

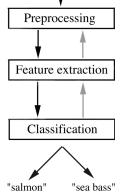
Classifiers, intro, evaluation

Tomáš Svoboda and Matěj Hoffmann
thanks to Daniel Novák and Filip Železný, Ondřej Drbohlav

Department of Cybernetics, Vision for Robotics and Autonomous Systems,
Center for Machine Perception (CMP)

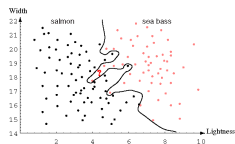
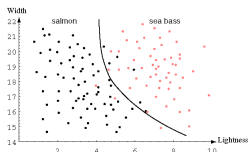
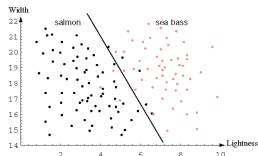
May 8, 2019

Classification example: What's the fish?



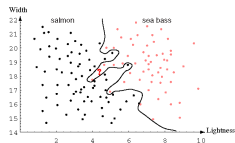
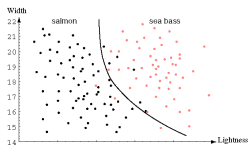
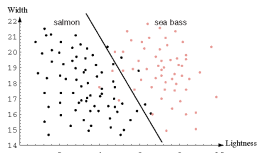
- ▶ Factory for fish processing
- ▶ 2 classes $s_{1,2}$:
 - ▶ salmon
 - ▶ sea bass
- ▶ Features \vec{x} : length, width, lightness etc. from a camera

Fish classification in feature space

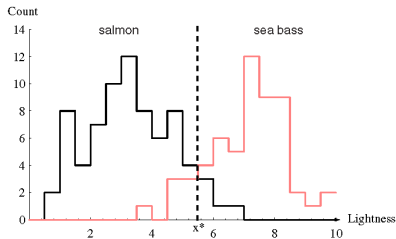
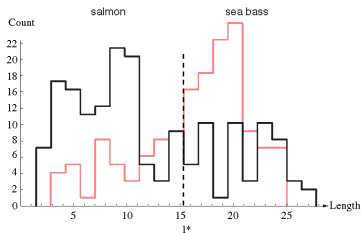


- ▶ Linear, quadratic, k-nearest neighbor classifier

Fish classification in feature space



► Linear, quadratic, k-nearest neighbor classifier



Fish – classification using probability

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$

- ▶ Notation for classification problem
 - ▶ Classes $s_j \in S$ (e.g., salmon, sea bass)
 - ▶ Features $x_i \in X$ or feature vectors (\vec{x}_i) (also called attributes)
- ▶ Optimal classification of \vec{x} :(?)

$$\delta^*(\vec{x}) = \arg \max_j P(s_j | \vec{x})$$

- ▶ We thus choose the most probable class for a given feature vector.
- ▶ Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j | \vec{x}) = \frac{P(\vec{x} | s_j) P(s_j)}{P(\vec{x})}$$

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Bayes classification in practice

- ▶ Usually we are not given $P(s|\vec{x})$
 - ▶ It has to be estimated from already classified examples – training data
 - ▶ For discrete \vec{x} , training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots, (\vec{x}_l, s_l)$
 - ▶ so-called i.i.d (independent, identically distributed) multiset
 - ▶ every (\vec{x}_i, s_i) is drawn independently from $P(\vec{x}, s)$
 - ▶ Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) \approx \frac{\# \text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\# \text{ examples where } \vec{x}_i = \vec{x}}$$

- ▶ Hard in practice:
 - ▶ To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of \vec{x} .
 - ▶ e.g. with the number of pixels in images
 - ▶ curse of dimensionality
 - ▶ denominator often 0

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Naïve Bayes classification

- ▶ For efficient classification we must thus rely on additional assumptions.
- ▶ In the exceptional case of **statistical independence** between components of \vec{x} for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

- ▶ Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots =$$

- ▶ No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s .
- ▶ No need to estimate $P(\vec{x})$. (Why?)
- ▶ $P(s)$ may be provided apriori.
- ▶ **naïve** = when used despite statistical dependence

Example: Digit recognition



- ▶ **Input:** 8-bit image 13×13 , pixel intensities 0 – 255.
- ▶ **Output:** Digit 0 – 9. Decision about the class, classification.
- ▶ **Features:** Pixel intensities ...

Collect data , ...

- ▶ $P(\vec{x})$. What is the dimension of \vec{x} ? How many possible images?
- ▶ Learn $P(\vec{x}|s)$ per each class (digit).
- ▶ Classify $s^* = \operatorname{argmax}_s P(s|\vec{x})$.

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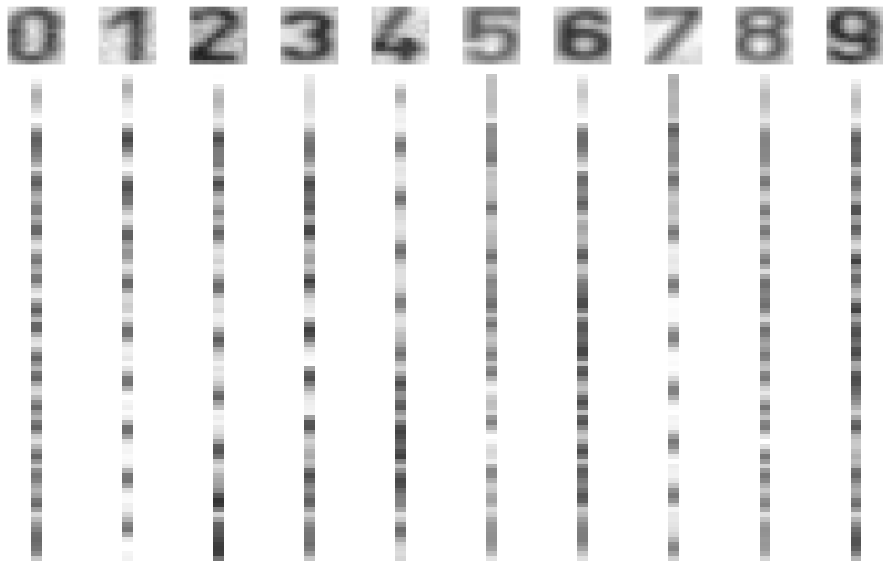


