# Sequential decisions under uncertainty
# Markov Decision Processes (MDP)

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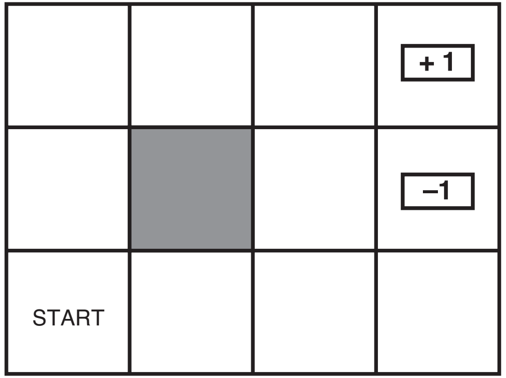
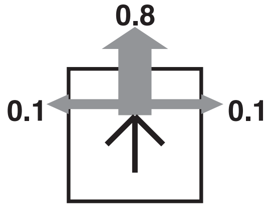
## Unreliable actions in observable grid world





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Observable - agent knows where it is. However, it does not always obey the command.

There is a treasure (desired goal/end state) but there is also some danger (unwanted goal/end state).

The danger state - think about a mountaineous are with safer but longer and shorter but more dangerous paths - a dangerous node may represent a chasm.

Notation note: caligraphic letters like $\mathcal{S}, \mathcal{A}$ will denote the set(s) of all states/actions.

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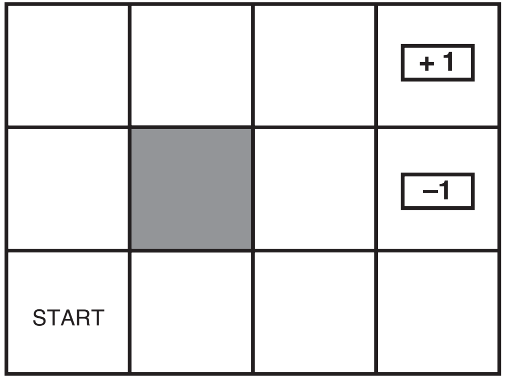

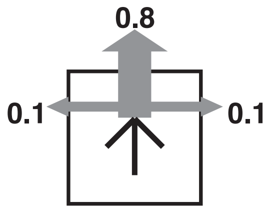

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## Unreliable actions

Actions: go over a glacier bridge or around?

## Plan? Policy

- In deterministic world: Plan – sequence of actions from Start to Goal.
- MDPs, we need a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$.
- An action for each possible state.
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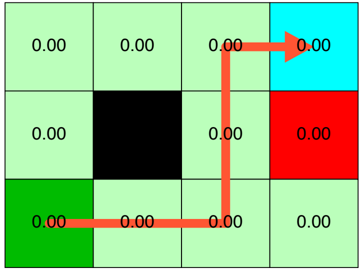

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What is the best policy, we will come to that in a minute, ...

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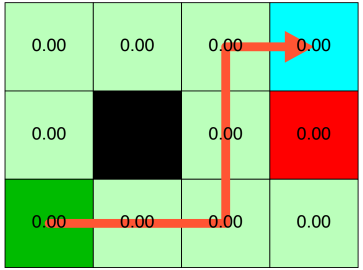

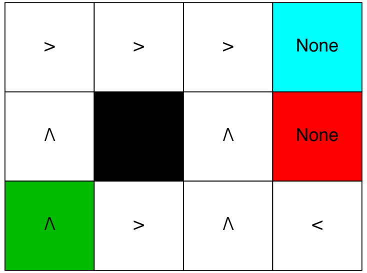

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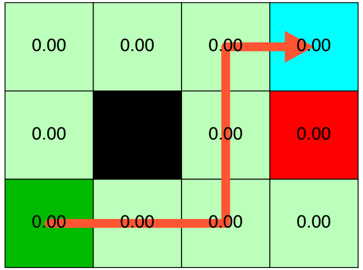

