Probability estimation

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Shortened presentation

- This is a shortened version of the presentation.
- It does not cover a gentle introduction into MLE principle.
- It assumes students know about Maximum likelihood estimation from a Probability and Statistics course.

Probability estimation

In previous two lectures:

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$

In practice:

- uknown quantities
- estimate from training data $\mathcal{T} = \{(x_1, s_1), (x_2, s_2), \dots (x_l, s_l)\}$

Problem: tossing coing, is it fair, how is the $P(\text{head})$?

Does ML solve it all?

- Tossing coing, $\mathcal{T} = \{\text{T},\text{T},\text{T}\}$
- What the ML estimate of p_H?
- Would you believe it?
- What is missing?

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Tossing coin, using priors

$$\mathcal{L}(p_H|\mathcal{T}) = p(\mathcal{T}|p_H) = \prod_{i=1}^{N} p(x_n|p_H) = \prod_{i=1}^{N} p_H^{x_n}(1-p_H)^{1-x_n}$$

$$p(h, N|p_H) = \binom{N}{h} p_H^h (1-p_H)^{N-h}; \;\; p_H = \frac{h}{N}$$

(Conjugate) Prior:

$$p(p_H|a, b) \sim p_H^a (1-p_H)^b$$

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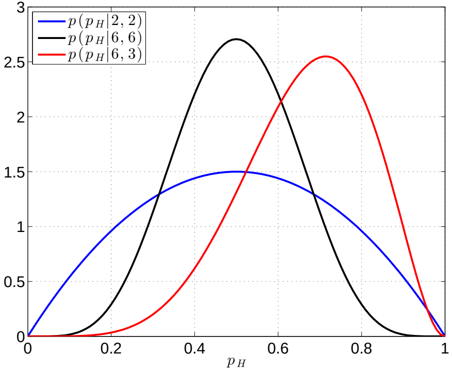


Using the prior

$$p(h, N|p_H) \sim p_H^h (1 - p_H)^{N-h}$$
$$p(p_H|a, b) \sim p_H^a (1 - p_H)^b$$

$$p(p_H|h, N) \sim p(h, N|p_H)p(p_H) \sim p_H^{h+a}(1 - p_H)^{N-h+b}$$

Looking for extremum

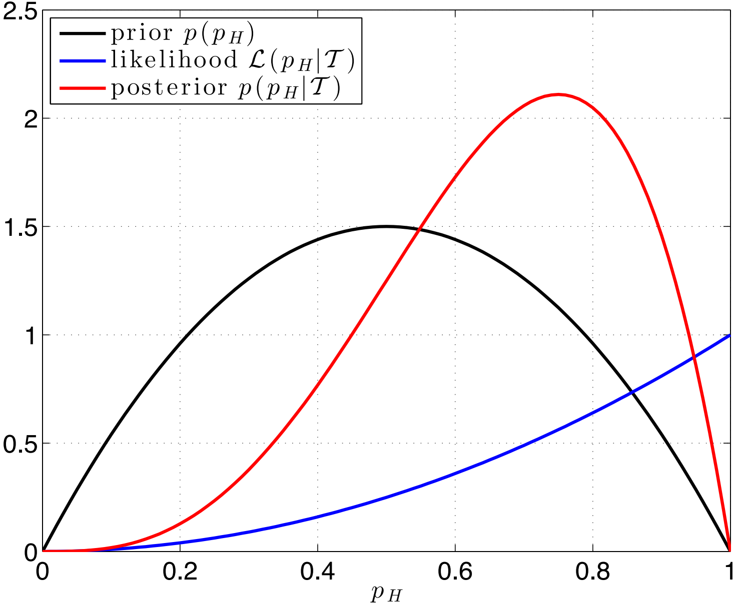
$$\frac{\partial p(p_H|h, N)}{\partial p_H} = 0$$

yields

$$p_H = \frac{h + a}{N + a + b}$$

Hyperparamaters a, b as regularization

Maximimum aposteriori estimate



Problem: Coins classification based on weight

s/x	5 g	10 g	15 g	20 g	25 g	$\sum$
1 CZK	15	10	3	0	0	**28**
2 CZK	7	13	16	6	1	**43**
5 CZK	0	1	2	11	15	**29**
$\sum$	22	24	21	17	16	**100**

- What if $x = 17$? Interpolate somehow?
- Two weighting devices A, B. $x_A = 16$, $x_B = 19$ what to do?

