Sequential decisions under uncertainty
Markov Decision Processes (MDP)

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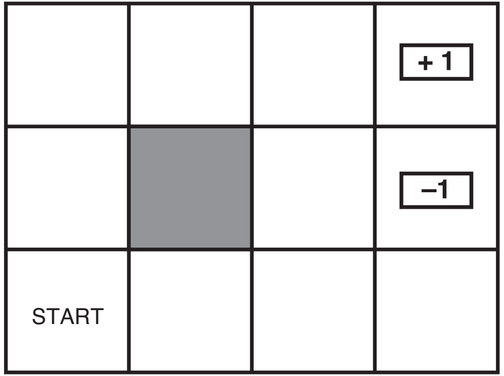
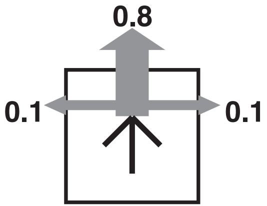
Unreliable actions in observable grid world





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Model $T(s, a, s') \equiv p(s'|s, a)$ = probability that a in s leads to s'

Observable - agent knows where it is. However, it does not always obey the command.
There is a treasure (desired goal/end state) but there is also some danger (unwanted goal/end state).
The danger state - think about a mountaineous are with safer but longer and shorter but more dangerous paths - a dangerous node may represent a chasm.

Notation note: caligraphic letters like $\mathcal{S}, \mathcal{A}$ will denote the set(s) of all states/actions.

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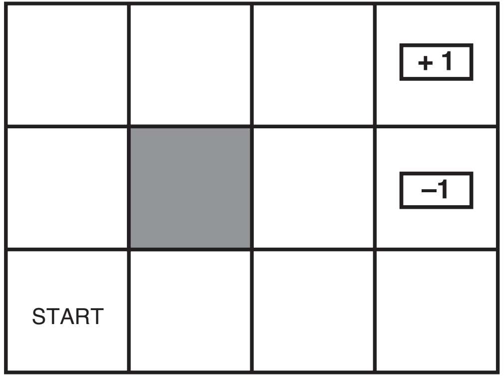

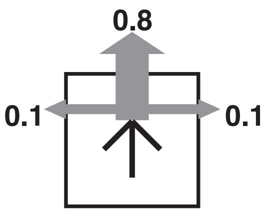

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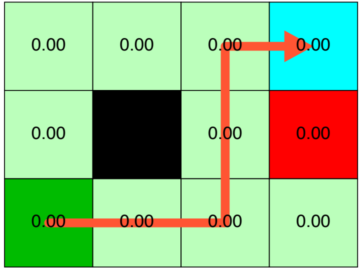
Unreliable actions

Actions: go over a glacier bridge or around?

Plan? Policy

- In deterministic world: Plan – sequence of actions from Start to Goal.
- MDPs, we need a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$.
- An action for each possible state.
- What is the best policy?



Remember, we can end up in any state. I any state, the robot/agent has to know what to do.

What is the best policy, we will come to that in a minute, ...

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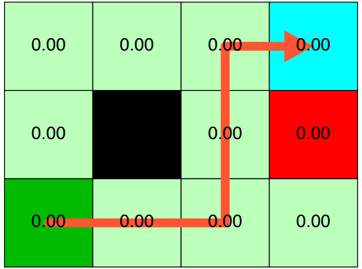

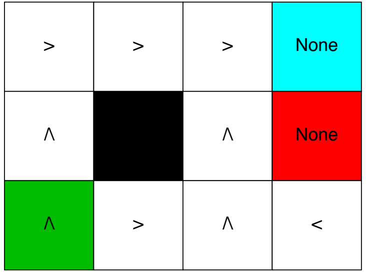

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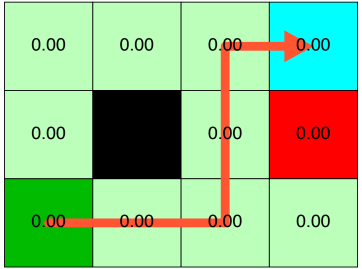

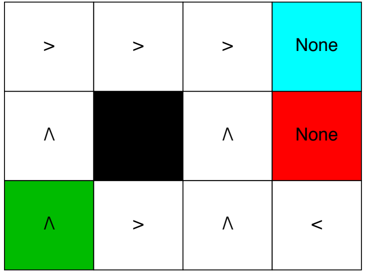

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Rewards

-0.04	-0.04	-0.04	1.00
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Reward : Robot/Agent takes an action a and it is **immediately** rewarded.