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From images to \vec{x}

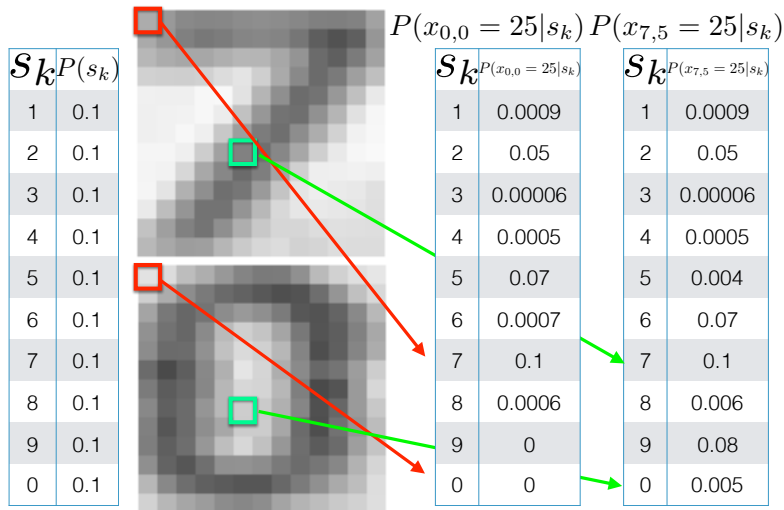


Conditional probabilities



- ▶ Apriori digit probabilities $P(s_k)$
- ▶ Likelihoods for pixels.
 $P(x_{u,v} = I_i | s_k)$

Conditional probabilities



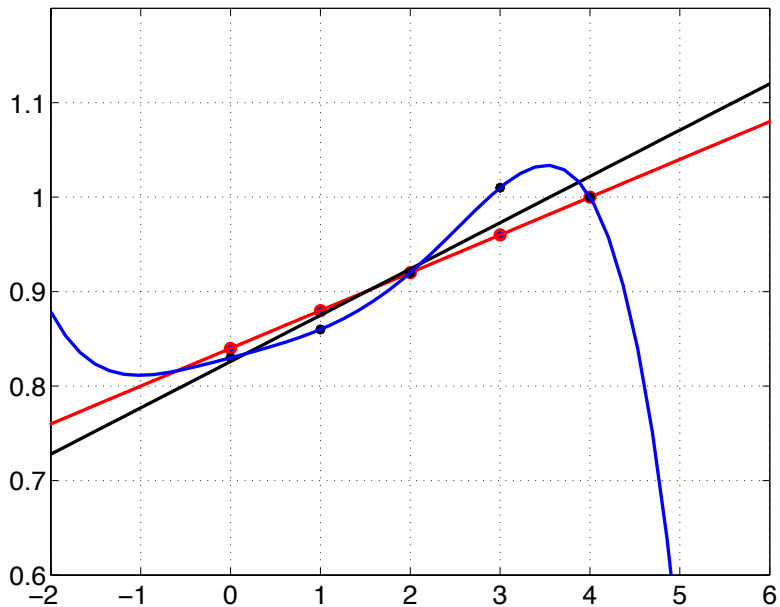
Generalization and overfitting

- ▶ Data: training, validation, testing. Wanted classifier performs well on what data?
- ▶ Overfitting: too close to training, poor on testing

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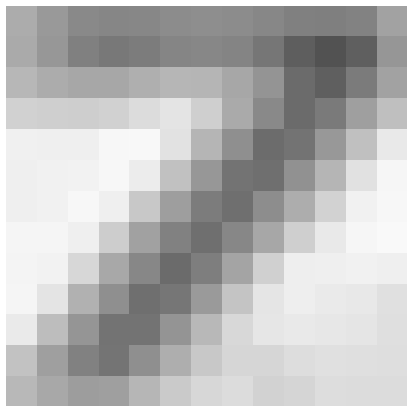
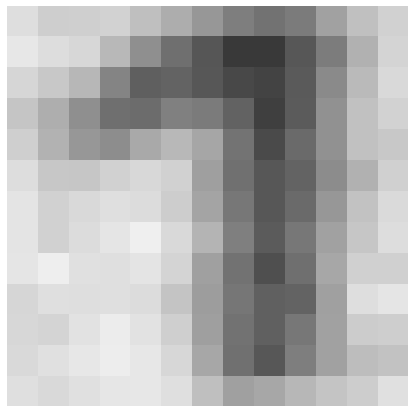
Overfitting



Unseen events



Images 13×13 , intensities 0 – 255, 100 exemplars per each class.



Laplace smoothing

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

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Laplace smoothing - as a hyperparameter k

Pretend you see every sample k extra times:

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{\sum_x [c(x) + k]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$

Product of many small numbers ...

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

$P(\vec{x})$ not needed,

$$\log(P(x[1]|s)P(x[2]|s) \dots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \dots$$

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Training and testing

Data labeled instances.

- ▶ Training set
- ▶ Held-out (validation) set
- ▶ Testing set.

Features : Attribute-value pairs.

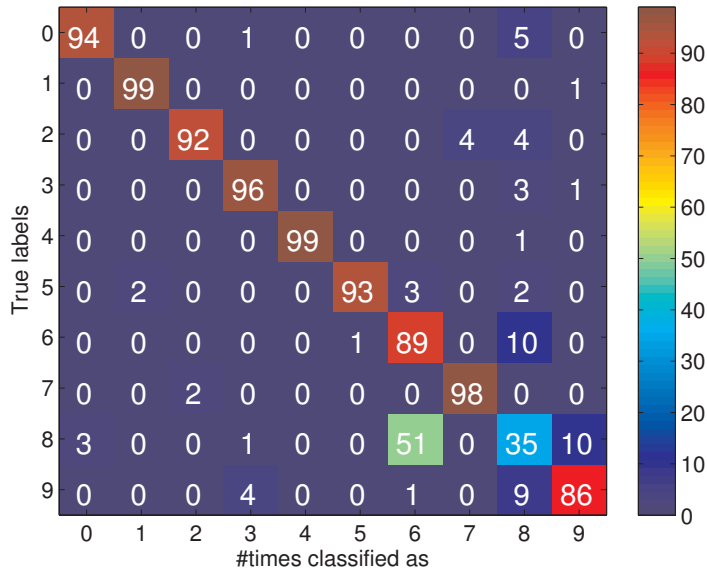
Learning cycle:

- ▶ **Learn** parameters (e.g. probabilities) on training set.
- ▶ **Tune** hyperparameters on held-out (validation) set.
- ▶ **Evaluate** performance on testing set.



How to evaluate a classifier? Confusion table

Matching table for test set



Precision and Recall, and ...