Maximum likelihood estimation of the weight

Two weighting devices A, B with some σ_A, σ_B measure $x_A = 16$, $x_B = 19$.
What is the ML estimate of the weight w?

- Devices independent:

$$\mathcal{L}(w) = p(x_A, x_B | w) = p(x_A|w) p(x_B|w)$$

- Sensors Gaussian:

$$\mathcal{L}(w) = \frac{1}{\sigma_A \sqrt{2\pi}} \exp\left[-\frac{(x_A - w)^2}{2\sigma_A^2}\right] \times \frac{1}{\sigma_B \sqrt{2\pi}} \exp\left[-\frac{(x_B - w)^2}{2\sigma_B^2}\right]$$

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Estimation methods

Parametric

- Distribution is a function with (a few) parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_D)$
- Example: the normal distribution $\mathcal{N}(x|\mu, \sigma^2)$.

Non-parametric

- Function of *many* parameters.
- But parameters disappear from estimation methods.
- Examples: K-nearest neighbours, histogram, Parzen window.

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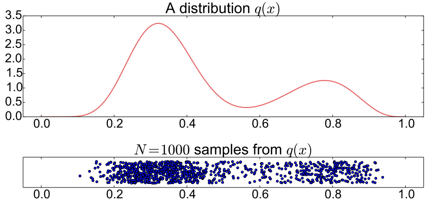
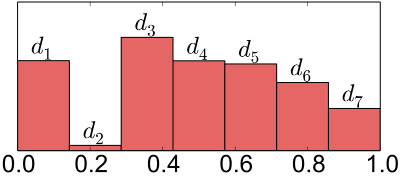
Histogram as piecewise constant density estimate

Histogram with B bins.

For a given B, the parameters of this piecewise-constant function are the heights $d_1, d_2, ..., d_B$ of the individual bins. This function is denoted $p(x|\{d_1, d_2, ..., d_B\})$.





For the given number of bins B, $d_1, d_2, ..., d_B$ must conform to the constraint that the area under the function must sum up to one,

$$1 = \int_{-\infty}^{\infty} p(x|\{d_1, d_2, ..., d_B\})\mathrm{d}x = \sum_{i=1}^{B} \int_{\frac{i-1}{B}}^{\frac{i}{B}} d_i \,\mathrm{d}x = \sum_{i=1}^{B} d_i \overset{\substack{\text{bin width} \\ \downarrow}}{w} = \sum_{i=1}^{B} \frac{d_i}{B}.$$

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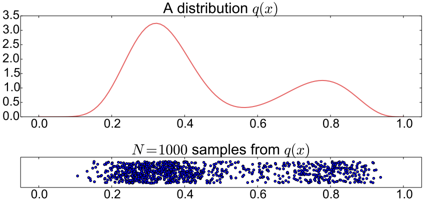

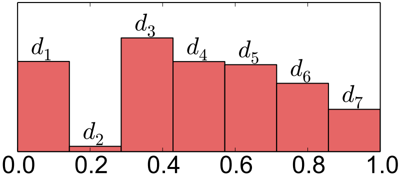

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Finding d_i using ML

$$L(\mathcal{T}) = p(\mathcal{T}\,|\,\boldsymbol{\theta}) = \prod_{i=1}^{N} p(x_i\,|\,\boldsymbol{\theta}) = \prod_{j=1}^{B} \overbrace{\left(\prod_{k=1}^{N_j} d_j\right)}^{\text{points in } j\text{-th bin}} = \prod_{j=1}^{B} d_j^{N_j}\,.$$

