

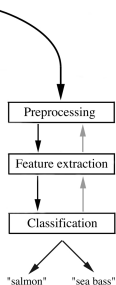
Classifiers, intro, evaluation

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thanks to Daniel Novák and Filip Železný, Ondřej Drbohlav

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Center for Machine Perception (CMP)

May 9, 2019

Classification example: What's the fish?

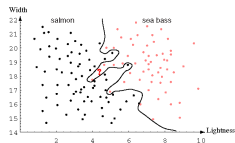
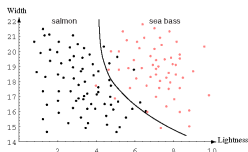
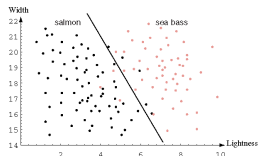


- ▶ Factory for fish processing
- ▶ 2 classes $s_{1,2}$:
 - ▶ salmon
 - ▶ sea bass
- ▶ Features \vec{x} : length, width, lightness etc. from a camera

Classification example:

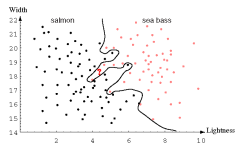
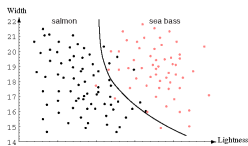
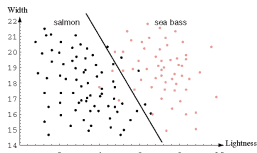
<http://robotics.fel.cvut.cz/cras/darpa-subt/>

Fish classification in feature space

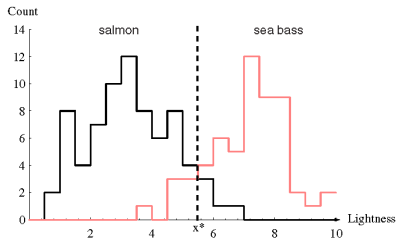
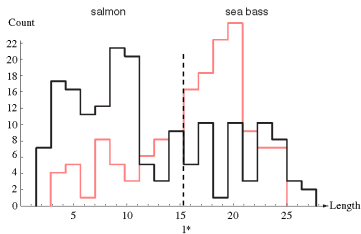


- ▶ Linear, quadratic, k-nearest neighbor classifier

Fish classification in feature space



► Linear, quadratic, k-nearest neighbor classifier



Fish – classification using probability

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$

- ▶ Notation for classification problem
 - ▶ Classes $s_j \in S$ (e.g., salmon, sea bass)
 - ▶ Features $x_i \in X$ or feature vectors (\vec{x}_i) (also called attributes)
- ▶ Optimal classification of \vec{x} :(?)

$$\delta^*(\vec{x}) = \arg \max_j P(s_j | \vec{x})$$

- ▶ We thus choose the most probable class for a given feature vector.
- ▶ Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j | \vec{x}) = \frac{P(\vec{x} | s_j) P(s_j)}{P(\vec{x})}$$

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Bayes classification in practice

- ▶ Usually we are not given $P(s|\vec{x})$
 - ▶ It has to be estimated from already classified examples – training data
 - ▶ For discrete \vec{x} , training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots, (\vec{x}_l, s_l)$
 - ▶ so-called i.i.d (independent, identically distributed) multiset
 - ▶ every (\vec{x}_i, s_i) is drawn independently from $P(\vec{x}, s)$
 - ▶ Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) \approx \frac{\# \text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\# \text{ examples where } \vec{x}_i = \vec{x}}$$

- ▶ Hard in practice:
 - ▶ To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of \vec{x} .
 - ▶ e.g. with the number of pixels in images
 - ▶ curse of dimensionality
 - ▶ denominator often 0

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Naïve Bayes classification

- ▶ For efficient classification we must thus rely on additional assumptions.
- ▶ In the exceptional case of **statistical independence** between components of \vec{x} for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

- ▶ Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots =$$

- ▶ No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s .
- ▶ No need to estimate $P(\vec{x})$. (Why?)
- ▶ $P(s)$ may be provided apriori.
- ▶ **naïve** = when used despite statistical dependence

Example: Digit recognition



- ▶ **Input:** 8-bit image 13×13 , pixel intensities 0 – 255.
- ▶ **Output:** Digit 0 – 9. Decision about the class, classification.
- ▶ **Features:** Pixel intensities ...

Collect data , ...

- ▶ $P(\vec{x})$. What is the dimension of \vec{x} ? How many possible images?
- ▶ Learn $P(\vec{x}|s)$ per each class (digit).
- ▶ Classify $s^* = \operatorname{argmax}_s P(s|\vec{x})$.

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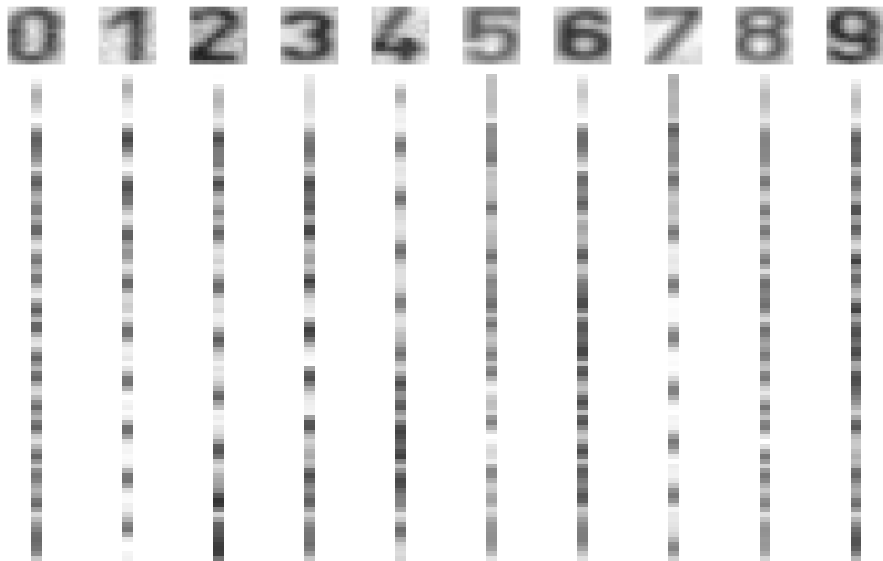


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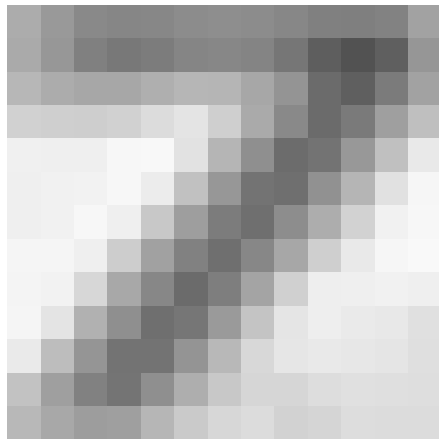
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From images to \vec{x}