Consider digit **detection** (is there a digit?) or SPAM/HAM classification.

Recall :

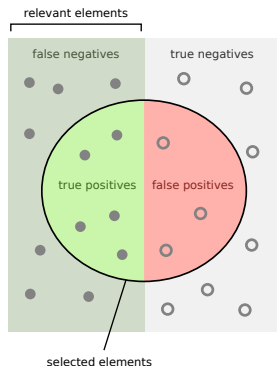
- ▶ How many relevant items are selected?
- ▶ Are we missing some items?
- ▶ Also called: **True positive rate** (TPR), sensitivity, hit rate ...

Precision

- ▶ How many selected items are relevant?
- ▶ Also called: Positive predictive value

False positive rate (FPR)

- ▶ Probability of false alarm



How many selected items are relevant?

Precision = $\frac{\text{true positives}}{\text{true positives} + \text{false positives}}$



How many relevant items are selected?

Recall = $\frac{\text{true positives}}{\text{true positives} + \text{false negatives}}$



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Inference and decision

Inference stage - learning models/function/parameters from data.

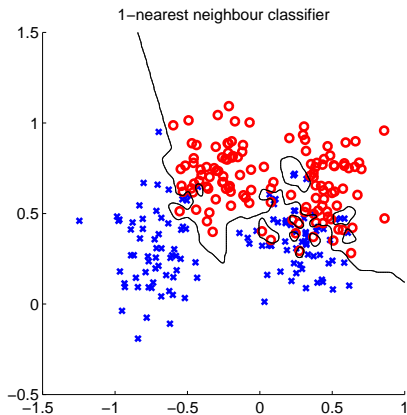
Decision stage - decide about a query \vec{x} .

- ▶ **Generative model** : Learn (infer) $P(\vec{x}, s)$. Decide by computing $P(s|\vec{x})$.
- ▶ **Discriminative model** : Learn $P(s|\vec{x})$
- ▶ **Discriminant function** : Learn $f(\vec{x})$ which maps \vec{x} directly into class labels.

K -Nearest neighbors classification

For a query \vec{x} :

- ▶ Find K nearest \vec{x} from the training (labeled) data.
- ▶ Classify to the class with the most exemplars in the set above.



K – Nearest Neighbor and Bayes

Assume data:

- ▶ N points \vec{x} in total.
- ▶ N_j points in s_j class. Hence, $\sum_j N_j = N$.

We want classify \vec{x} . We draw a sphere centered at \vec{x} containing K points irrespective of class. V is the volume of this sphere.

$$P(\vec{x}) = \frac{K}{NV}$$

$$P(\vec{x}|s_j) = \frac{K_j}{N_j V}$$

$$P(s_j) = \frac{N_j}{N}$$

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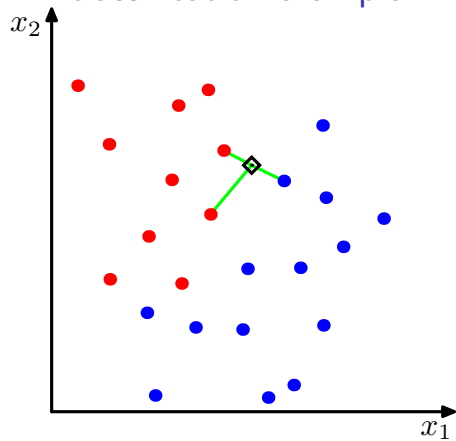
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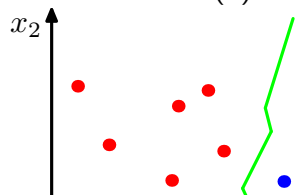
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NN classification example

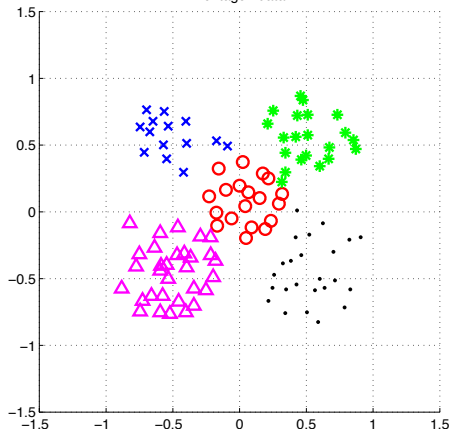


(a)

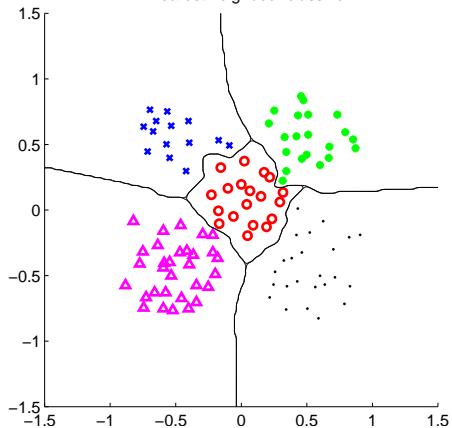


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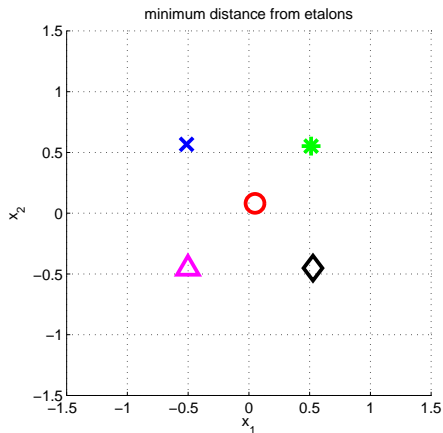
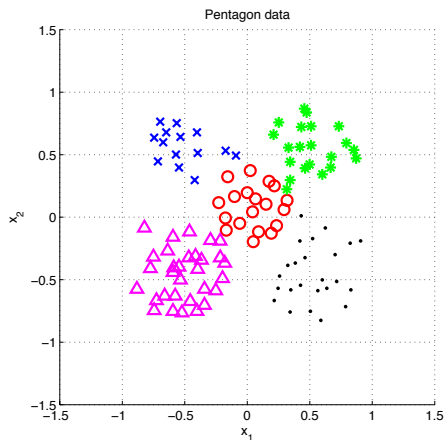
Pentagon data



