Uncertainty, Chances, and Utilities

Tomáš Svoboda

Vision for Robots and Autonomous Systems, Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering, Czech Technical University in Prague

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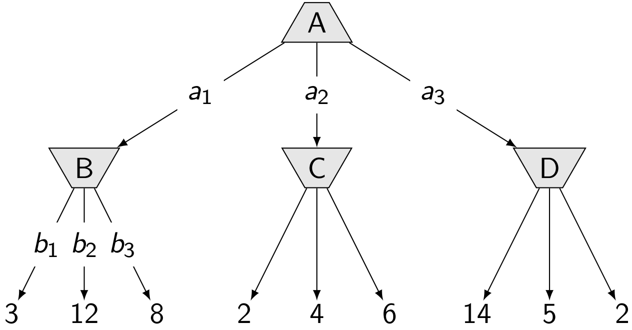
Deterministic opponent → stochastic environment



b_1, b_2, b_3 - probable branches, uncertain outcomes of a_1 action.

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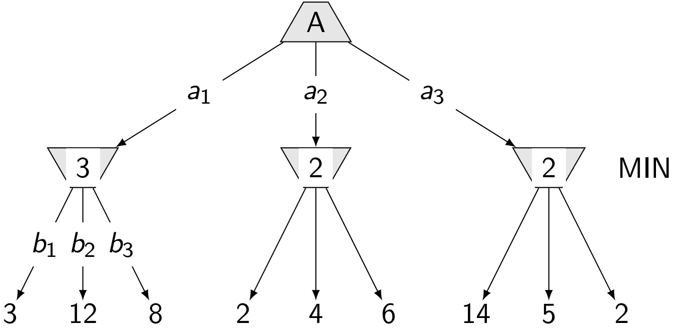

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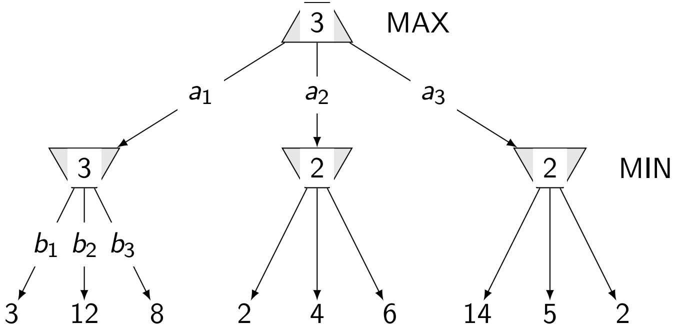

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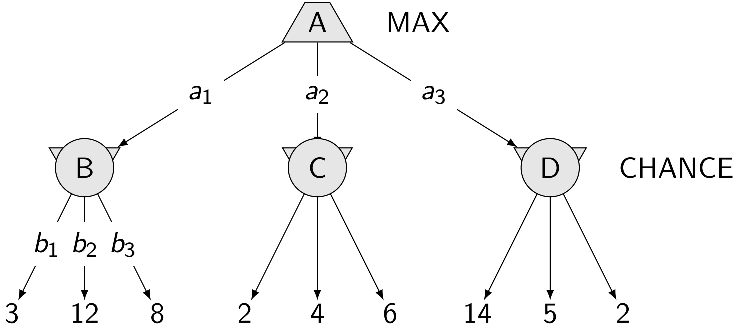

b_1, b_2, b_3 - probable branches, uncertain outcomes of a_1 action.

Why? Actions may fail, ...

Video: Slipping robot

Vision for Robotics and Autonomous Systems, `http://cyber.felk.cvut.cz/vras`

Why? Actions may fail, . . . , getting to work

A At home

tram *bike* *car*

Random variable: Situation on rails R

r_1 free rails

r_2 accident

r_3 congestion

MAX/MIN depends on what the $r_?$ otions and terminal numbers mean.

The goal may be to get to work as fast as possible.