Reward function $r(s)$ (or $r(s,a)$, $r(s,a,s')$)

$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

What do the rewards express? Reward to an agent to be/dwell in that state? Obviously we want the robot to go to the goal and do not stay too long in the maze.

Thinking about Reward: Robot/Agent takes an action a and it is immediately rewarded for this. The reward may depend on

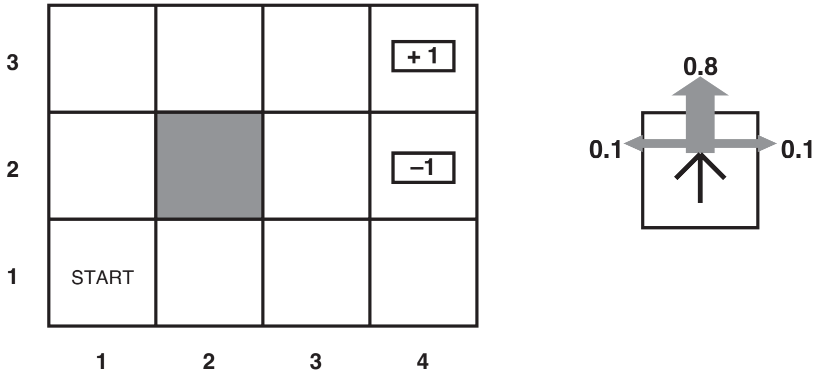
- current state s,
- the action taken a
- the next state s' - result of the action.

Rewards for terminal states can be understood in a way: there is only one action: $a =$ exit. We will come to this soon.

The **reward function** is a property of (is related to) the problem.

Notation remark: lowercase letters will be used for functions like $p, r, v, f, \ldots$

Markov Decision Processes (MDPs)



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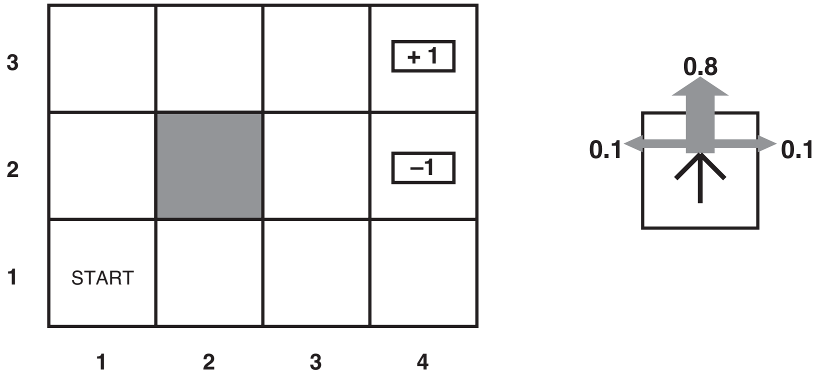

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Markovian property

- Given the present state, the future and the past are independent.
- MDP: Markov means action depends only on the current state.
- In search: successor function (transition model) depends on the current state only.

Optimal(?) policies

On-line demos.

- $r(s) = \{-0.04, 1, -1\}$
- $r(s) = \{-2, 1, -1\}$
- $r(s) = \{-0.01, 1, -1\}$

How to measure quality of a policy?

We run `mdp_agents.py` changing reward functions.

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Utilities of sequences

- State reward reward value at time/step t, R_t.
- State at time t, S_t. State sequence $[S_0, S_1, S_2, \ldots,]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \ldots] \succ [R, R'_1, R'_2, R'_3, \ldots] \Leftrightarrow [R_1, R_2, R_3, \ldots] \succ [R'_1, R'_2, R'_3, \ldots]$$

If stationary preferences:
Utility (h-history)
$U_h([S_0, S_1, S_2, \ldots,]) = R_1 + R_2 + R_3 + \cdots$

We consdier discrete time t. S_t, R_t notation emphasises the time sequence - not a sequence of particular states. The reward is for an action (transition)

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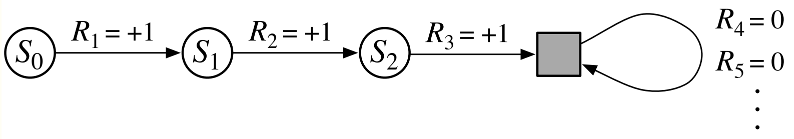
Returns and Episodes

- Executing policy - sequence of states and **rewards**.
- Episode starts at t, ends at T (ending in a terminal state).
- Return (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$$



Comparing policies; Finite vs infinite horizon

Problem: Infinite lifetime ⇒ additive utilities are infinite.