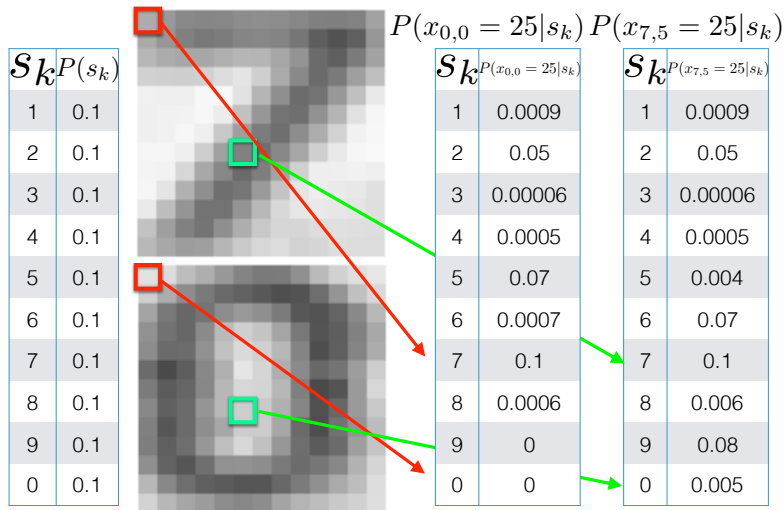


Conditional probabilities



- ▶ Apriori digit probabilities $P(s_k)$
- ▶ Likelihoods for pixels.
 $P(x_{u,v} = I_i | s_k)$

Conditional probabilities



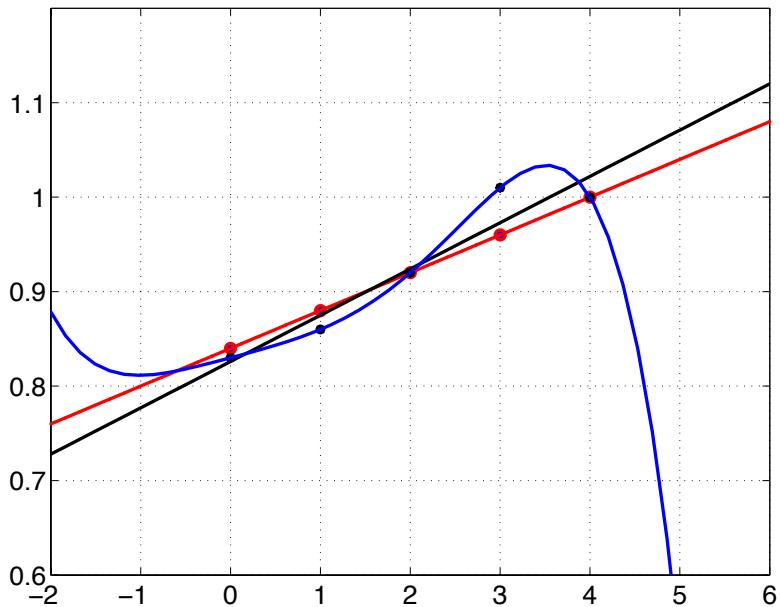
Generalization and overfitting

- ▶ Data: training, validation, testing. Wanted classifier performs well on what data?
- ▶ Overfitting: too close to training, poor on testing

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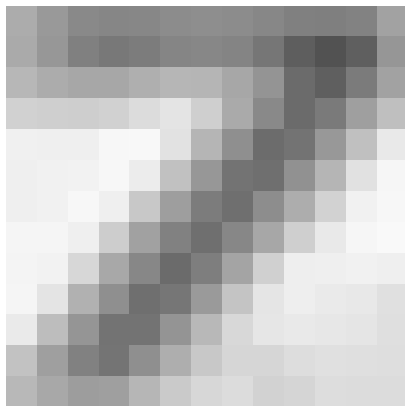
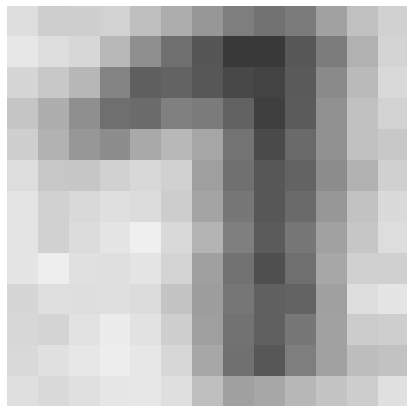
Overfitting



Unseen events



Images 13×13 , intensities 0 – 255, 100 exemplars per each class.



Laplace smoothing

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

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Laplace smoothing - as a hyperparameter k

Pretend you see every sample k extra times:

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{\sum_x [c(x) + k]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$

Product of many small numbers ...

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

$P(\vec{x})$ not needed,

$$\log(P(x[1]|s)P(x[2]|s)\dots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \dots$$

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Training and testing

Data labeled instances.

- ▶ Training set
- ▶ Held-out (validation) set
- ▶ Testing set.

Features : Attribute-value pairs.

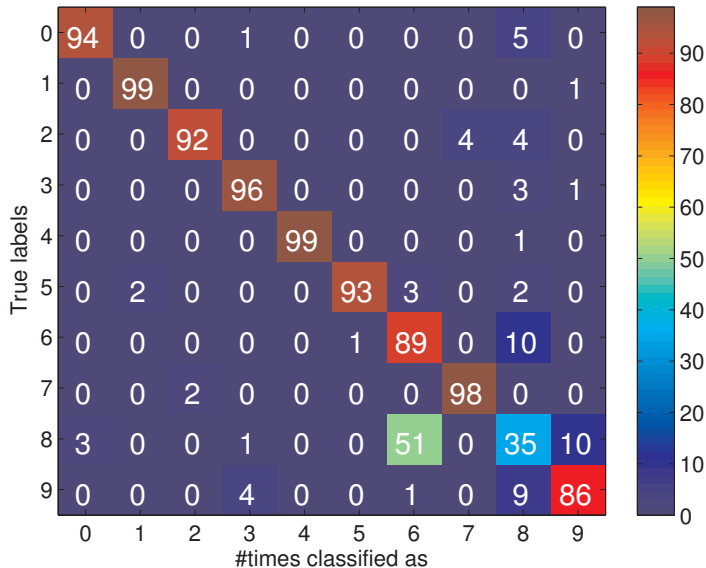
Learning cycle:

- ▶ **Learn** parameters (e.g. probabilities) on training set.
- ▶ **Tune** hyperparameters on held-out (validation) set.
- ▶ **Evaluate** performance on testing set.