1-nearest neighbour classifier



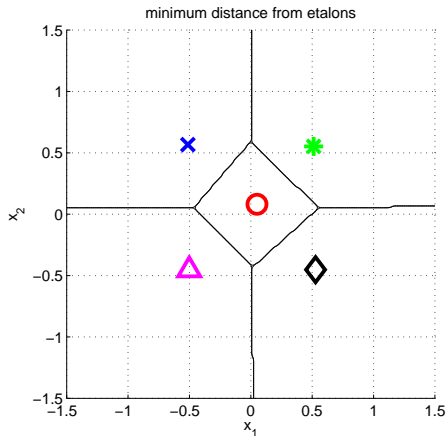
Etalon based classification



Represent \vec{x} by **etalon**, \vec{e}_s per each class $s \in S$

Separate etalons

$$f(\vec{x}) = \arg \min_{s \in S} (||\vec{x} - \vec{e}_s||^2 + o_s)$$

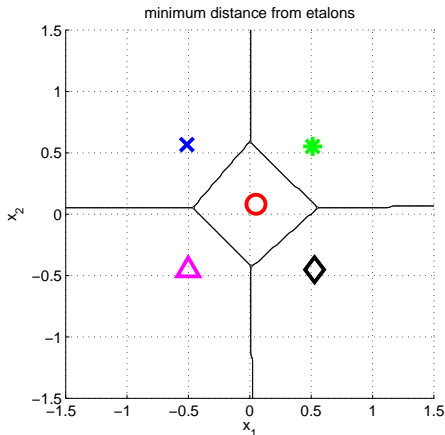


What etalons?

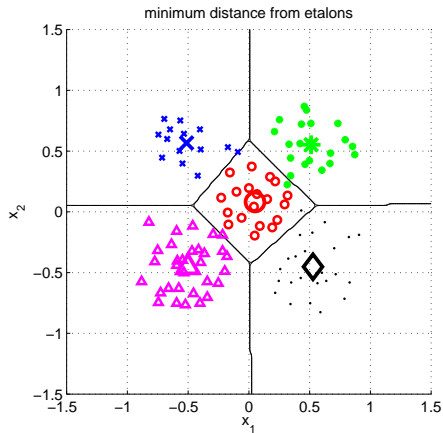
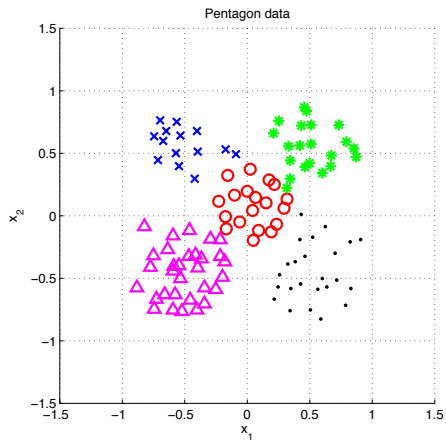
If $\mathcal{N}(\vec{x}|\vec{\mu}, \Sigma)$; all classes same covariance matrices, then

$$\vec{e}_s \stackrel{\text{def}}{=} \vec{\mu}_s = \frac{1}{|\mathcal{X}^s|} \sum_{i \in \mathcal{X}^s} \vec{x}_i^s$$

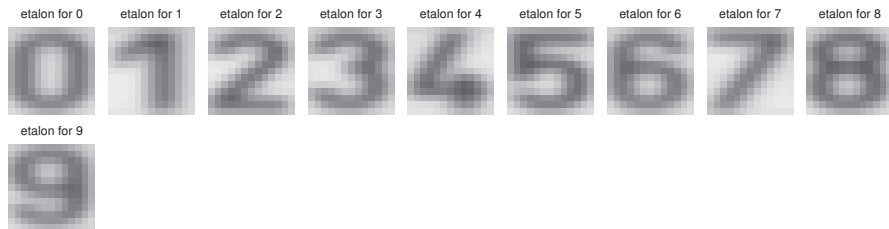
and separating hyperplanes halve distances between pairs.



Etalon based classification, $\vec{e}_s = \vec{\mu}_s$

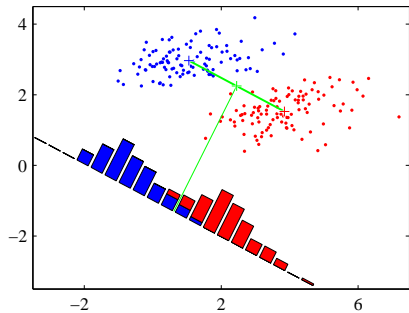


Digit recognition - etalons $\vec{e}_s = \vec{\mu}_s$

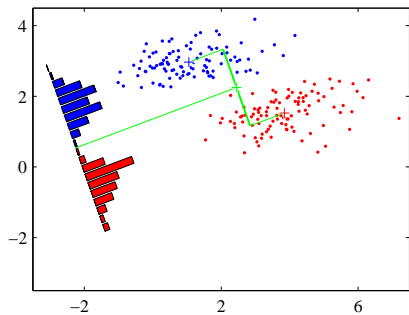
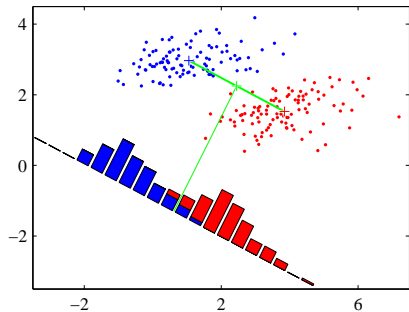


Figures from [5]

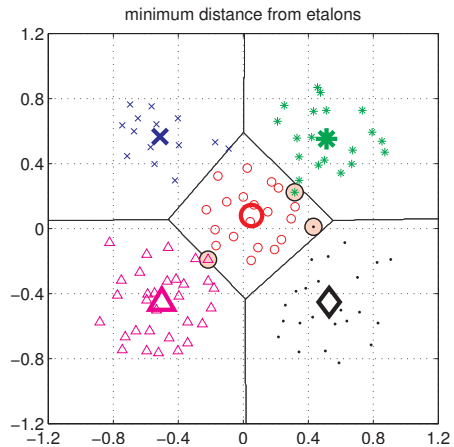
Better etalons – Fischer linear discriminant



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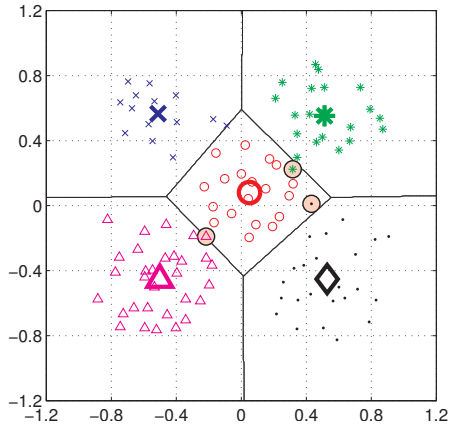