- Finite horizon: termination at a fixed time ⇒ nonstationary policy, $\pi(s)$ depends on the time left.
- Discounted return , $\gamma < 1$, $R_t \leq R_{\max}$

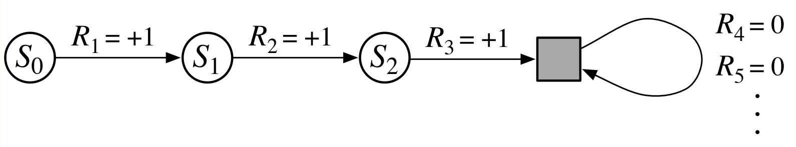
$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

- Absorbing (terminal) state.

Returns are successive steps related to each other

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma(R_{t+2} + \gamma^1 R_{t+3} + \gamma^2 R_{t+4} + \cdots) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

Discounting is quite natural choice. Think about your preferences/rewards. Go to pub with friends tonight, studying (for the far future reward of getting A in the course)?



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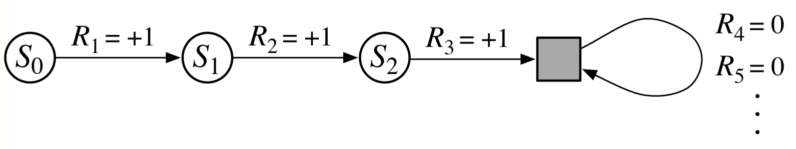

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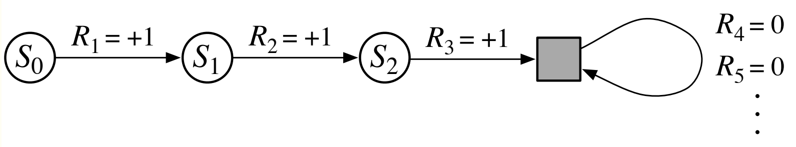

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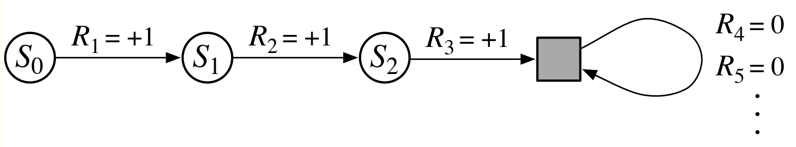

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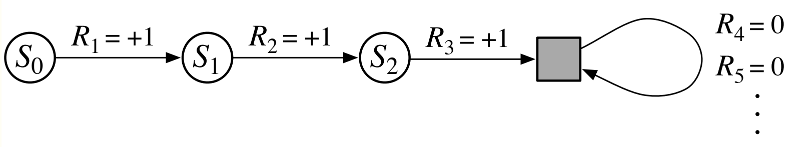

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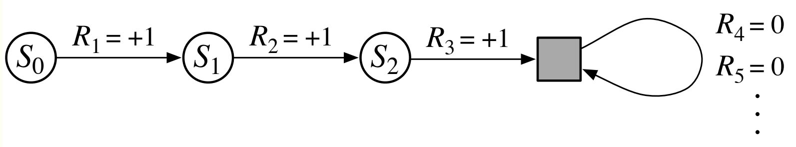

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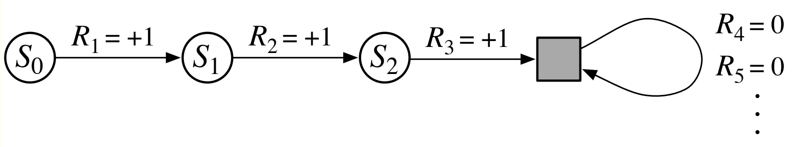

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MDPs recap

Markov decition processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s, a)$ or $T(s, a, s')$
- Reward function $r(s, a, s')$; and discount γ

MDP quantities:

- (deterministic) Policy $\pi(s)$ – choice of action for each state
- Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

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Value functions

- Executing policy π - sequence of states (and rewards).
- Utility of a state sequence.
- But actions are unreliable - environment is stochastic.
- Expected return of a policy π.