How to evaluate a classifier? Confusion table

Matching table for test set



Precision and Recall, and ...

Consider digit **detection** (is there a digit?) or SPAM/HAM classification.

Recall :

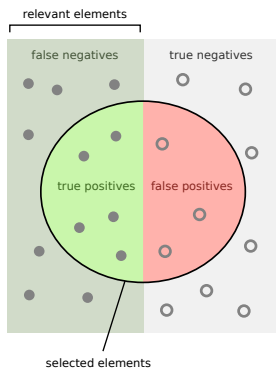
- ▶ How many relevant items are selected?
- ▶ Are we missing some items?
- ▶ Also called: **True positive rate** (TPR), sensitivity, hit rate ...

Precision

- ▶ How many selected items are relevant?
- ▶ Also called: Positive predictive value

False positive rate (FPR)

- ▶ Probability of false alarm



How many selected items are relevant?

Precision = $\frac{\text{true positives}}{\text{true positives} + \text{false positives}}$

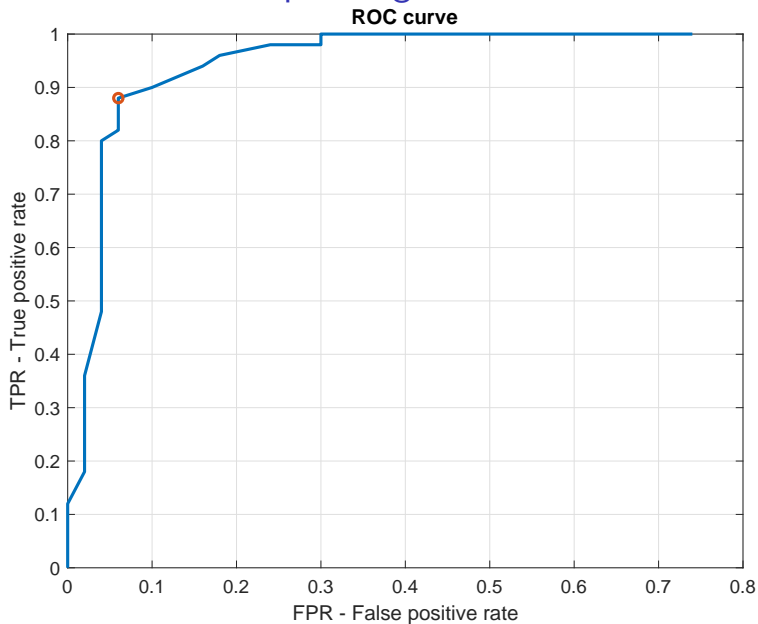
How many relevant items are selected?

Recall = $\frac{\text{true positives}}{\text{true positives} + \text{false negatives}}$

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<https://commons.wikimedia.org/w/index.php?curid=36926283>

ROC – Receiver operating characteristics curve



Inference and decision

Inference stage - learning models/function/parameters from data.

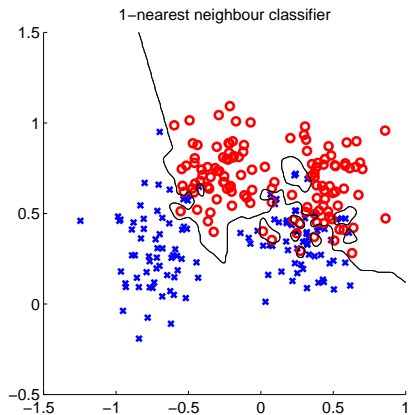
Decision stage - decide about a query \vec{x} .

- ▶ **Generative model** : Learn (infer) $P(\vec{x}, s)$. Decide by computing $P(s|\vec{x})$.
- ▶ **Discriminative model** : Learn $P(s|\vec{x})$
- ▶ **Discriminant function** : Learn $f(\vec{x})$ which maps \vec{x} directly into class labels.

K -Nearest neighbors classification

For a query \vec{x} :

- ▶ Find K nearest \vec{x} from the training (labeled) data.
- ▶ Classify to the class with the most exemplars in the set above.



K – Nearest Neighbor and Bayes

Assume data:

- ▶ N points \vec{x} in total.
- ▶ N_j points in s_j class. Hence, $\sum_j N_j = N$.

We want classify \vec{x} . We draw a sphere centered at \vec{x} containing K points irrespective of class. V is the volume of this sphere. $P(s_j|\vec{x}) = ?$

$$P(\vec{x}) = \frac{K}{NV}$$

$$P(\vec{x}|s_j) = \frac{K_j}{N_j V}$$

$$P(s_j) = \frac{N_j}{N}$$

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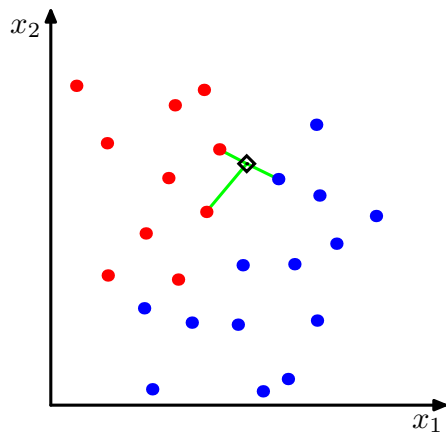
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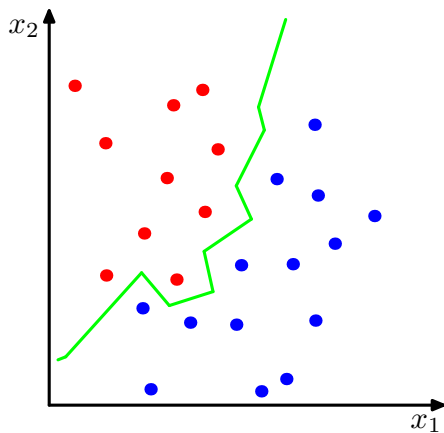
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NN classification example



(a)

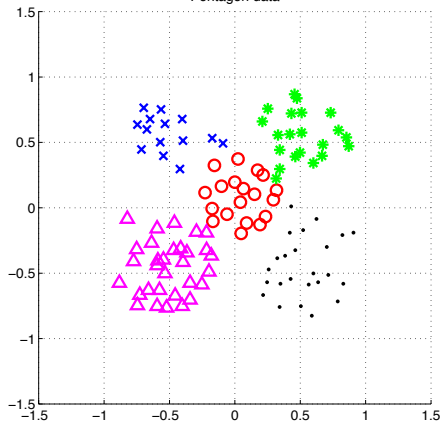


(b)