Better etalons - Perceptron

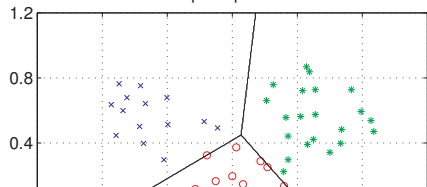


Better etalons - Perceptron

minimum distance from etalons



perceptron



Etalon classifier – Linear classifier

$$\begin{aligned} f(\vec{x}) &= \arg \min_{s \in S} (\|\vec{x} - \vec{e}_s\|^2 + o_s) = \arg \min_{s \in S} (\vec{x}^\top \vec{x} - 2 \vec{e}_s^\top \vec{x} + \vec{e}_s^\top \vec{e}_s + o_s) = \\ &= \arg \min_{s \in S} \left(\vec{x}^\top \vec{x} - 2 (\vec{e}_s^\top \vec{x} - \frac{1}{2} (\vec{e}_s^\top \vec{e}_s + o_s)) \right) = \\ &= \arg \min_{s \in S} (\vec{x}^\top \vec{x} - 2 (\vec{e}_s^\top \vec{x} + b_s)) = \\ &= \boxed{\arg \max_{s \in S} (\vec{e}_s^\top \vec{x} + b_s)} = \arg \max_{s \in S} f_s(\vec{x}). \end{aligned} \quad b_s = -\frac{1}{2} (\vec{e}_s^\top \vec{e}_s + o_s)$$

Linear function (plus offset)

$$f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Etalon classifier – Linear classifier

$$\begin{aligned} f(\vec{x}) &= \arg \min_{s \in S} (\|\vec{x} - \vec{e}_s\|^2 + o_s) = \arg \min_{s \in S} (\vec{x}^T \vec{x} - 2 \vec{e}_s^T \vec{x} + \vec{e}_s^T \vec{e}_s + o_s) = \\ &= \arg \min_{s \in S} \left(\vec{x}^T \vec{x} - 2 (\vec{e}_s^T \vec{x} - \frac{1}{2} (\vec{e}_s^T \vec{e}_s + o_s)) \right) = \\ &= \arg \min_{s \in S} (\vec{x}^T \vec{x} - 2 (\vec{e}_s^T \vec{x} + b_s)) = \\ &= \boxed{\arg \max_{s \in S} (\vec{e}_s^T \vec{x} + b_s)} = \arg \max_{s \in S} f_s(\vec{x}). \end{aligned} \quad b_s = -\frac{1}{2} (\vec{e}_s^T \vec{e}_s + o_s)$$

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Linear function (plus offset)

$$f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Perceptron learning - problem set up

We seek $\mathcal{K} = \{(\mathbf{w}_s, w_{0_s}) \mid s \in S\}$

$$f(\mathbf{x}) = \arg \max_{s \in S} (\mathbf{w}_s^\top \mathbf{x} + w_{0_s})$$

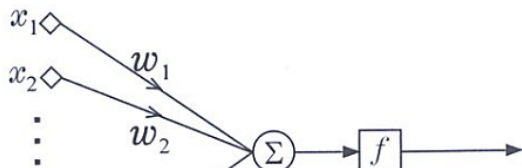
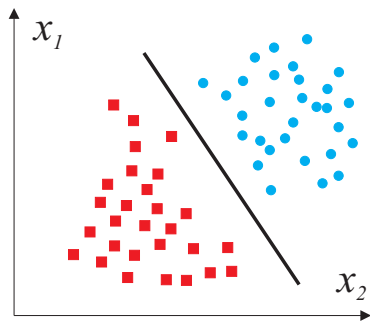
achieves no error on training set $\mathcal{T} = \{(\mathbf{x}^i, s^i), i = 0, 1, \dots, m\}$

$$\epsilon_{tr} = \frac{1}{m} \sum_{j=1}^m \mathbf{1}(s^j \neq f(\mathbf{x}^j)), \quad \mathbf{1}(s) = \begin{cases} 1 & s \text{ True} \\ 0 & s \text{ False} \end{cases}$$

Perceptron, two classes linearly separable

$|S| = 2$, i.e. two states (typically also classes)

$$f(\mathbf{x}) = \begin{cases} s = 1, & \text{if } \mathbf{w}^\top \mathbf{x} + w_0 > 0, \\ s = -1, & \text{if } \mathbf{w}^\top \mathbf{x} + w_0 < 0. \end{cases}$$



Perceptron learning – Algorithm

$\mathbf{x}'_j = s_j \begin{bmatrix} 1 \\ \mathbf{x}_j \end{bmatrix}$, $\mathbf{w}' = \begin{bmatrix} w_0 \\ \mathbf{w} \end{bmatrix}$ drop the dashes to avoid notation clutter.

Goal: Find a weight vector $\mathbf{w} \in \Re^{D+1}$ (original feature space dimensionality is D) such that:

$$\mathbf{w}^\top \mathbf{x}_j > 0 \quad (\forall j \in \{1, 2, \dots, m\})$$

Perceptron algorithm (Rosenblatt 1962):

1. $t \leftarrow 0$, $\mathbf{w}^{(t)} \leftarrow 0$.
2. Find a wrongly classified observation \mathbf{x}_j :

$$\mathbf{w}^{(t)\top} \mathbf{x}_j \leq 0, \quad (j \in \{1, 2, \dots, m\}.)$$

3. If there is no misclassified observation then terminate. Otherwise,

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbf{x}_j.$$

4. Goto 2.

Perceptron learning – Algorithm

$\mathbf{x}'_j = s_j \begin{bmatrix} 1 \\ \mathbf{x}_j \end{bmatrix}$, $\mathbf{w}' = \begin{bmatrix} w_0 \\ \mathbf{w} \end{bmatrix}$ drop the dashes to avoid notation clutter.