Maximization task:

$$\ell(\mathcal{T}) = \sum_{j=1}^{B} N_j \log d_j \to \max\,, \qquad \text{subject to } \frac{1}{B}\sum_{j=1}^{B} d_j = 1\,,$$

$$\text{Lagrangian: } \sum_{j=1}^{B} N_j \log d_j + \lambda\left(\frac{1}{B}\sum_{j=1}^{B} d_j - 1\right)$$

$$\frac{N_j}{d_j} + \frac{\lambda}{B} = 0 \Rightarrow \frac{d_j}{N_j} = \text{const.} \Rightarrow \; d_j = B\frac{N_j}{N}\,.$$

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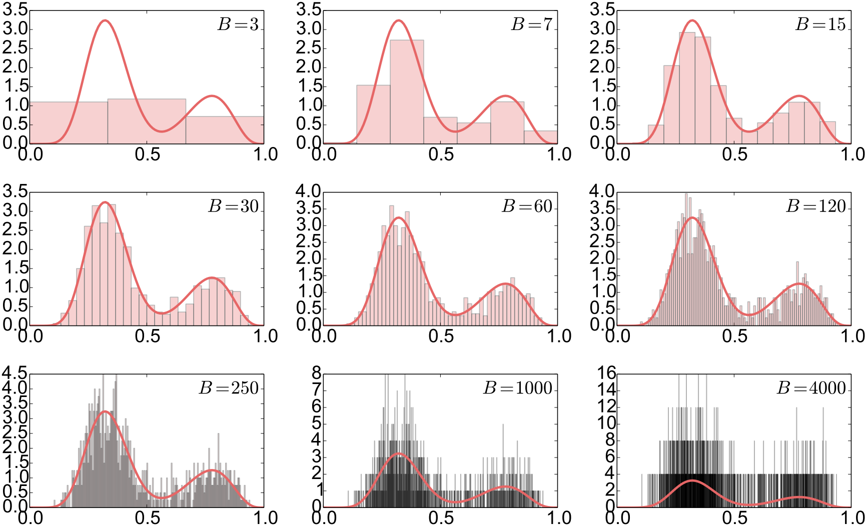
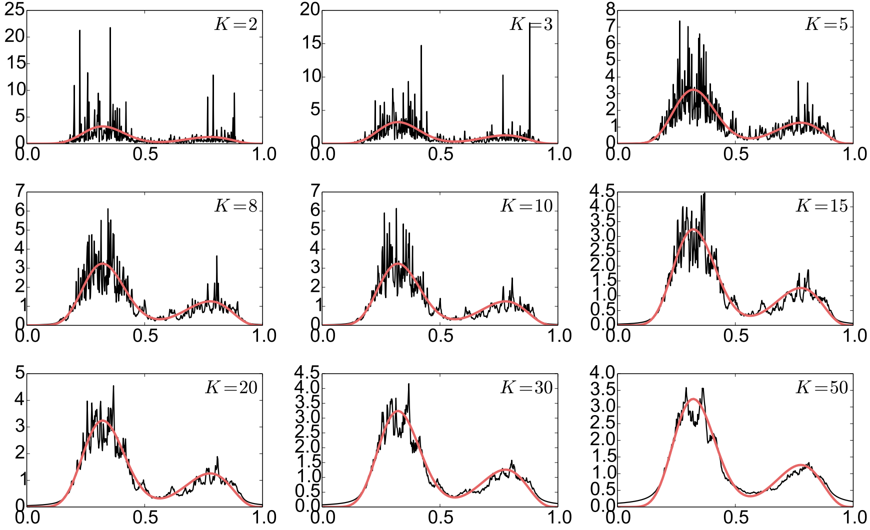
Different number of bins



K-Nearest neighbors density estimates

Find K neighbors, the density estimate is then $p \sim 1/V$ where V is the volume of a minimum cell containing K NNs.



References I

Further reading: Chapter 13 and 14 of [3]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. The lecture has been greatly inspired by the 4th and 5th lecture of the Machine Learning and Pattern Recognition course (B4B33RPZ)

[1] Christopher M. Bishop.
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[2] Richard O. Duda, Peter E. Hart, and David G. Stork.
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[3] Stuart Russell and Peter Norvig.
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