Starting at time t, i.e. S_t,

$$U^{\pi}(S_t) = \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty}\gamma^k R_{t+k+1}\right]$$

Value function

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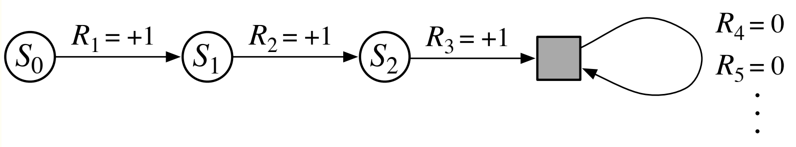

Expected value can be also computed by running (executing) the policy many times and then computing average - Monte Carlo simulation methods.

Value functions

- Executing policy π - sequence of states (and rewards).
- Utility of a state sequence.
- But actions are unreliable - environment is stochastic.
- Expected return of a policy π.

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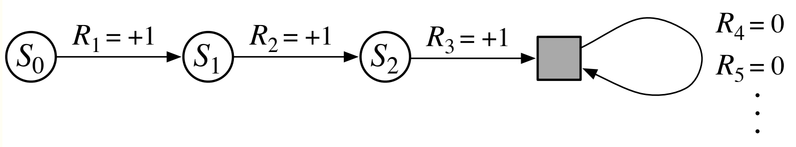

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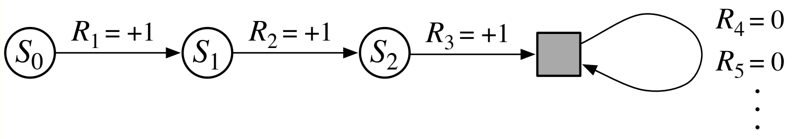

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Optimal policy π^*, and optimal value $v^*(s)$

$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

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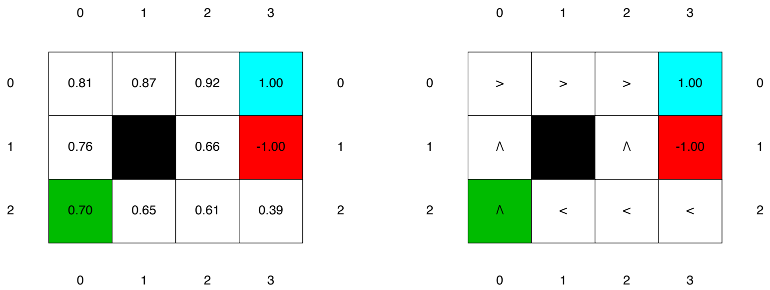

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	0	1	2	3	
0	0.95	0.96	0.98	1.00	0
1	0.94		0.89	-1.00	1
2	0.92	0.91	0.90	0.80	2
	0	1	2	3	

	0	1	2	3	
0	>	>	>	1.00	0
1	∧		<	-1.00	1
2	∧	<	<	∨	2
	0	1	2	3	

We still do not know how to compute the optimality, ... right?

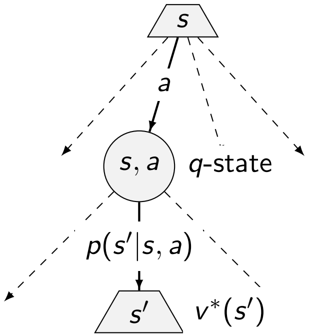
MDP search tree

The value of a q-state (s, a):

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right]$$

The value of a state s:

$$v^*(s) = \max_a q^*(s,a)$$



$$\begin{aligned} v^\pi(s) &= \mathrm{E}^\pi\left[G_t \mid S_t = s\right] \\ &= \mathrm{E}^\pi\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\ &= \sum_{s'} p(s'|a,s) r(s,a,s') + \gamma \mathrm{E}^\pi\left[G_{t+1} \,\middle|\, S_{t+1} = s'\right] \end{aligned}$$

Remind Expectimax algorithm from the last lecture.
How to compute $V(s)$? Well, we could solve the expectimax search - but it grows quickly. We can see $R(s)$ as the price for leaving the state s just anyhow.

