Sequential decisions under uncertainty
Policy iteration

Tomáš Svoboda & Matej Hoffmann

Vision for Robots and Autonomous Systems, Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering, Czech Technical University in Prague

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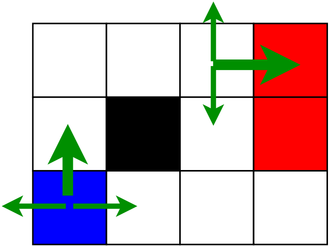
Unreliable actions in observable grid world

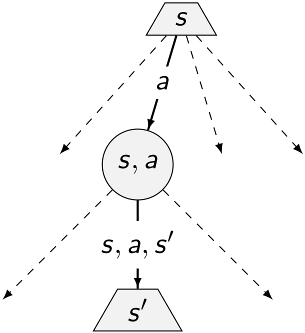
- Walls block movement – agent/robot stays in place.
- Actions do not always go as planned.
- Agent receives rewards each time step:
  - Small "living" reward/penalty.
  - Big rewards/penalties at the end.
- Goal: maximize sum of (discounted) rewards



MDPs recap

Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s, a)$ or $T(s, a, s')$
- Rewards $r(s, a, s')$; and discount γ



MDP quantities:

- Policy $\pi(s) : \mathcal{S} \to \mathcal{A}$
- Utility – sum of (discounted) rewards.
- Values – expected future utility from a state (max-node), $v(s)$
- Q-Values – expected future utility from a q-state (chance-node), $q(s, a)$

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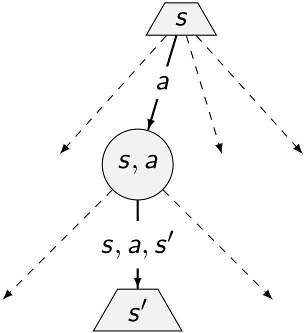


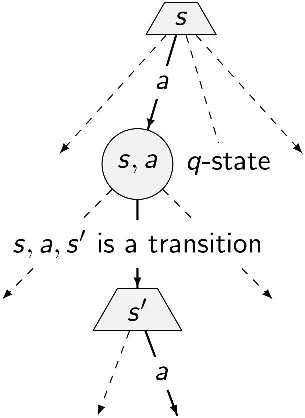
Optimal quantities

- The optimal policy: $\pi^*(s)$ – optimal action from state s
- Expected utility/return of a policy.

$$U^\pi(S_t) = \mathrm{E}^\pi\left[\sum_{k=0}^{\infty} \gamma^t R_{t+k+1}\right]$$

Best policy π^* maximizes above.

- The value of a state s: $v^*(s)$ – expected utility starting in s and acting optimally.
- The value of a q-state (s, a): $q^*(s, a)$ - expected utility having taken a from state s and acting optimally thereafter.



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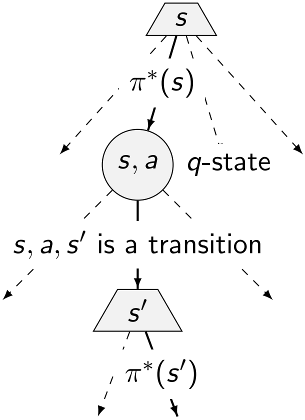

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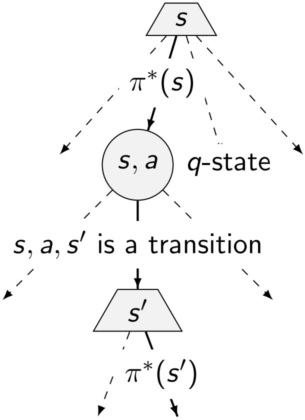

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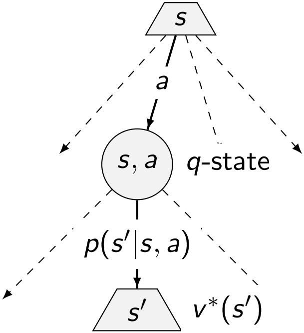
V^* and Q^*