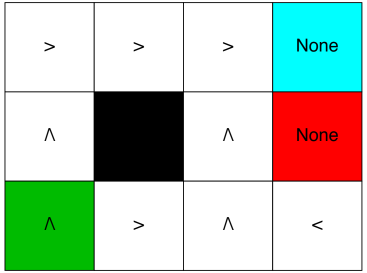

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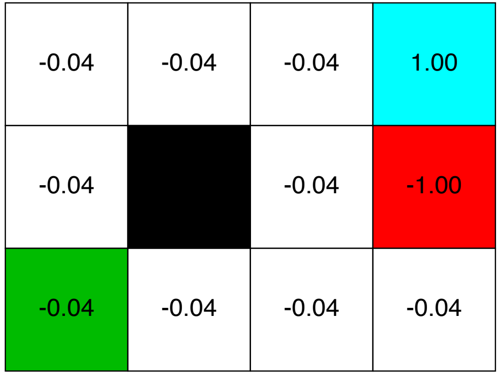
## Rewards



Reward : Robot/Agent takes an action *a* and it is **immediately** rewarded.

Reward function $r(s)$ (or $r(s,a)$, $r(s,a,s')$)

$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

What do the rewards express? Reward to an agent to be/dwell in that state? Obviously we want the robot to go to the goal and do not stay too long in the maze.

**Thinking about Reward**: Robot/Agent takes an action *a* and it is immediately rewarded for this. The reward may depend on

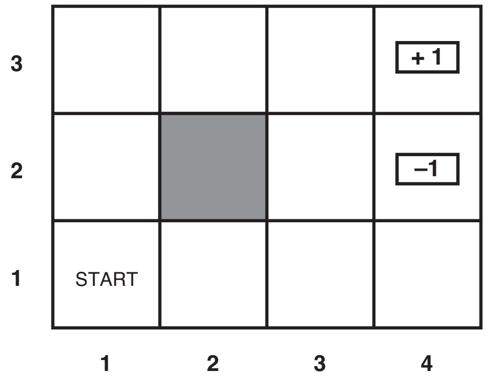
- current state $s$,
- the action taken $a$
- the next state $s'$ - result of the action.

**Rewards for terminal states** can be understood in a way: there is only one action: $a =$ exit. We will come to this soon.

The **reward function** is a property of (is related to) the problem.

**Notation remark:** lowercase letters will be used for functions like $p, r, v, f, \ldots$

## Markov Decision Processes (MDPs)



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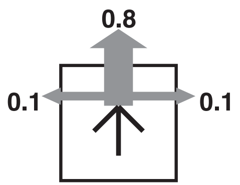

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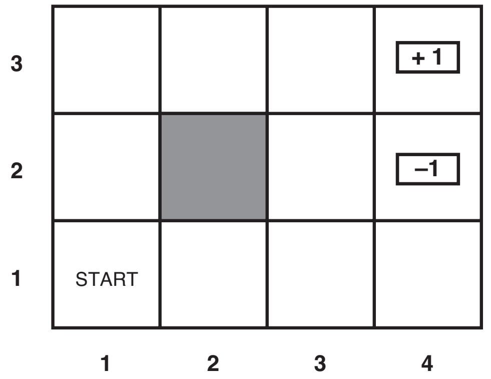

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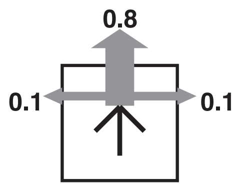

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## Markovian property

- Given the present state, the future and the past are independent.
- MDP: Markov means action depends only on the current state.
- In search: successor function (transition model) depends on the current state only.

## Optimal(?) policies

On-line demos.

- $r(s) = \{-0.04, 1, -1\}$
- $r(s) = \{-2, 1, -1\}$
- $r(s) = \{-0.01, 1, -1\}$

How to measure quality of a policy?