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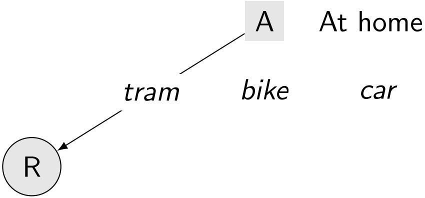

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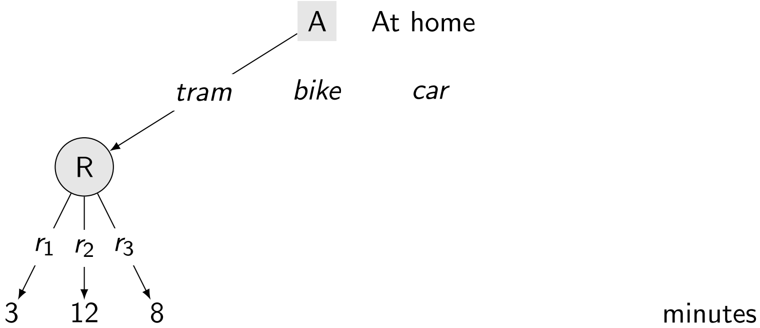

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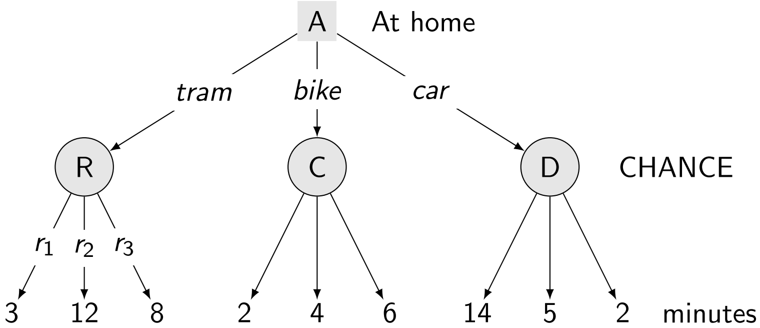

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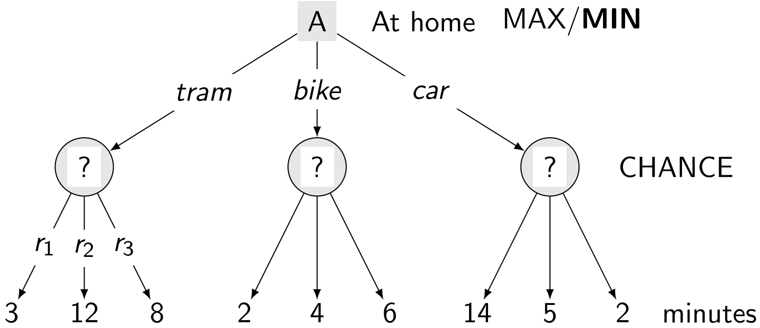

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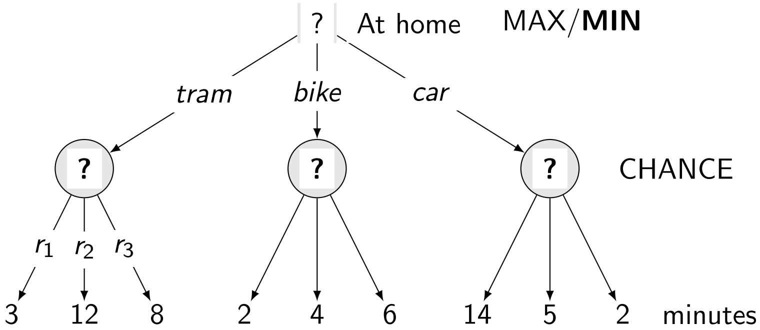
Chance nodes values



- Average case, not the worst case.
- Calculate expected utilities . . .
- i.e. take weighted average (expectation) of successors