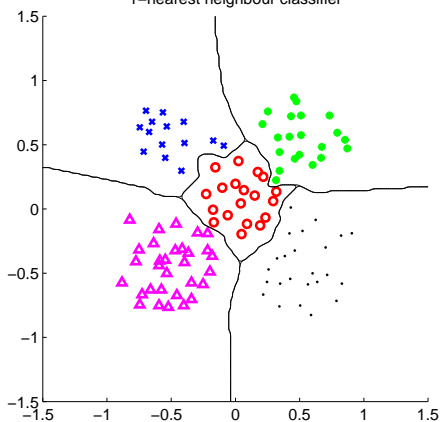
1

NN classification example

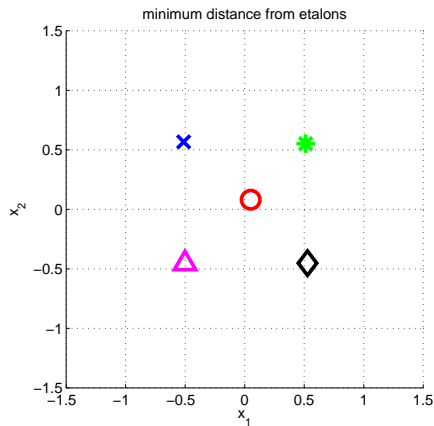
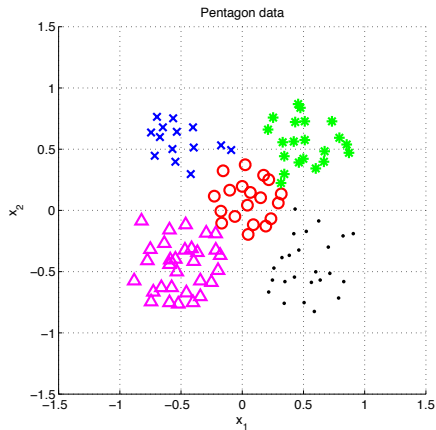
Pentagon data



1-nearest neighbour classifier



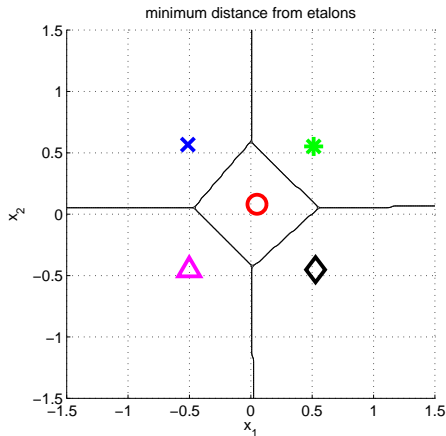
Etalon based classification



Represent \vec{x} by **etalon** , \vec{e}_s per each class $s \in S$

Separate etalons

$$f(\vec{x}) = \arg \min_{s \in S} (||\vec{x} - \vec{e}_s||^2 + o_s)$$

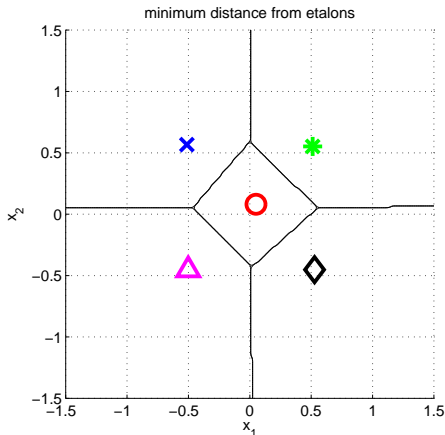


What etalons?

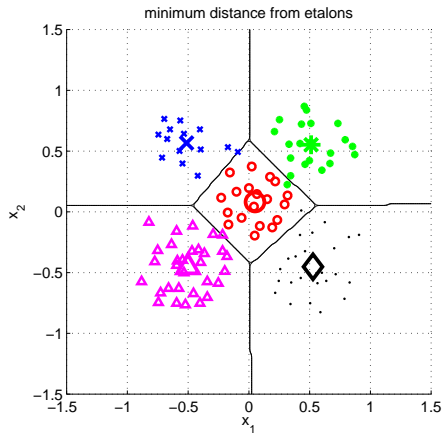
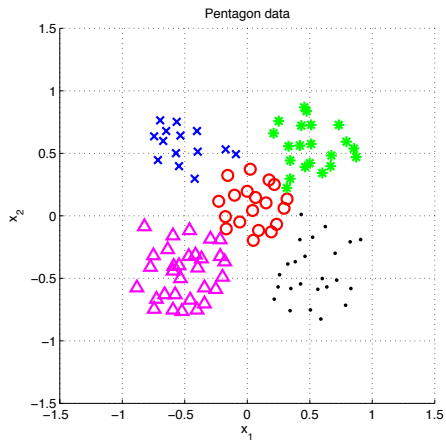
If $\mathcal{N}(\vec{x}|\vec{\mu}, \Sigma)$; all classes same covariance matrices, then

$$\vec{e}_s \stackrel{\text{def}}{=} \vec{\mu}_s = \frac{1}{|\mathcal{X}^s|} \sum_{i \in \mathcal{X}^s} \vec{x}_i^s$$

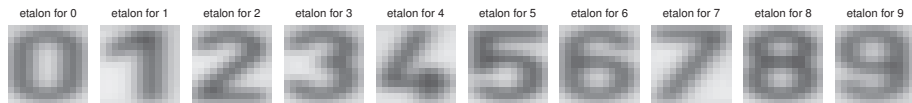
and separating hyperplanes halve distances between pairs.



Etalon based classification, $\vec{e}_s = \vec{\mu}_s$

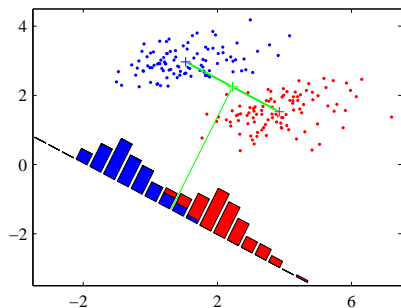


Digit recognition - etalons $\vec{e}_s = \vec{\mu}_s$



Figures from [5]

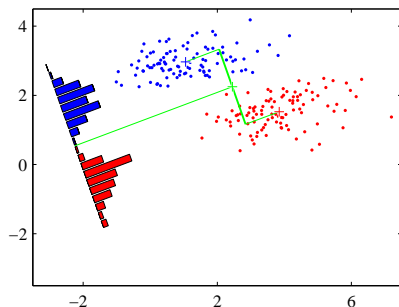
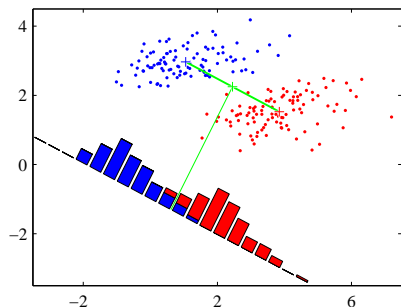
Better etalons – Fischer linear discriminant