We run `mdp_agents.py` changing reward functions.

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## Utilities of sequences

- State reward reward value at time/step $t$, $R_t$.
- State at time $t$, $S_t$. State sequence $[S_0, S_1, S_2, \ldots,]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \ldots] \succ [R, R'_1, R'_2, R'_3, \ldots] \Leftrightarrow [R_1, R_2, R_3, \ldots] \succ [R'_1, R'_2, R'_3, \ldots]$$

If stationary preferences:
Utility ($h$-history)
$U_h([S_0, S_1, S_2, \ldots,]) = R_1 + R_2 + R_3 + \cdots$

We consdier discrete time $t$. $S_t, R_t$ notation emphasises the time sequence - not a sequence of particular states. The reward is for an action (transition)

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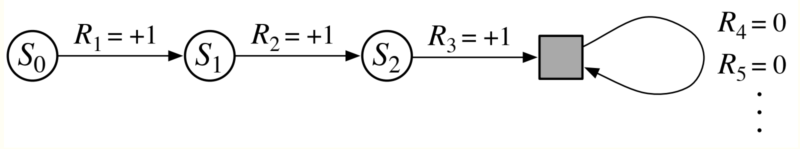
## Returns and Episodes



- Executing policy - sequence of states and **rewards**.
- Episode starts at $t$, ends at $T$ (ending in a terminal state).
- Return (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$$

## Comparing policies; Finite vs infinite horizon

Problem: Infinite lifetime $\Rightarrow$ additive utilities are infinite.

- Finite horizon: termination at a fixed time $\Rightarrow$ nonstationary policy, $\pi(s)$ depends on the time left.
- Discounted return , $\gamma < 1$, $R_t \leq R_{\max}$

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- Absorbing (terminal) state.

Returns are successive steps related to each other

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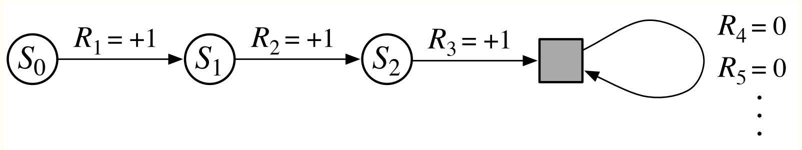

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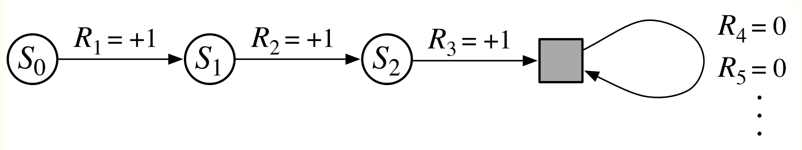

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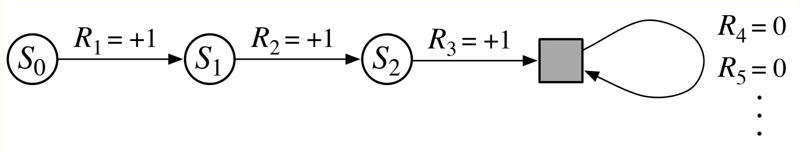

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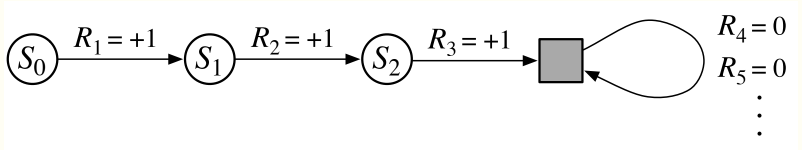

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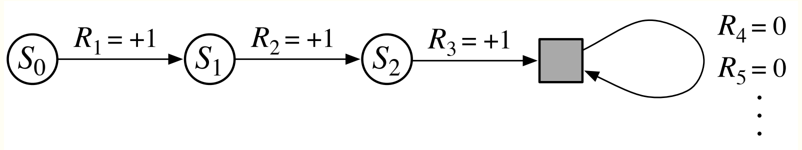