The value of a q-state (s, a):

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s'))\right]$$

The value of a state s:

$$v^*(s) = \max_a q^*(s,a)$$



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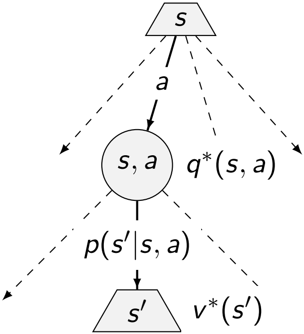

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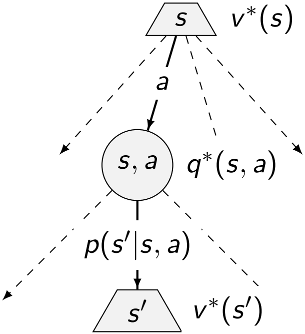

Maze: v^* vs. q^*

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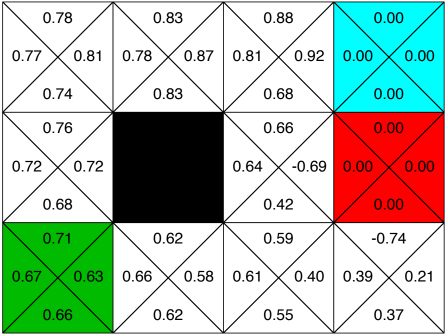

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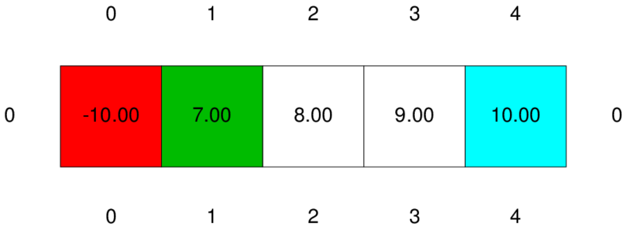
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7.00
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0
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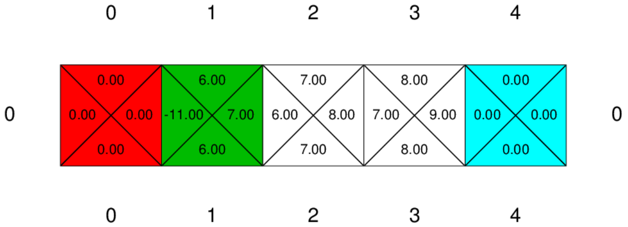
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0
0.00 0.00 0.00 0.00
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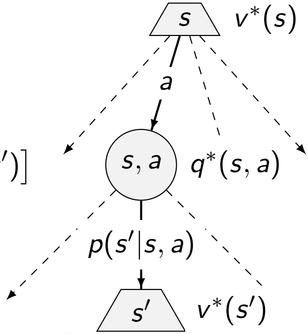
Value iteration

- Bellman equations **characterize** the optimal values

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma v^*(s')\right]$$



- Value iteration **computes** them:

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma V_k(s')\right]$$

Value iteration is a fixed point solution method.

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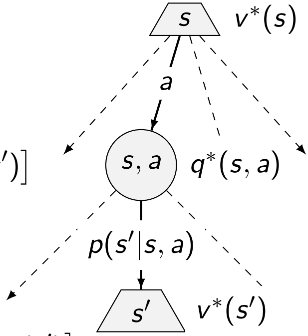

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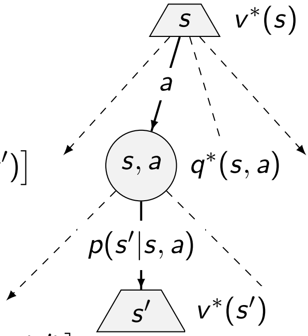

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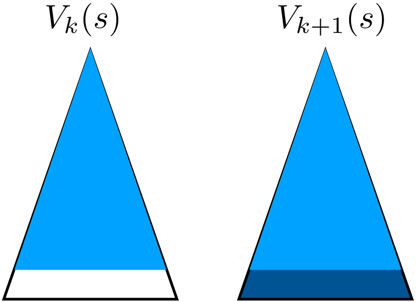
Convergence