- ▶ Dimensionality reduction
- ▶ Maximize distance between means, ...
- ▶ ... and minimize within class variance. (minimize overlap)

Figures from [1]

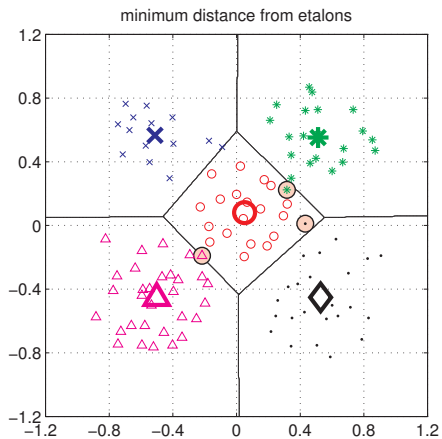
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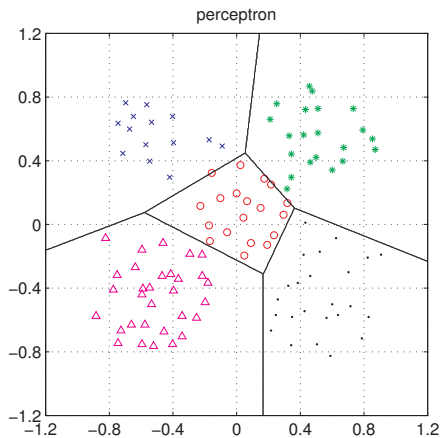
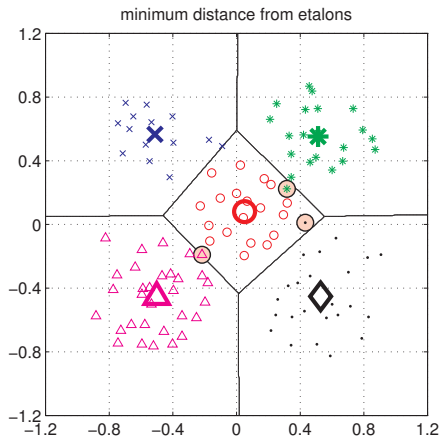
Figures from [1]

Better etalons - Perceptron



Figures from [5]

Better etalons - Perceptron



Figures from [5]

Etalon classifier – Linear classifier

$$\begin{aligned} f(\vec{x}) &= \arg \min_{s \in S} (\|\vec{x} - \vec{e}_s\|^2 + o_s) = \arg \min_{s \in S} (\vec{x}^\top \vec{x} - 2 \vec{e}_s^\top \vec{x} + \vec{e}_s^\top \vec{e}_s + o_s) = \\ &= \arg \min_{s \in S} \left(\vec{x}^\top \vec{x} - 2 (\vec{e}_s^\top \vec{x} - \frac{1}{2} (\vec{e}_s^\top \vec{e}_s + o_s)) \right) = \\ &= \arg \min_{s \in S} (\vec{x}^\top \vec{x} - 2 (\vec{e}_s^\top \vec{x} + b_s)) = \\ &= \boxed{\arg \max_{s \in S} (\vec{e}_s^\top \vec{x} + b_s)} = \arg \max_{s \in S} f_s(\vec{x}). \end{aligned} \quad b_s = -\frac{1}{2} (\vec{e}_s^\top \vec{e}_s + o_s)$$

Linear function (plus offset)

$$f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Etalon classifier – Linear classifier

$$\begin{aligned} f(\vec{x}) &= \arg \min_{s \in S} (\|\vec{x} - \vec{e}_s\|^2 + o_s) = \arg \min_{s \in S} (\vec{x}^T \vec{x} - 2 \vec{e}_s^T \vec{x} + \vec{e}_s^T \vec{e}_s + o_s) = \\ &= \arg \min_{s \in S} \left(\vec{x}^T \vec{x} - 2 (\vec{e}_s^T \vec{x} - \frac{1}{2} (\vec{e}_s^T \vec{e}_s + o_s)) \right) = \\ &= \arg \min_{s \in S} (\vec{x}^T \vec{x} - 2 (\vec{e}_s^T \vec{x} + b_s)) = \\ &= \boxed{\arg \max_{s \in S} (\vec{e}_s^T \vec{x} + b_s)} = \arg \max_{s \in S} f_s(\vec{x}). \end{aligned} \quad b_s = -\frac{1}{2} (\vec{e}_s^T \vec{e}_s + o_s)$$

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Linear function (plus offset)

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Perceptron learning - problem set up

We seek $\mathcal{K} = \{(\mathbf{w}_s, w_{0_s}) \mid s \in S\}$

$$f(\mathbf{x}) = \arg \max_{s \in S} (\mathbf{w}_s^\top \mathbf{x} + w_{0_s})$$

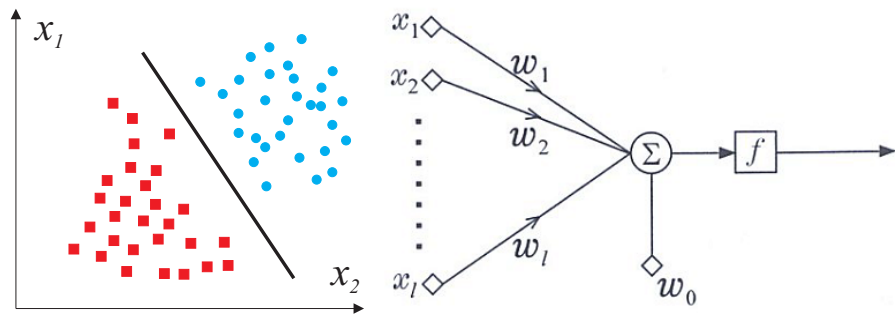
achieves no error on training set $\mathcal{T} = \{(\mathbf{x}^i, s^i), i = 0, 1, \dots, m\}$

$$\epsilon_{tr} = \frac{1}{m} \sum_{j=1}^m \mathbf{1}(s^j \neq f(\mathbf{x}^j)), \quad \mathbf{1}(s) = \begin{cases} 1 & s \text{ True} \\ 0 & s \text{ False} \end{cases}$$

Perceptron, two classes linearly separable

$|S| = 2$, i.e. two states (typically also classes)

$$f(\mathbf{x}) = \begin{cases} s = 1, & \text{if } \mathbf{w}^\top \mathbf{x} + w_0 > 0, \\ s = -1, & \text{if } \mathbf{w}^\top \mathbf{x} + w_0 < 0. \end{cases}$$



Perceptron learning – Algorithm

$\mathbf{x}'_j = s_j \begin{bmatrix} 1 \\ \mathbf{x}_j \end{bmatrix}$, $\mathbf{w}' = \begin{bmatrix} w_0 \\ \mathbf{w} \end{bmatrix}$ drop the dashes to avoid notation clutter.