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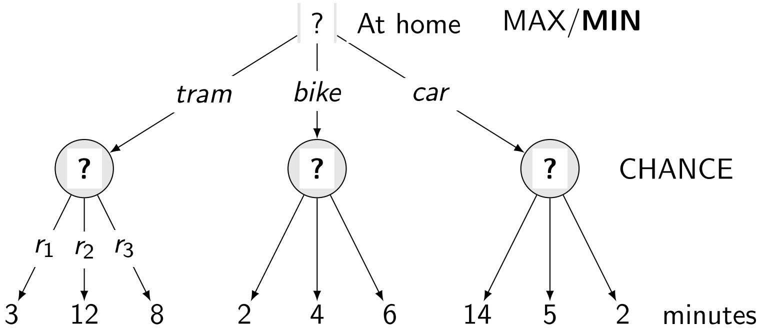

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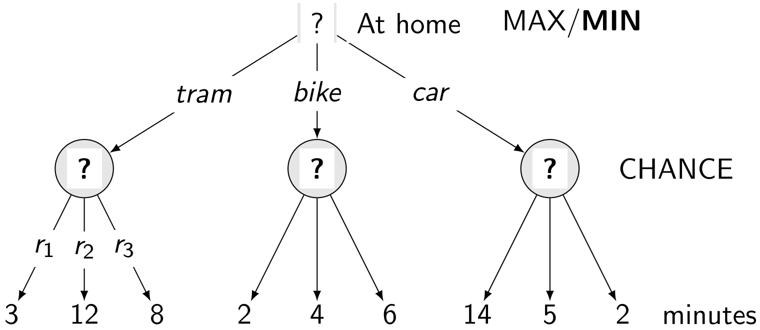

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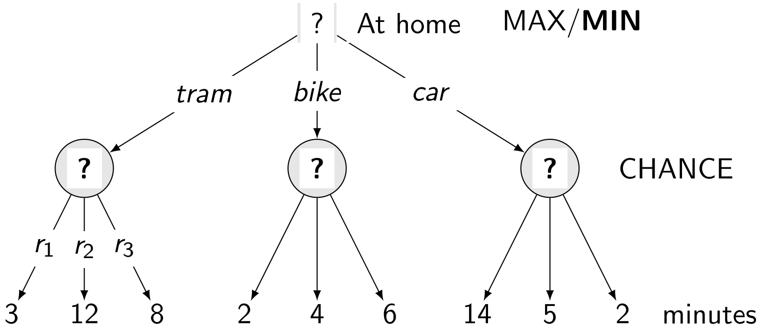

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Expectimax

function EXPECTIMAX(state) **return** a value
    if TERMINAL-TEST(state): **return** UTILITY(state)
    if state (next agent) is **MAX**: **return** MAX-VALUE(state)
    if state (next agent) is **CHANCE**: **return** EXP-VALUE(state)
end function

---

function MAX-VALUE(state) **return** value v
    $v \leftarrow -\infty$
    for a **in** ACTIONS(state) **do**
        $v \leftarrow$ **max**(v, EXPECTIMAX(RESULT(state,a)))
    end for
end function

---

function EXP-VALUE(state) **return** value v
    $v \leftarrow 0$
    for all $r \in$ random events **do**
        $v \leftarrow v + P(r)$ EXPECTIMAX(RESULT(state,r))
    end for
end function

Expectimax

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function EXPECTIMAX(state) return a value
    if TERMINAL-TEST(state): return UTILITY(state)
    if state (next agent) is MAX: return MAX-VALUE(state)
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function MAX-VALUE(state) return value v
    v ← −∞
    for a in ACTIONS(state) do
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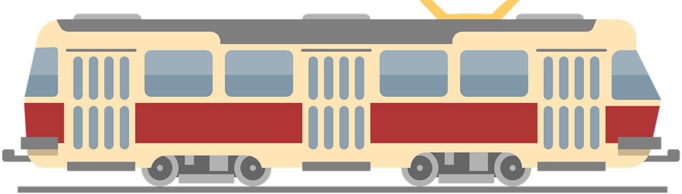
Random variables, probability distribution, . . .