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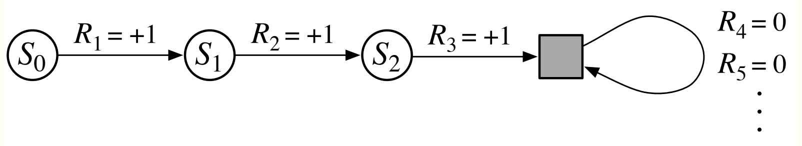

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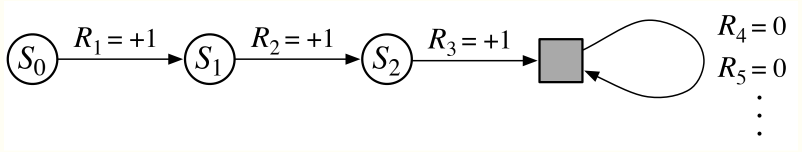

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## MDPs recap

Markov decition processes (MDPs):

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- Set of actions $\mathcal{A}$
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- (deterministic) Policy $\pi(s)$ – choice of action for each state
- Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

## MDPs recap

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## Value functions

- Executing policy $\pi$ - sequence of states (and rewards).
- Utility of a state sequence.
- But actions are unreliable - environment is stochastic.
- Expected return of a policy $\pi$.

Starting at time $t$, i.e. $S_t$,

$$U^{\pi}(S_t) = \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

**Value function**

$$v^{\pi}(s) = \mathrm{E}^{\pi}\left[G_t \mid S_t = s\right] = \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

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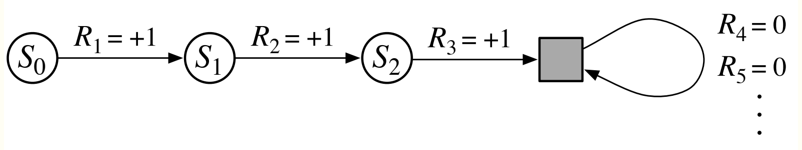

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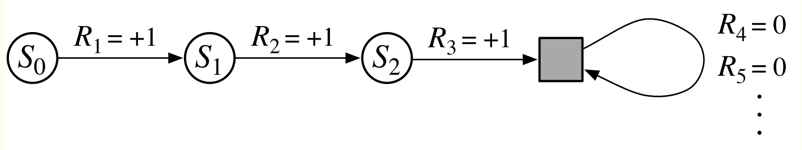

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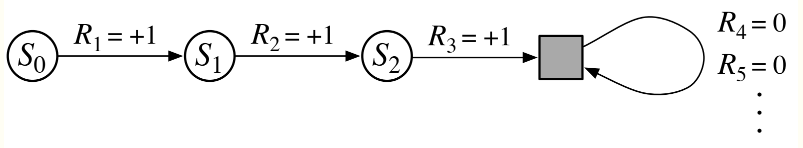

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We still do not know how to compute the optimality, ... right?

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|  | 0 | 1 | 2 | 3 |  |
|---|---|---|---|---|---|
| 0 | 0.81 | 0.87 | 0.92 | 1.00 | 0 |
| 1 | 0.76 | ⬛ | 0.66 | -1.00 | 1 |
| 2 | 0.70 | 0.65 | 0.61 | 0.39 | 2 |
|  | 0 | 1 | 2 | 3 |  |