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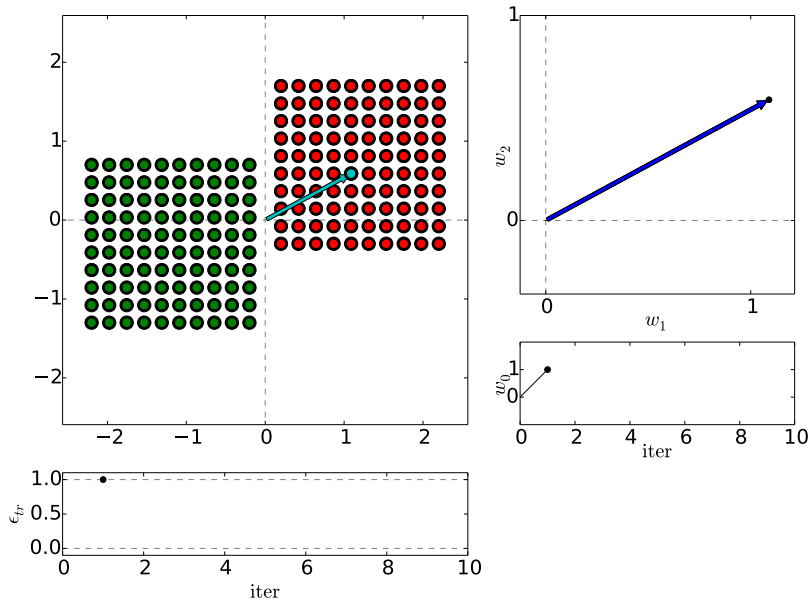
$$\mathbf{w}^{(t)\top} \mathbf{x}_j \leq 0, \quad (j \in \{1, 2, \dots, m\}.)$$

3. If there is no misclassified observation then terminate. Otherwise,

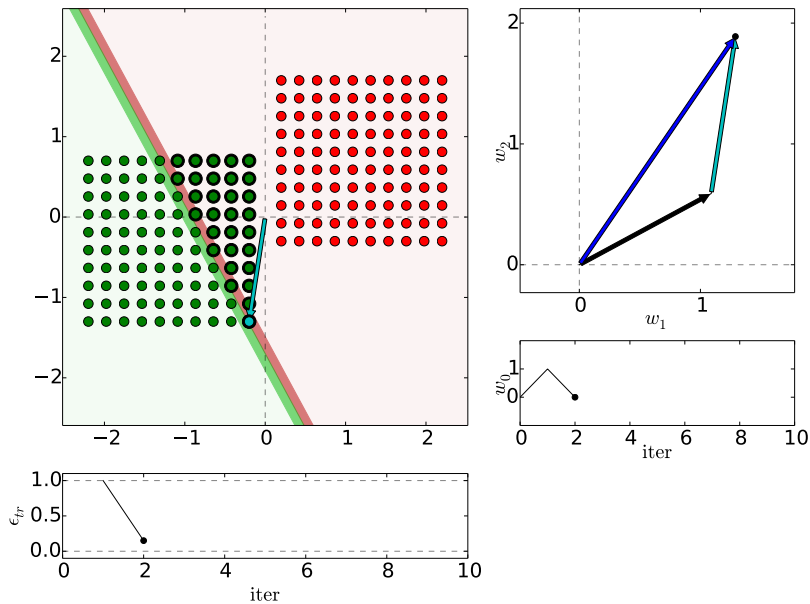
$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbf{x}_j .$$