- Random variable - an event with unknown outcome
- Probability distribution - assignment of weights to the outcomes



- Random variable: R - situation on rails
- Outcomes/events: $r \in \{\text{free rails, accindent, congestion}\}$
- Probability distribution: $P(R = \text{free rails}) = 0.3$, $P(R = \text{accident}) = 0.1$, $P(R = \text{congestion}) = 0.6$

Few reminders from laws of probability, Probabilities:

- always non-negative,
- sum over all possible outcomes is equal to 1.

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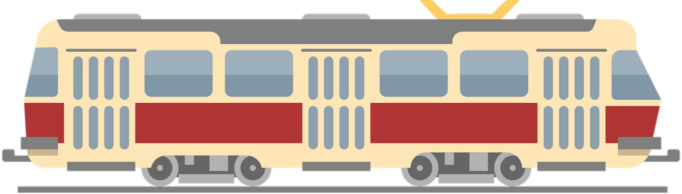

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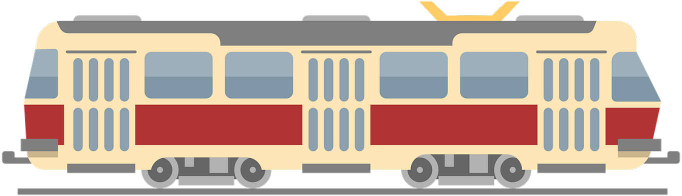

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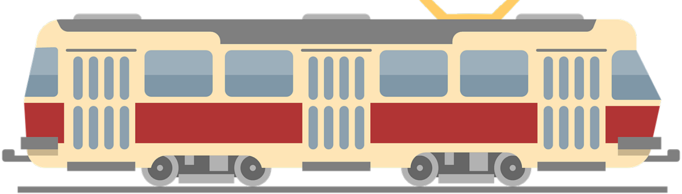

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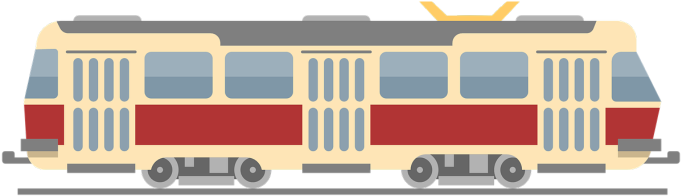

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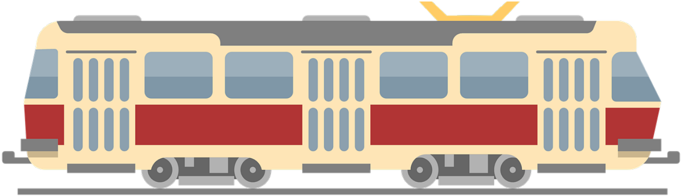

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Expectations, ...

How long does it take to go to work by tram?

- Depends on the random variable R - situation on rails with possible events r_1, r_2, r_3.
- What is the expectation of the time?

$$t = P(r_1)t_1 + P(r_2)t_2 + P(r_3)t_3$$

Weighted average.

