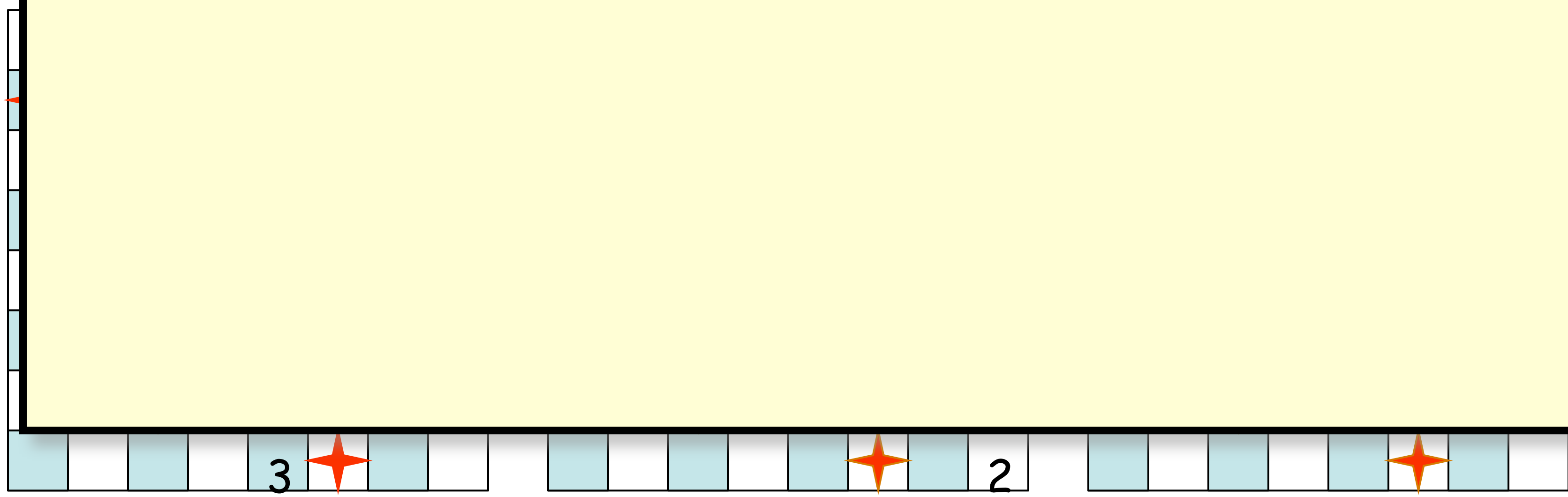
Why does it work ???



Application: 8-Queen

Why does it work ???

1) There are **many** goal states that are well-distributed over the state space



Application: 8-Queen

Why does it work ???

1) There are **many** goal states that are well-distributed over the state space

2) If no solution has been found after a few steps, it's better to start it all over again. Building a search tree would be much less efficient because of the high branching factor

Application: 8-Queen

Why does it work ???

1) There are **many** goal states that are well-distributed over the state space
2) If no solution has been found after a few steps, it's better to start it all over again. Building a search tree would be much less efficient because of the high branching factor
3) Running time almost independent of the number of queens

Steepest Descent

1) S $\leftarrow$ initial state
2) Repeat:
   a) S' $\leftarrow$ arg $\min_{S' \in SUCCESSORS(S)}$ {h(S')}
   b) if GOAL?(S') return S'
   c) if h(S') < h(S) then S $\leftarrow$ S' else return failure

may easily get stuck in local minima

→ Random restart (as in n-queen example)

→ Monte Carlo descent

Monte Carlo Descent

1) S $\leftarrow$ initial state
2) Repeat k times:
   a) If GOAL?(S) then return S
   b) S' $\leftarrow$ successor of S picked at random
   c) if $h(S') \leq h(S)$ then S $\leftarrow$ S'
   d) else
      - $\Delta h = h(S') - h(S)$
      - with probability ~ $\exp(-\Delta h/T)$, where T is called the "temperature", do: S $\leftarrow$ S' [Metropolis criterion]
3) Return failure