Goal: Find a weight vector $\mathbf{w} \in \Re^{D+1}$ (original feature space dimensionality is D) such that:

$$\mathbf{w}^\top \mathbf{x}_j > 0 \quad (\forall j \in \{1, 2, \dots, m\})$$

Perceptron algorithm (Rosenblatt 1962):

1. $t \leftarrow 0$, $\mathbf{w}^{(t)} \leftarrow 0$.
2. Find a wrongly classified observation \mathbf{x}_j :

$$\mathbf{w}^{(t)\top} \mathbf{x}_j \leq 0, \quad (j \in \{1, 2, \dots, m\}.)$$

3. If there is no misclassified observation then terminate. Otherwise,

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbf{x}_j.$$

4. Goto 2.

Perceptron learning – Algorithm

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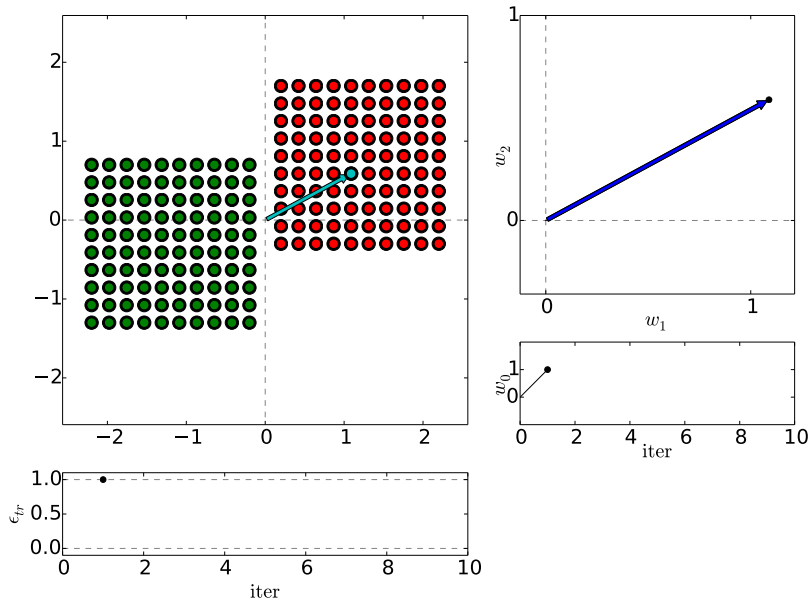
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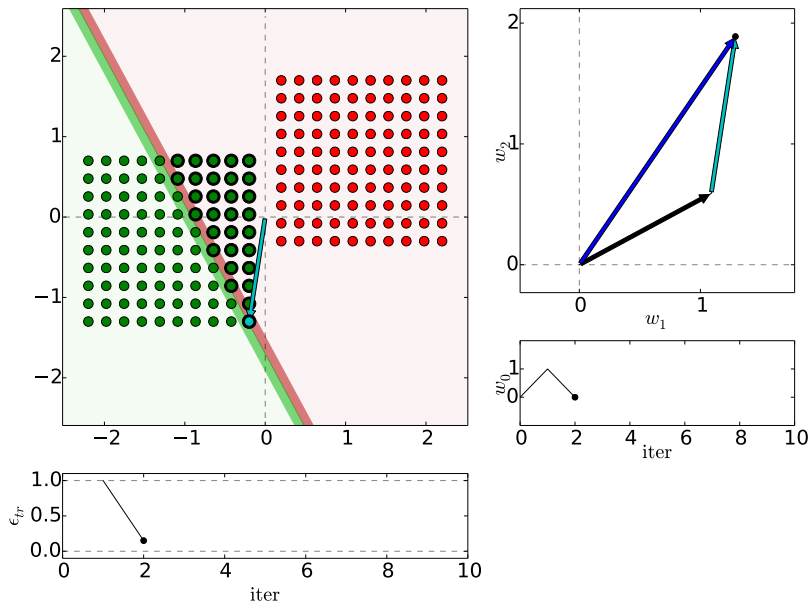
$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbf{x}_j .$$