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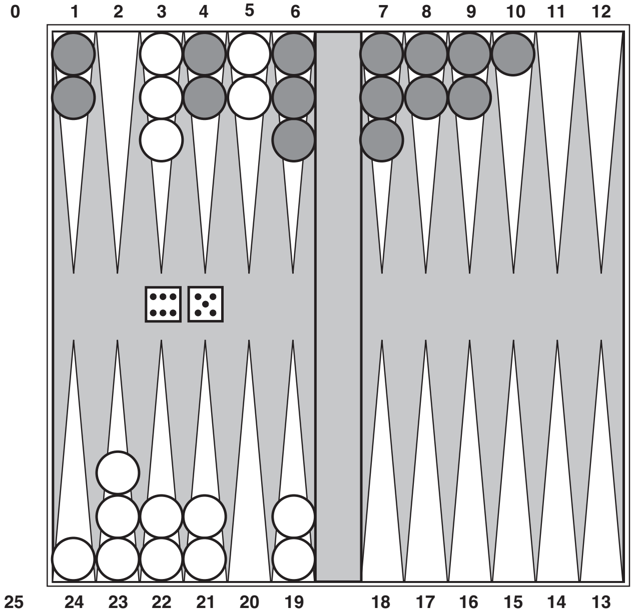
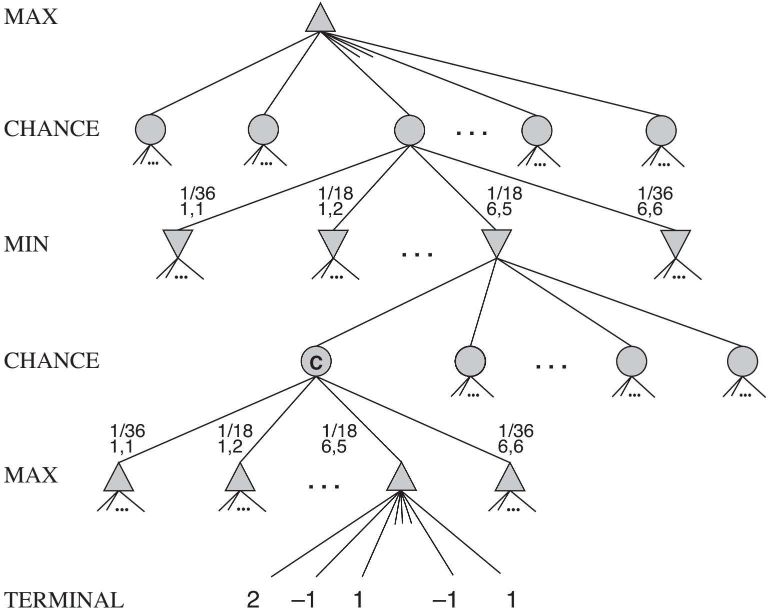
How about the Reversi game?

- Dangerous optimism
- Dangerous pessimism

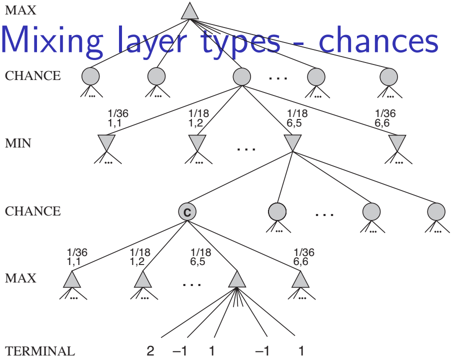
Games with chances **and** strategy

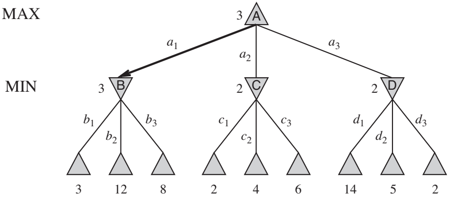


Mixing MAX, CHANCE, and MIN nodes



Mixing layer types - chances inserted





Extra random agent that moves after each MAX and MIN agent

$$\text{EXPECTIMINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if} \quad \text{TERMINAL-TEST}(s) \\ \max_a \text{EXPECTIMINIMAX}(\text{RESULT}(s,a)) & \text{if} \quad \text{PLAYER}(s) = \text{MAX} \\ \min_a \text{EXPECTIMINIMAX}(\text{RESULT}(s,a)) & \text{if} \quad \text{PLAYER}(s) = \text{MIN} \\ \sum_r P(r)\text{EXPECTIMINIMAX}(\text{RESULT}(s,r)) & \text{if} \quad \text{PLAYER}(s) = \text{CHANCE} \end{cases}$$

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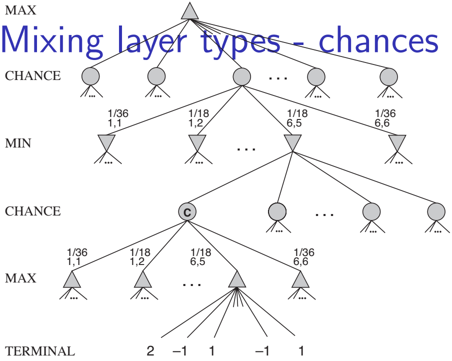