|  | 0 | 1 | 2 | 3 |  |
|---|---|---|---|---|---|
| 0 | > | > | > | 1.00 | 0 |
| 1 | ∧ | ⬛ | ∧ | -1.00 | 1 |
| 2 | ∧ | < | < | < | 2 |
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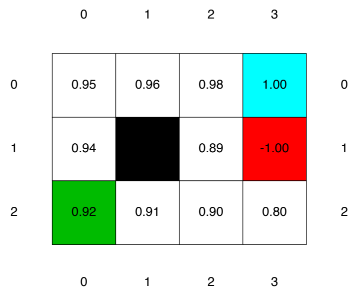
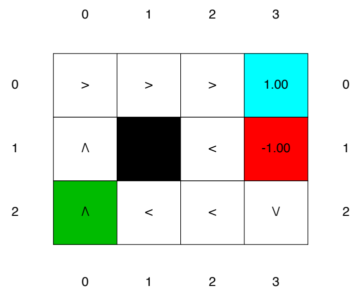


|  | 0 | 1 | 2 | 3 |  |
|---|---|---|---|---|---|
| 0 | 0.95 | 0.96 | 0.98 | 1.00 | 0 |
| 1 | 0.94 | ■ | 0.89 | -1.00 | 1 |
| 2 | 0.92 | 0.91 | 0.90 | 0.80 | 2 |
|  | 0 | 1 | 2 | 3 |  |



|  | 0 | 1 | 2 | 3 |  |
|---|---|---|---|---|---|
| 0 | > | > | > | 1.00 | 0 |
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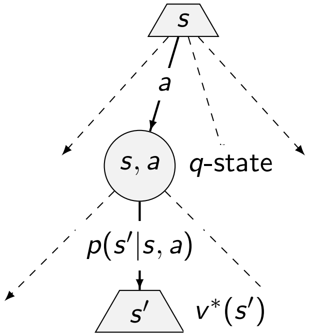
## MDP search tree

The value of a $q$-state $(s, a)$:

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s'))\right]$$

The value of a state $s$:

$$v^*(s) = \max_a q^*(s,a)$$



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Remind Expectimax algorithm from the last lecture.
How to compute $V(s)$? Well, we could solve the expectimax search - but it grows quickly. We can see $R(s)$ as the price for leaving the state $s$ just anyhow.

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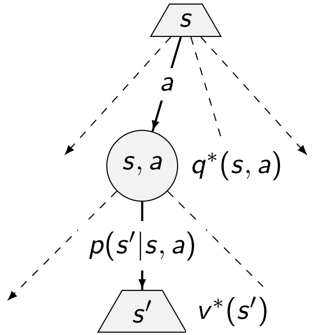

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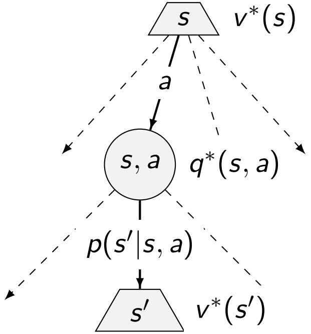

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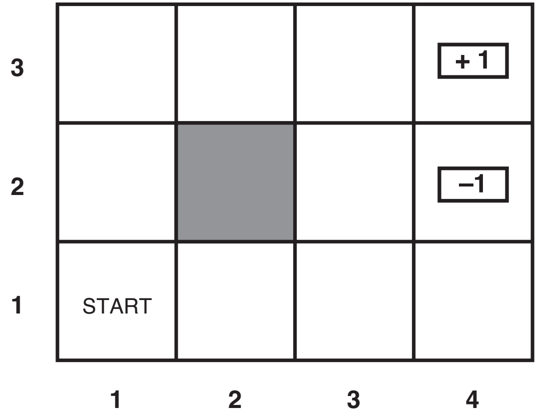
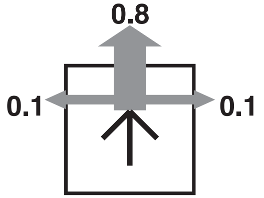
## Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma v^*(s')\right]$$



(a)



(b)

$v$ computation on the table - one row for each action. We got $n$ equations for $n$ unknown - $n$ states. But max is a non-linear operator!