4. Goto 2.

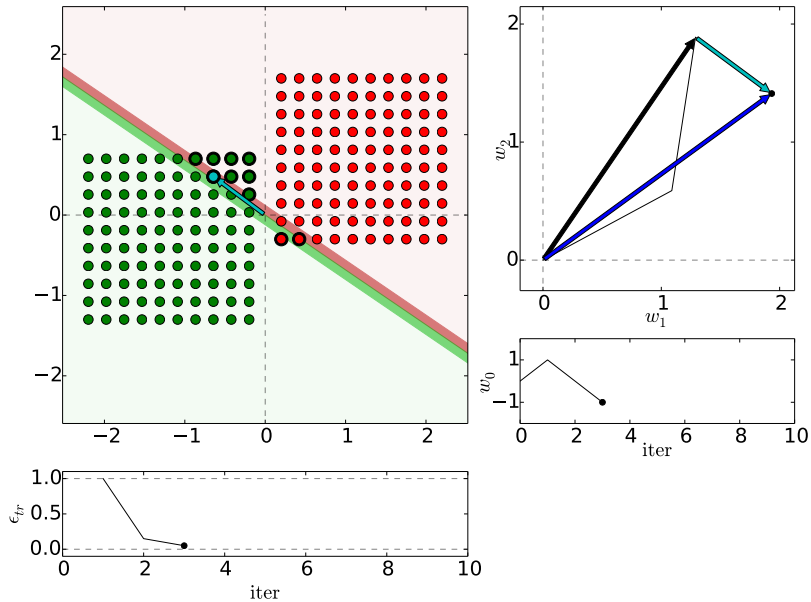
Perceptron iterations



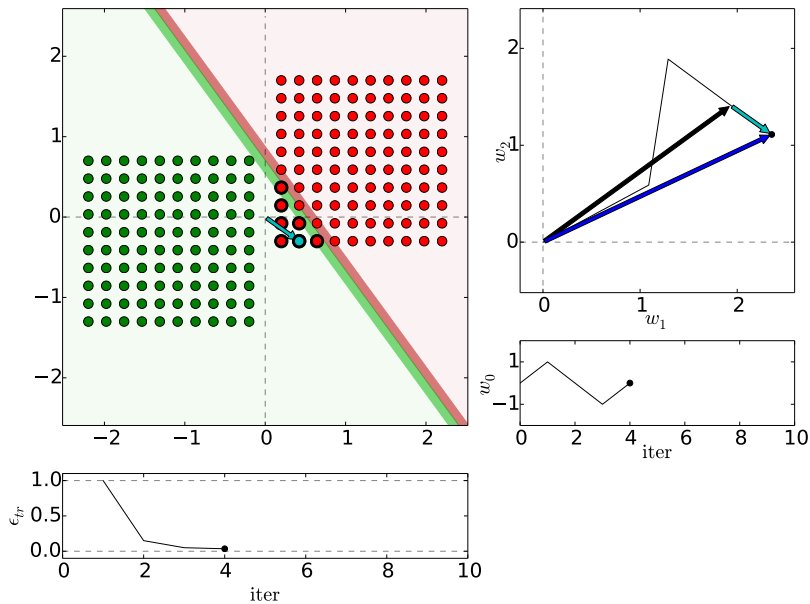
Perceptron iterations



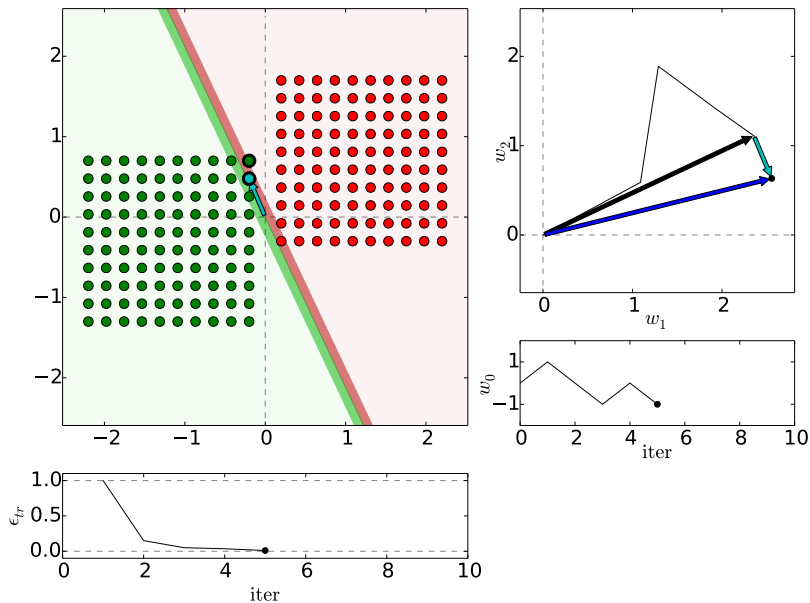
Perceptron iterations



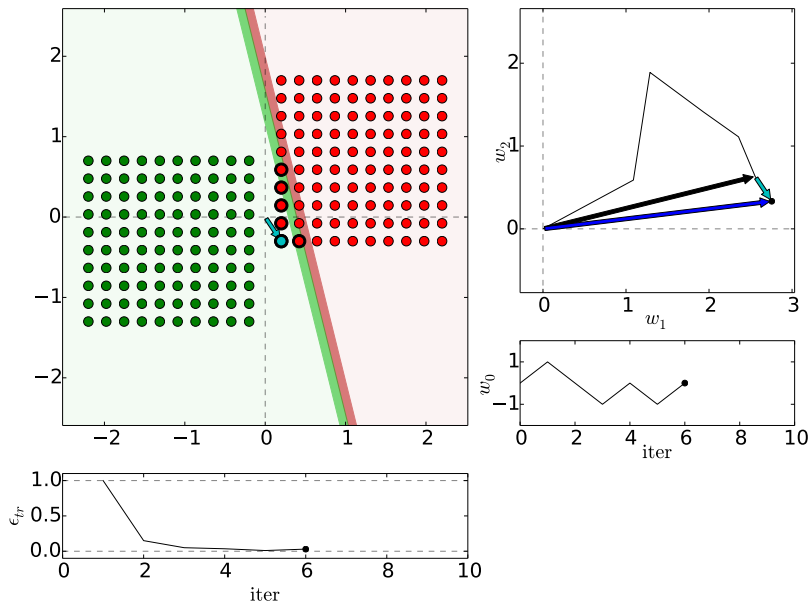
Perceptron iterations



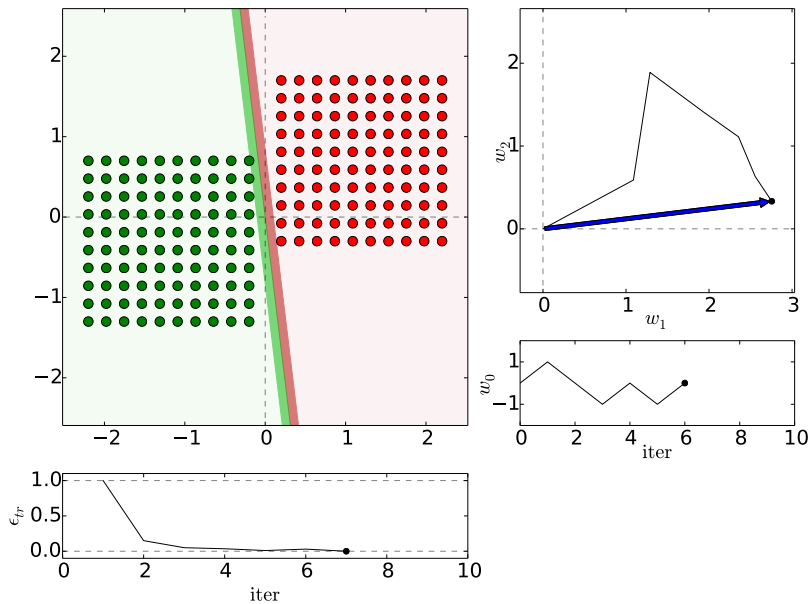
Perceptron iterations



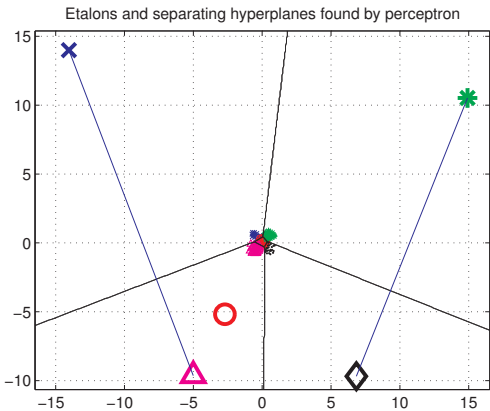
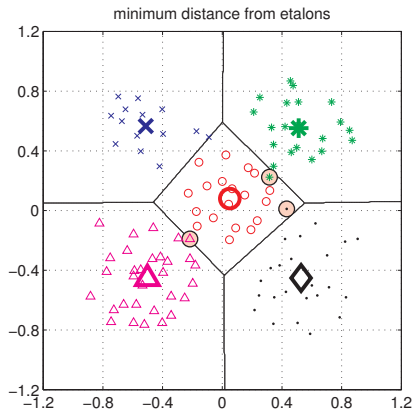
Perceptron iterations