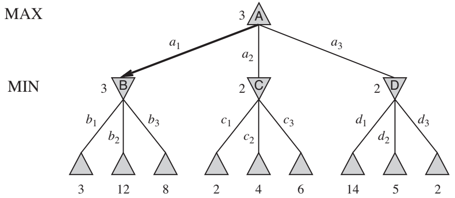

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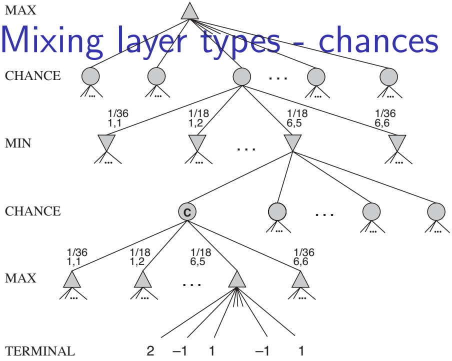

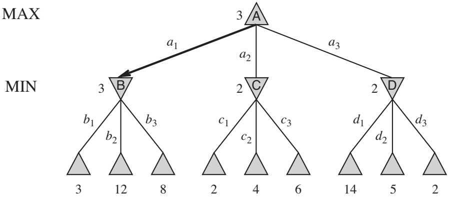

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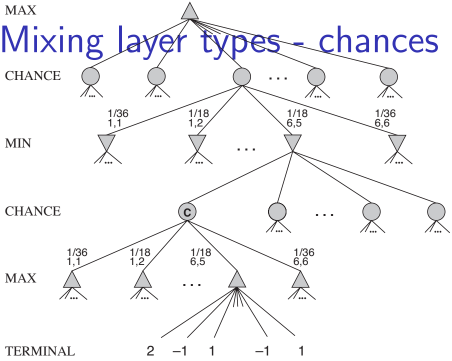

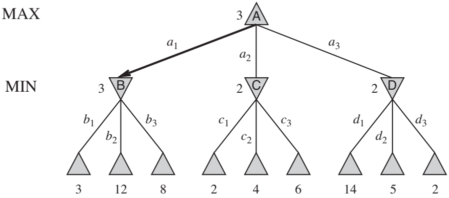

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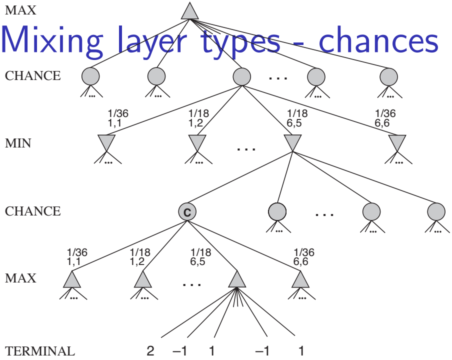

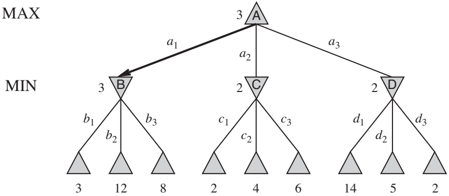

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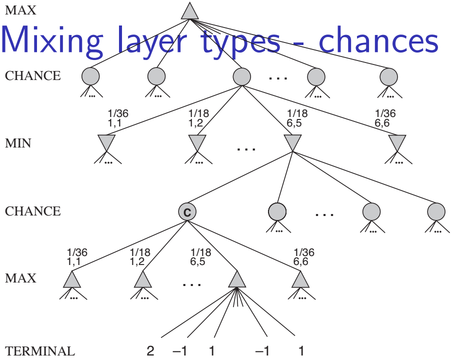