4. Goto 2.

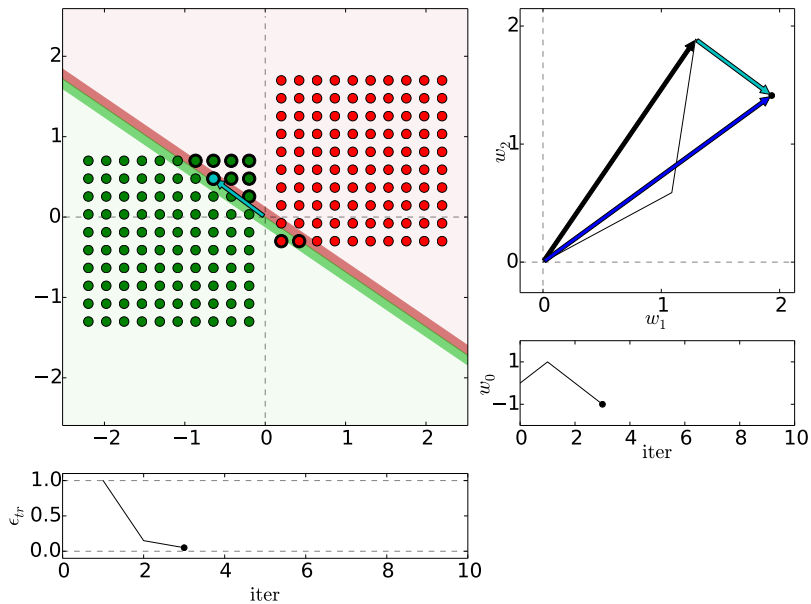
Perceptron iterations



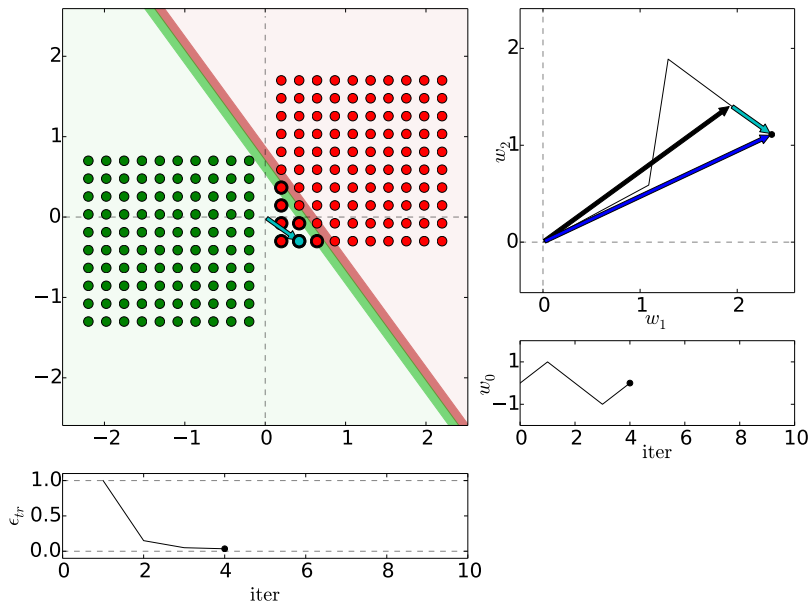
Perceptron iterations



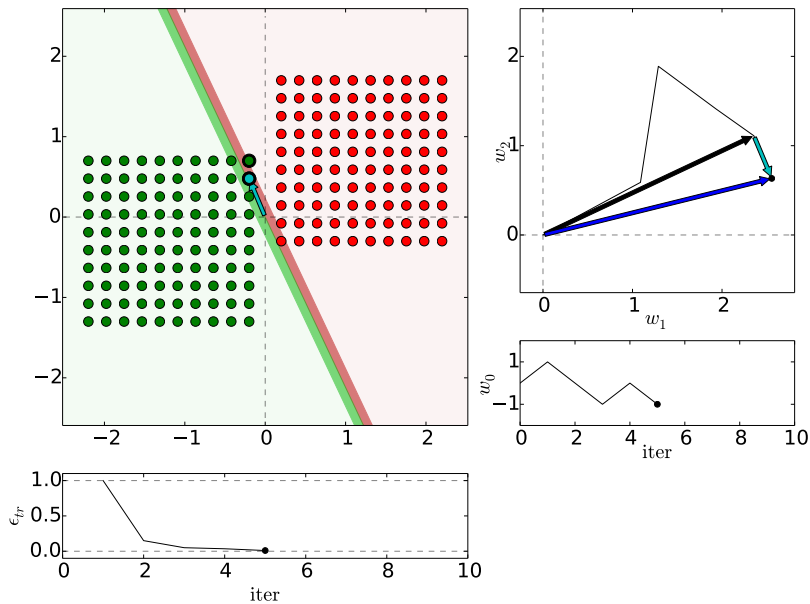
Perceptron iterations



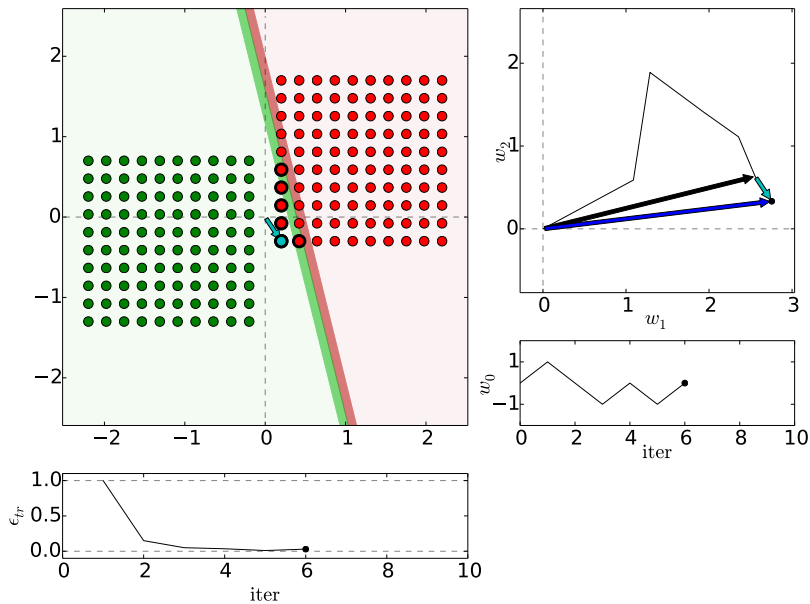
Perceptron iterations



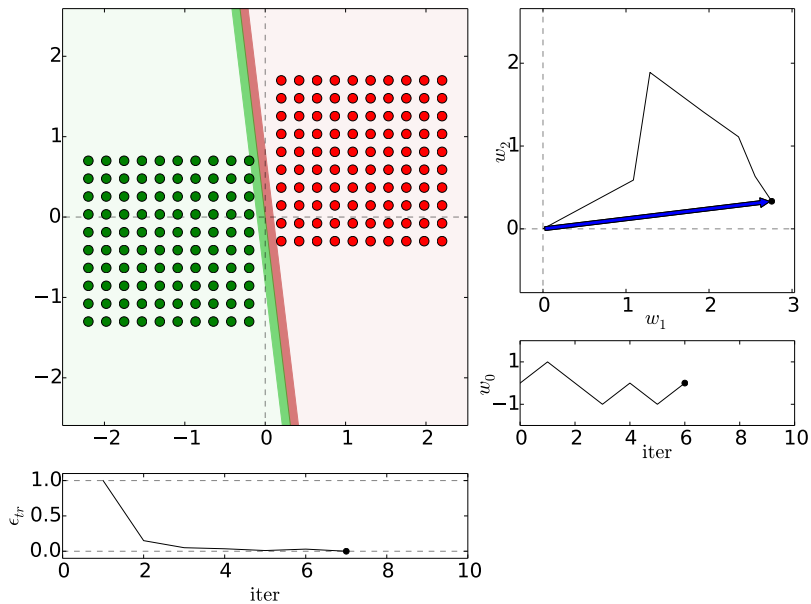
Perceptron iterations



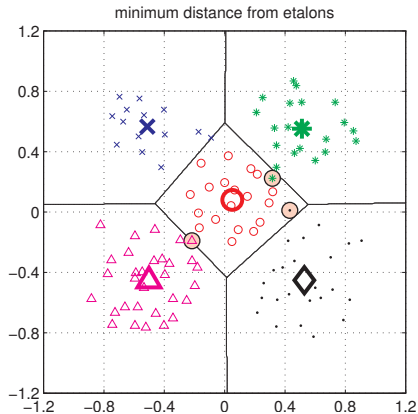
Perceptron iterations



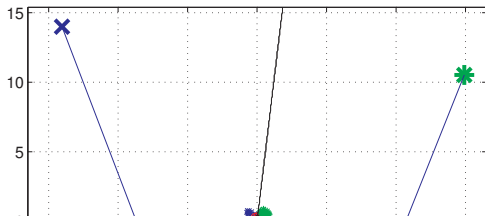
Perceptron iterations



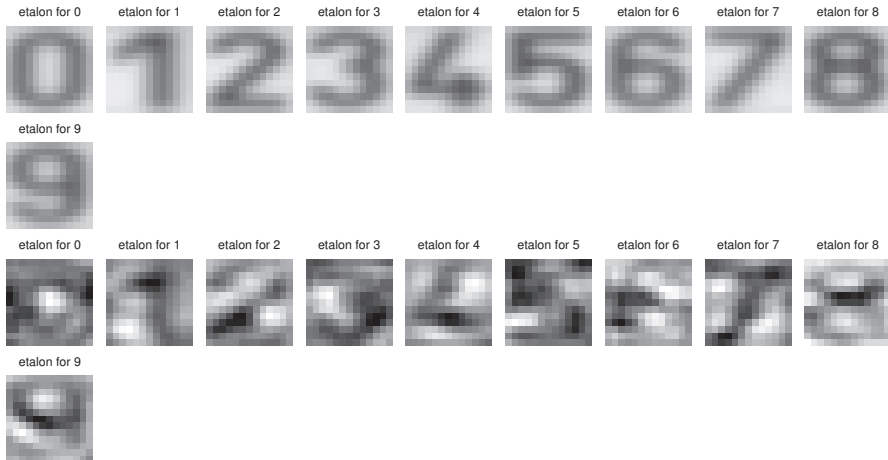
Etalons: means vs found by perceptron



Etalons and separating hyperplanes found by perceptron

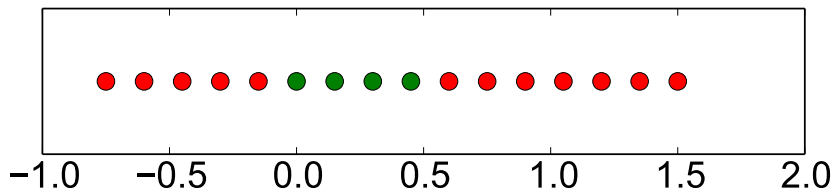


Digit recognition - etalons means vs. perceptron



Figures from [5]

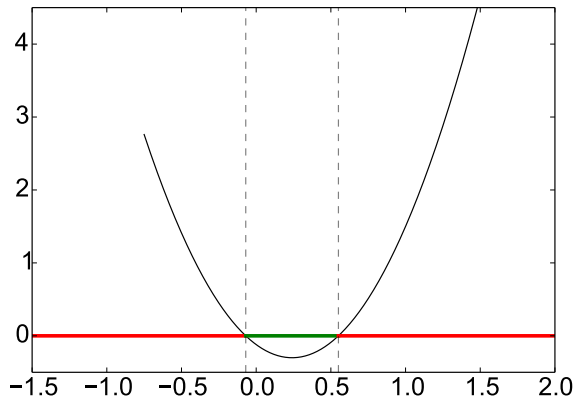
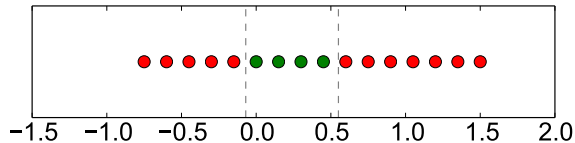
What if not lin separable?



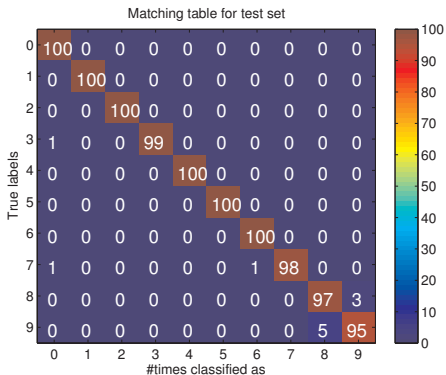
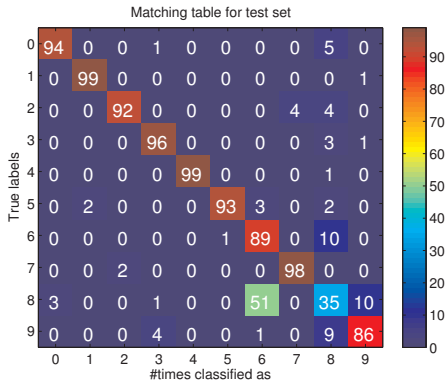
Dimension lifting

$$\mathbf{x} = [x, x^2]^T$$

Dimension lifting, $\mathbf{x} = [x, x^2]^\top$