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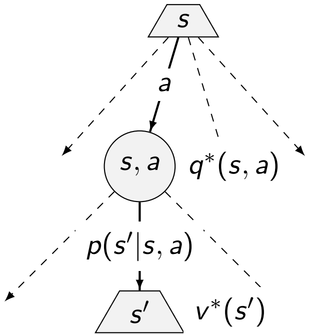

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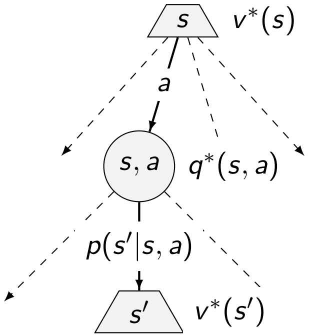

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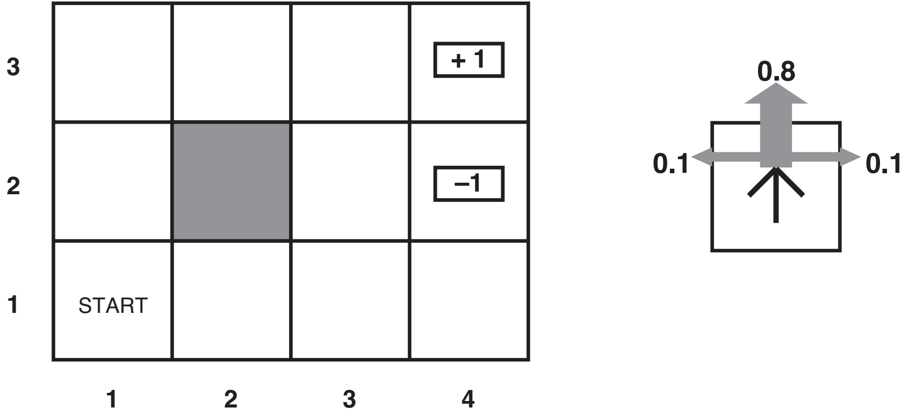
Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a,s) r(s,a,s') + \gamma v^*(s')$$



(a) (b)

v computation on the table - one row for each action. We got n equations for n unknown - n states. But max is a non-linear operator!

Value iteration

- Start with arbitrary $V_0(s)$ (except for terminals)
- Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

- Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

What is the complexity of each iteration? $O(S^2 A)$

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Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \leq R(s) \leq R_{\max}$$

Max norm:

$$\|V\| = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma}$$

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Convergence cont'd

$V_{k+1} \leftarrow BV_k$
$\|BV_k - BV'_k\| \leq \gamma \|V_k - V'_k\|$
$\|BV_k - V_{\text{true}}\| \leq \gamma \|V_k - V_{\text{true}}\|$
Rewards are bounded, at the beginning then Value error is
$\|V_0 - V_{true}\| \leq \frac{2R_{\max}}{1-\gamma}$
We run N iterations and reduce the error by factor γ in each and want to stop the error is below ϵ:
$\gamma^N 2R_{\max}/(1-\gamma) \leq \epsilon$ Taking logs, we find: $N \geq \frac{\log(2R_{\max}/\epsilon(1-\gamma))}{\log(1/\gamma)}$
To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.
We stop if

$$\|V_{k+1} - V_k\| \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also: $\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$ Proof on the next slide

Try to proove that:

$$\|\max f(a) - \max g(a)\| \leq \max \|f(a) - g(a)\|$$

Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$ is the same as $\|V_{k+1} - V_{\infty}\| \leq \epsilon$
Assume $\|V_{k+1} - V_k\| = \text{err}$
In each of the following iteration steps we reduce the error by the factor γ.
Till ∞, the total sum of reduced errors is:

$$\text{total} = \gamma\text{err} + \gamma^2\text{err} + \gamma^3\text{err} + \gamma^4\text{err} + \cdots = \frac{\gamma\text{err}}{(1-\gamma)}$$

We want to have total $< \epsilon$.

$$\frac{\gamma\text{err}}{(1-\gamma)} < \epsilon$$

From it follows that

$$\text{err} < \frac{\epsilon(1-\gamma)}{\gamma}$$

Hence we can stop if $\|V_{k+1} - V_k\| < \epsilon(1-\gamma)/\gamma$

Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
2	0.70	0.65	0.61	0.39	2
	0	1	2	3	

Run `mdp_agents.py` and try to compute next state value in advance. Remind the $R(s) = -0.04$ and $\gamma = 1$ in order to simplify computation. Then discuss the course of the Values.

Value iteration algorithm

function VALUE-ITERATION(env,ϵ) **returns:** state values V
    input: env - MDP problem, ϵ
    $V' \leftarrow 0$ in all states
    repeat ▷ iterate values until convergence
        $V \leftarrow V'$ ▷ keep the last known values
        $\delta \leftarrow 0$ ▷ reset the max difference
        for each state s **in** S **do**
            $V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a)V(s')$
            if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$
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References

Some figures from [1] (chapter 17) but notation slightly changed adapting notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course.

[1] Stuart Russell and Peter Norvig.
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[2] Richard S. Sutton and Andrew G. Barto.
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