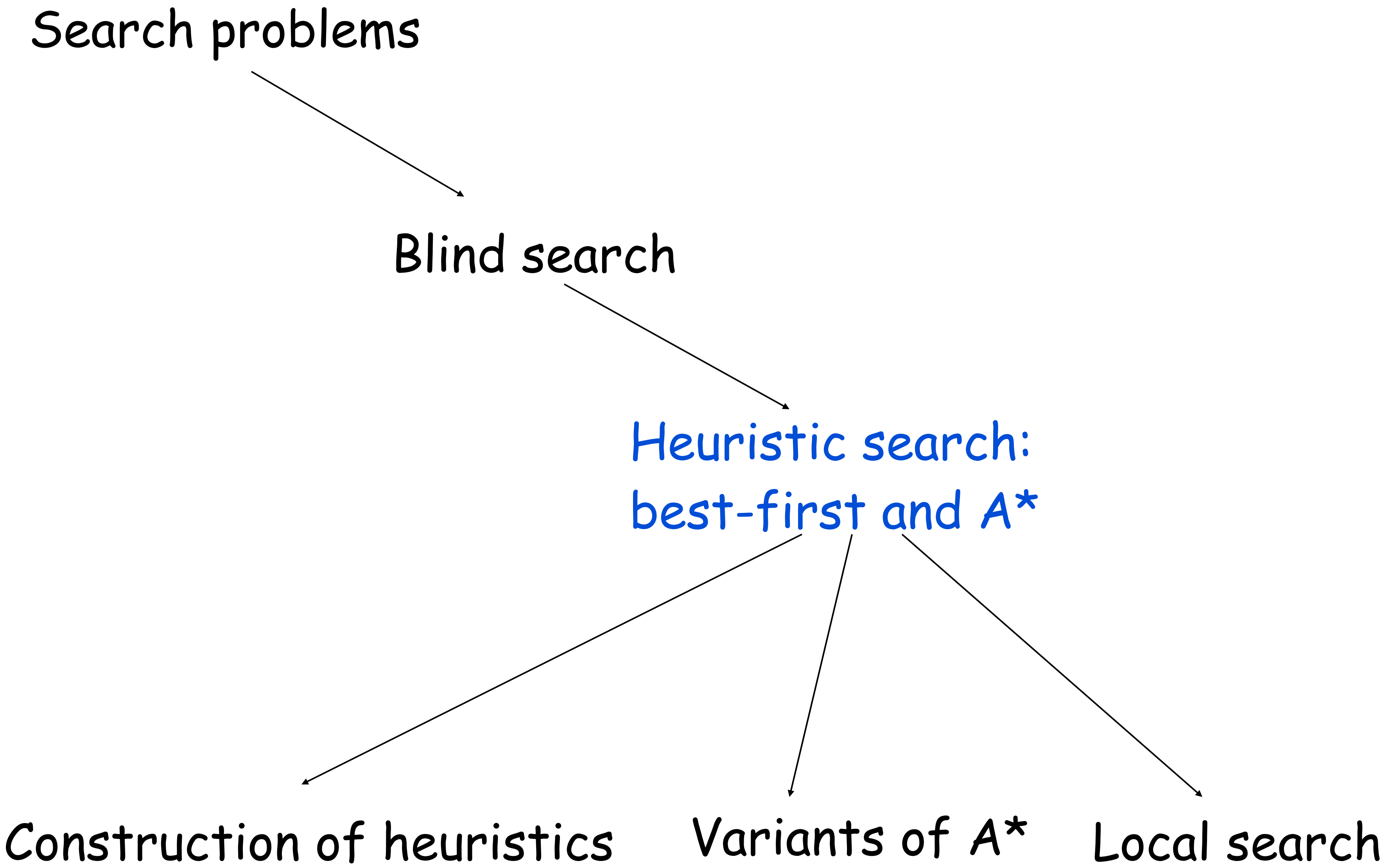
Simulated annealing lowers T over the k iterations.

“Parallel” Local Search Techniques

They perform several local searches concurrently, but not independently:

- Beam search
- Genetic algorithms

See R&N, pages 115-119


Search problems
Blind search
Heuristic search:
best-first and A*
Construction of heuristics
Variants of A*
Local search

When to Use Search Techniques?

1) The search space is small, and
   - No other technique is available, or
   - Developing a more efficient technique is not worth the effort

2) The search space is large, and
   - No other available technique is available, and
   - There exist "good" heuristics