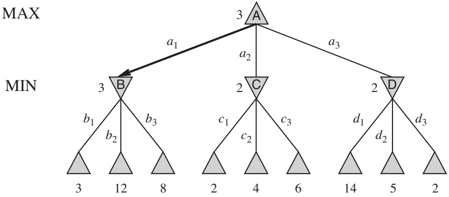

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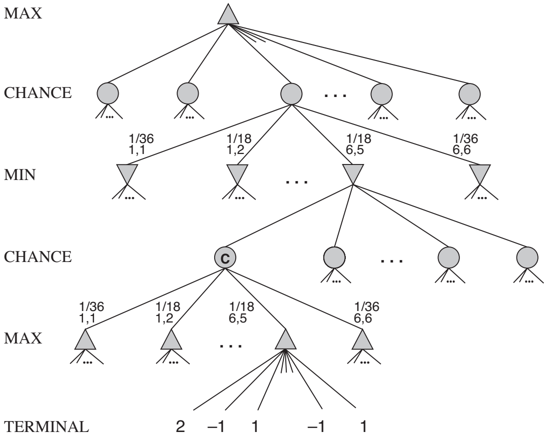
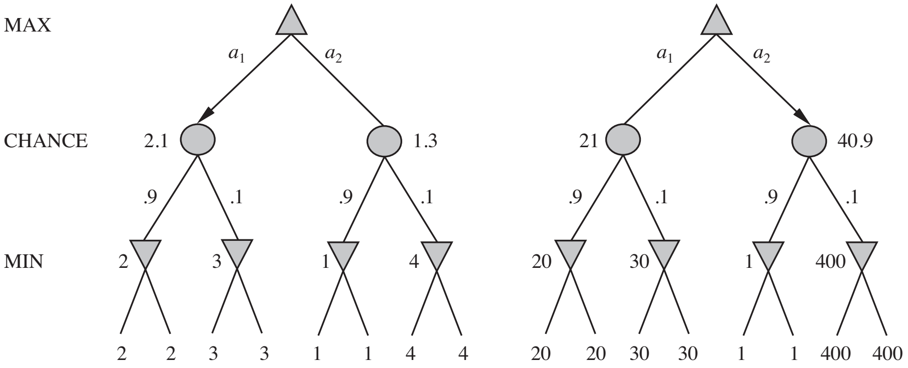
Mixing chances into min/max tree, how big?



- b branching factor
- m maximumm depth
- n number of distinct rolls

Evaluation function



- Scale matters! Not only ordering.
- Can we prune the tree? (α, β like?)

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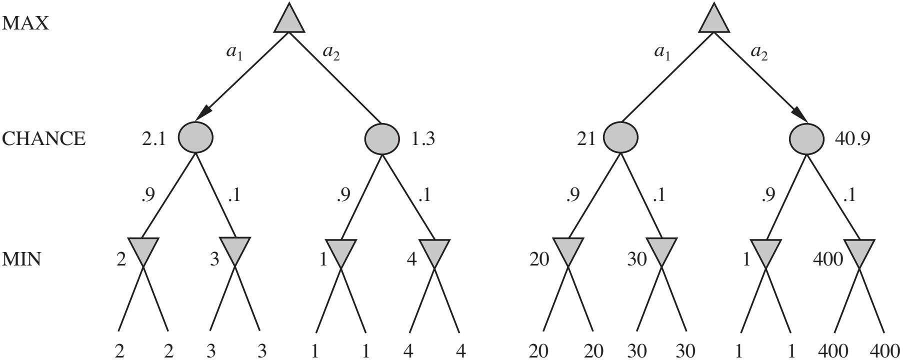