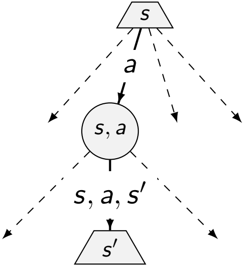
$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma V_k(s')\right]$$

- Thinking about special cases: deterministic world, $\gamma = 0$, $\gamma = 1$.
- For all s, $V_k(s)$ and $V_{k+1}(s)$ can be seen as expectimax search trees of depth k and $k+1$



From Values to Policy

Policy extraction - computing actions from Values



- Assume we have $v^*(s)$
- What is the optimal action?
- We need a one-step expectimax:

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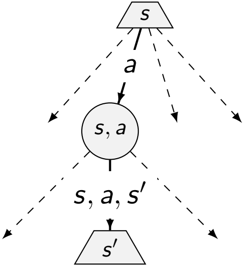

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Policy extraction - computing actions from q-Values

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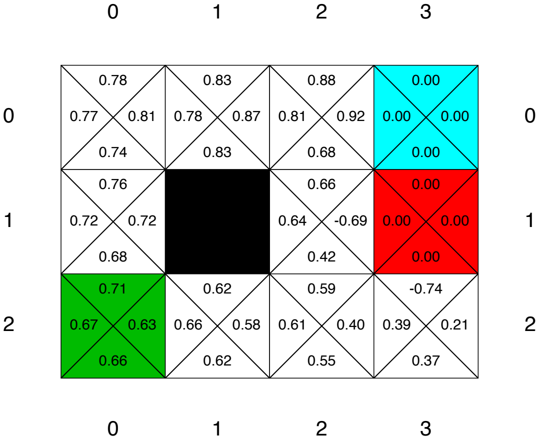

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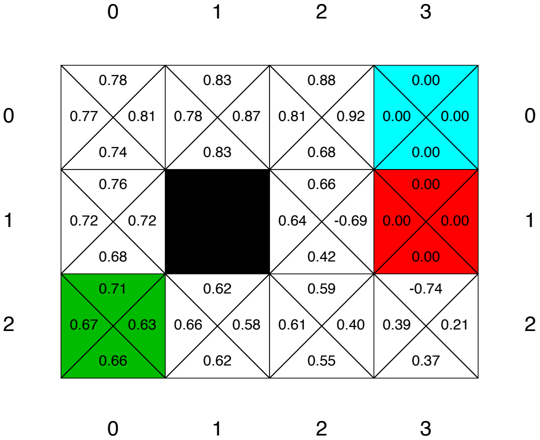

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$$V_{k+1}(s) \leftarrow \max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V_k(s') \right]$$

- What is complexity of one iteration - over all S states?
- Does the "max" change often?
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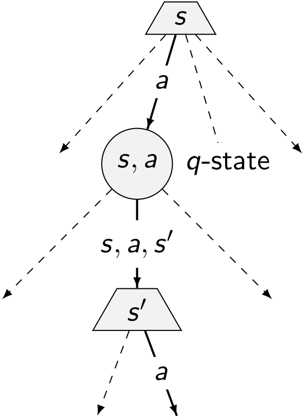
Policy evaluation

- Assume $\pi(s)$ given.
- How to evaluate (compare)?