## Value iteration

- Start with arbitrary $V_0(s)$ (except for terminals)
- Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

- Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

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## Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \le R(s) \le R_{\max}$$

Max norm:

$$\|V\| = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \le \frac{R_{\max}}{1-\gamma}$$

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## Convergence cont'd

$V_{k+1} \leftarrow BV_k$
$\|BV_k - BV'_k\| \leq \gamma \|V_k - V'_k\|$
$\|BV_k - V_{\text{true}}\| \leq \gamma \|V_k - V_{\text{true}}\|$
Rewards are bounded, at the beginning then Value error is
$\|V_0 - V_{true}\| \leq \frac{2R_{\max}}{1-\gamma}$
We run $N$ iterations and reduce the error by factor $\gamma$ in each and want to stop the error is below $\epsilon$:
$\gamma^N 2R_{\max}/(1-\gamma) \leq \epsilon$ Taking logs, we find: $N \geq \frac{\log(2R_{\max}/\epsilon(1-\gamma))}{\log(1/\gamma)}$
To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.
We stop if

$$\|V_{k+1} - V_k\| \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also: $\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$ Proof on the next slide

Try to proove that:

$$\|\max f(a) - \max g(a)\| \leq \max \|f(a) - g(a)\|$$

## Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\| \le \epsilon$ is the same as $\|V_{k+1} - V_{\infty}\| \le \epsilon$
Assume $\|V_{k+1} - V_k\| = \text{err}$
In each of the following iteration steps we reduce the error by the factor $\gamma$.
Till $\infty$, the total sum of reduced errors is:

$$\text{total} = \gamma\text{err} + \gamma^2\text{err} + \gamma^3\text{err} + \gamma^4\text{err} + \cdots = \frac{\gamma\text{err}}{(1-\gamma)}$$

We want to have total $< \epsilon$.

$$\frac{\gamma\text{err}}{(1-\gamma)} < \epsilon$$

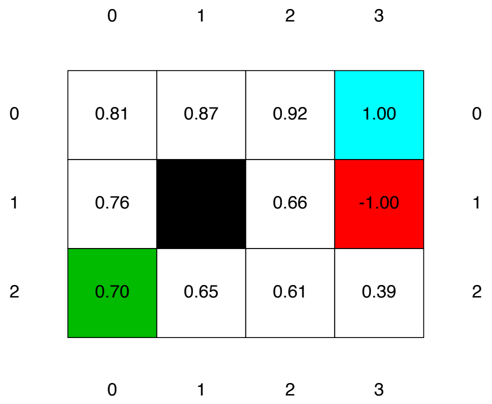
From it follows that

$$\text{err} < \frac{\epsilon(1-\gamma)}{\gamma}$$

Hence we can stop if $\|V_{k+1} - V_k\| < \epsilon(1-\gamma)/\gamma$

## Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$



Run `mdp_agents.py` and try to compute next state value in advance. Remind the $R(s) = -0.04$ and $\gamma = 1$ in order to simplify computation. Then discuss the course of the Values.

## Value iteration algorithm

```
function VALUE-ITERATION(env,$\epsilon$) returns: state values $V$
    input: env - MDP problem, $\epsilon$
    $V' \leftarrow 0$ in all states
    repeat                                          ▷ iterate values until convergence
        $V \leftarrow V'$                                     ▷ keep the last known values
        $\delta \leftarrow 0$                                    ▷ reset the max difference
        for each state $s$ in $S$ do
            $V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a)V(s')$
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## References

Some figures from [1] (chapter 17) but notation slightly changed adapting notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course.

[1] Stuart Russell and Peter Norvig.
*Artificial Intelligence: A Modern Approach*.
Prentice Hall, 3rd edition, 2010.
`http://aima.cs.berkeley.edu/`.

[2] Richard S. Sutton and Andrew G. Barto.
*Reinforcement Learning; an Introduction*.
MIT Press, 2nd edition, 2018.
`http://www.incompleteideas.net/book/the-book-2nd.html`.