Perceptron iterations

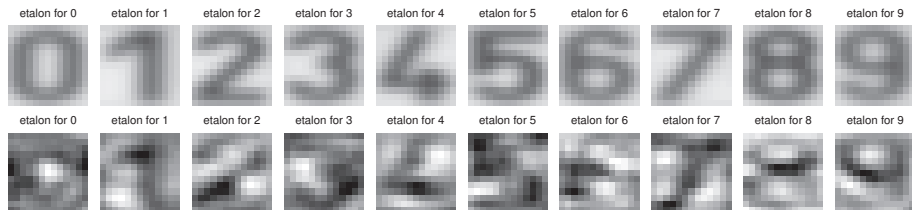


Etalons: means vs found by perceptron



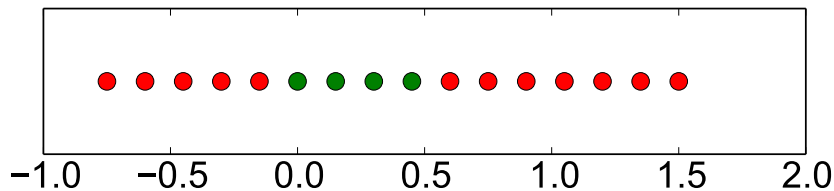
Figures from [5]

Digit recognition - etalons means vs. perceptron



Figures from [5]

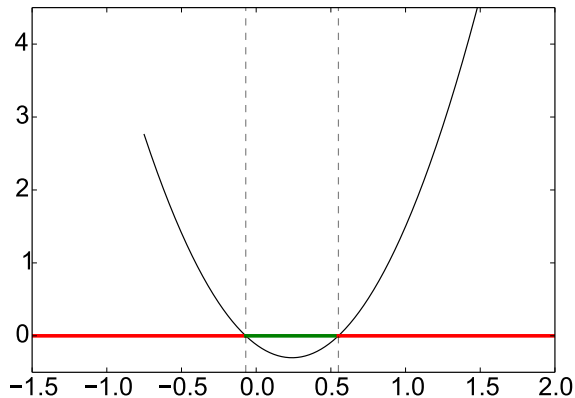
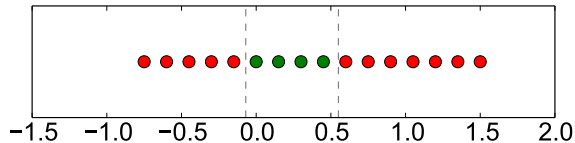
What if not lin separable?



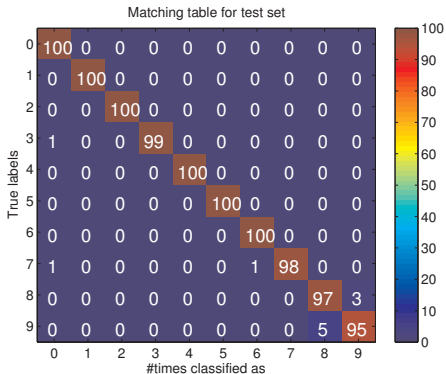
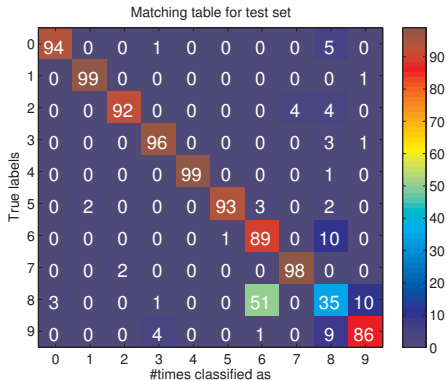
Dimension lifting

$$\mathbf{x} = [x, x^2]^T$$

Dimension lifting, $\mathbf{x} = [x, x^2]^\top$

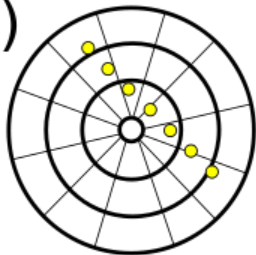


Performance comparison, parameters fixed

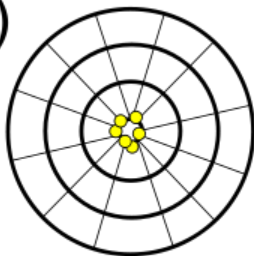


Accuracy vs precision

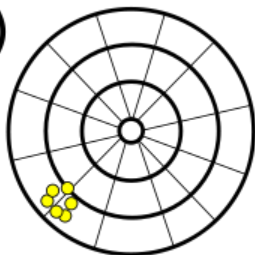
(a)



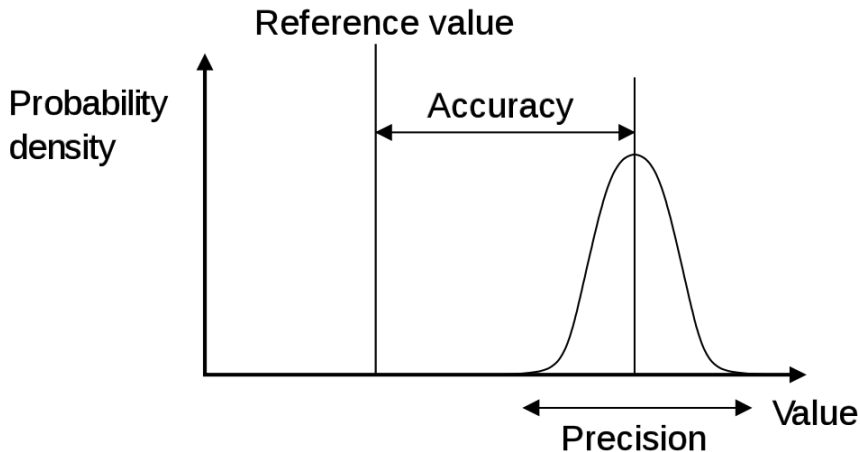
(b)



(c)



Accuracy vs precision



https://en.wikipedia.org/wiki/Accuracy_and_precision

References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. Many Matlab figures created with the help of [3]

[1] Christopher M. Bishop.

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[2] Richard O. Duda, Peter E. Hart, and David G. Stork.

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[3] Votjěch Franc and Václav Hlaváč.

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- [4] Stuart Russell and Peter Norvig.
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- [5] Tomáš Svoboda, Jan Kybic, and Hlaváč Václav.
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