Fixed policy, do what π says



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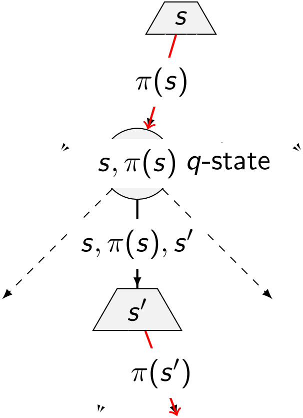

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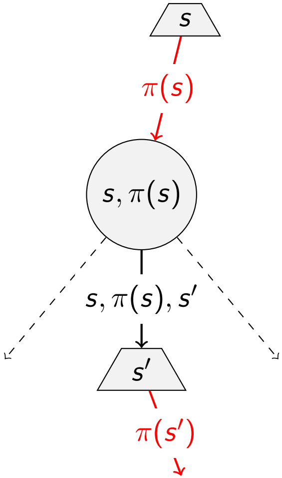

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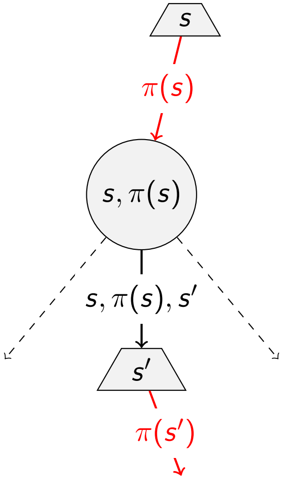

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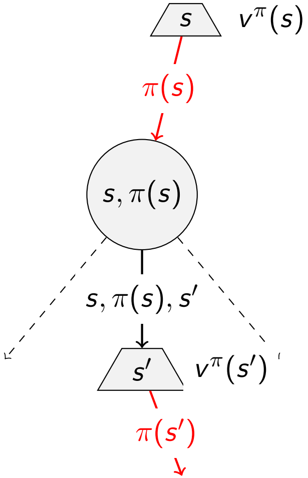

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How to compute $v^{\pi}(s)$?

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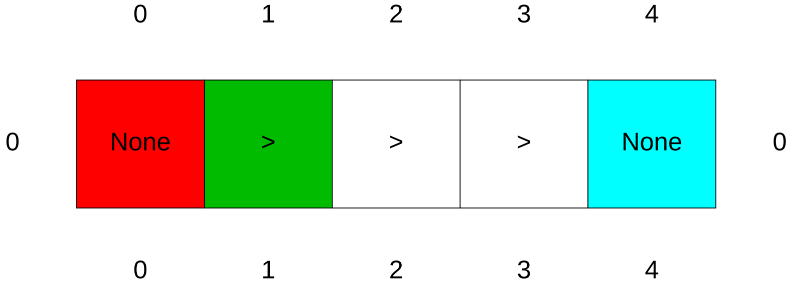

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- Start with a random policy.
- Step 1: Evaluate it.
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- Repeat steps until policy converges.

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Policy iteration

- Policy π evaluation. Solve equations or iterate until convergence.

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, a, s') + \gamma V_k^{\pi_i}(s') \right]$$

- Policy improvement. Look-ahead and keep optimality. Policy extraction from fixed values.

$$\pi_{i+1}(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, a, s') + \gamma V_k^{\pi_i}(s') \right]$$

Policy iteration algorithm

function POLICY-ITERATION(env) **returns:** policy π
  input: env - MDP problem
  $\pi(s) \leftarrow$ random $a \in A(s)$ in all states
  $V(s) \leftarrow 0$ in all states
  repeat ▷ iterate values until no change
    $V \leftarrow$ POLICY-EVALUATION(π, V, env)
    unchanged $\leftarrow$ True
    for each state s **in** S **do**
      if $\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ **then**
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- Policy iteration.
  - Update utilities is fast – only one action per state.
  - New policy from values (slower)
  - New policy is better or done.
- Both methods belong to Dynamic programming realm.

References

Further reading: Chapter 17 of [1] however, policy iteration is quite compact there. More detailed discussion can be found in chapter Dynamic programming in [2] with slightly different notation, though. This lecture has been also greatly inspired by the 9th lecture of CS 188 at http://ai.berkeley.edu as it convincingly motivates policy search and offers an alternative convergence proof of the value iteration method.

[1] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
http://aima.cs.berkeley.edu/.

[2] Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning; an Introduction.
MIT Press, 2nd edition, 2018.
http://www.incompleteideas.net/book/bookdraft2018jan1.pdf.

Bandits