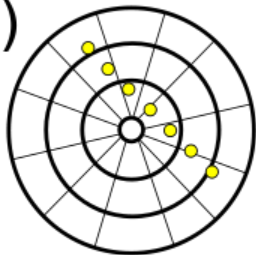


Performance comparison, parameters fixed

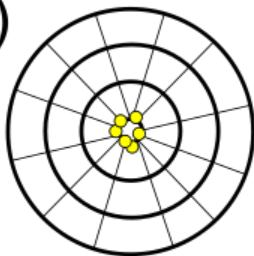


Accuracy vs precision

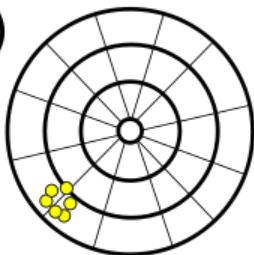
(a)



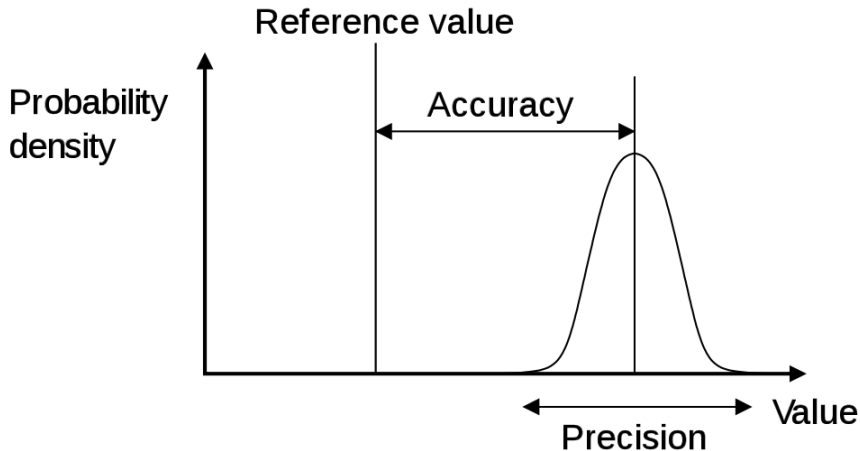
(b)



(c)



Accuracy vs precision



https://en.wikipedia.org/wiki/Accuracy_and_precision

References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. Many Matlab figures created with the help of [3]

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[2] Richard O. Duda, Peter E. Hart, and David G. Stork.

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[3] Votjěch Franc and Václav Hlaváč.

Statistical pattern recognition toolbox.

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- [4] Stuart Russell and Peter Norvig.
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- [5] Tomáš Svoboda, Jan Kybic, and Hlaváč Václav.
Image Processing, Analysis and Machine Vision — A MATLAB Companion.
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