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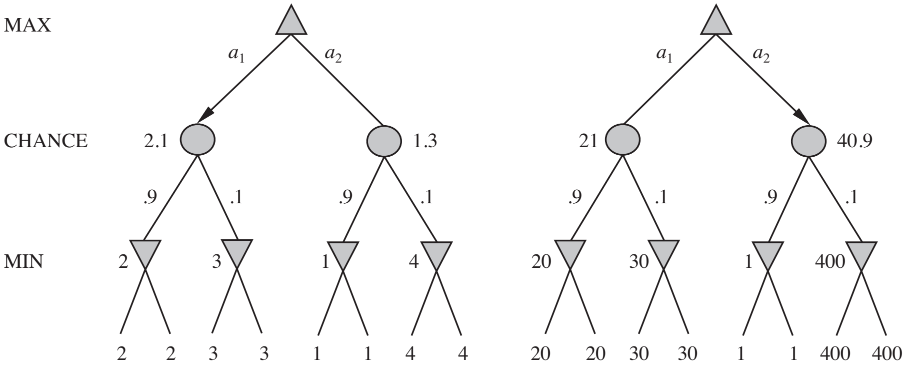

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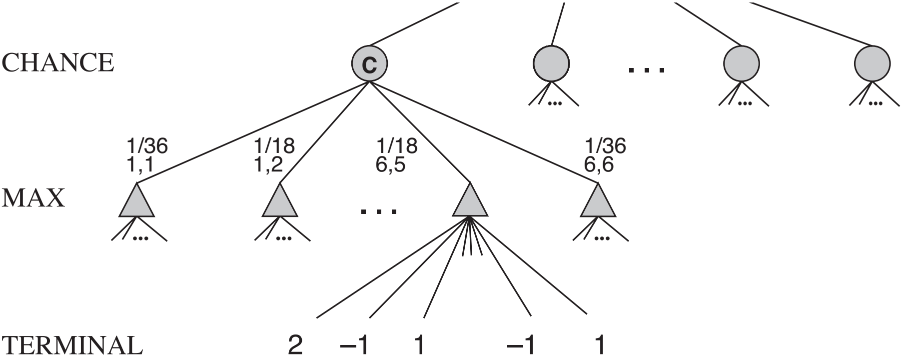
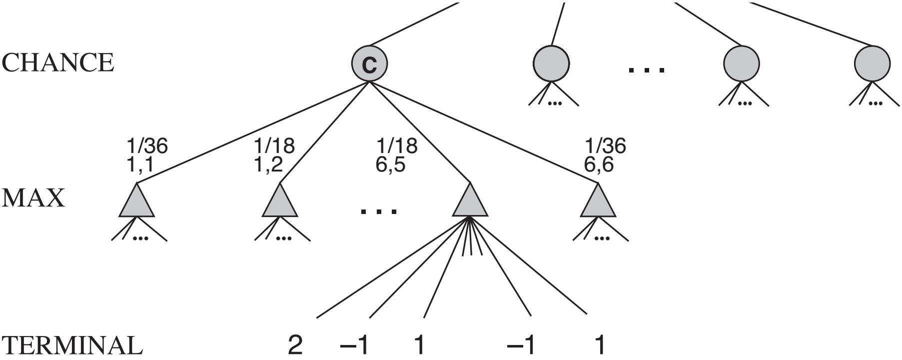
Prunning expectiminimax tree



Bounds on terminal utilities needed.

Prunning expectiminimax tree



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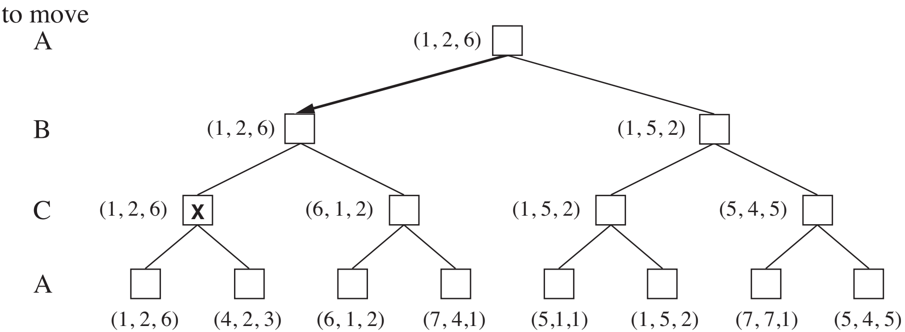
Multi player games



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- Each player maximizes its own
- Coalitions, cooperations, competions may be dynamic

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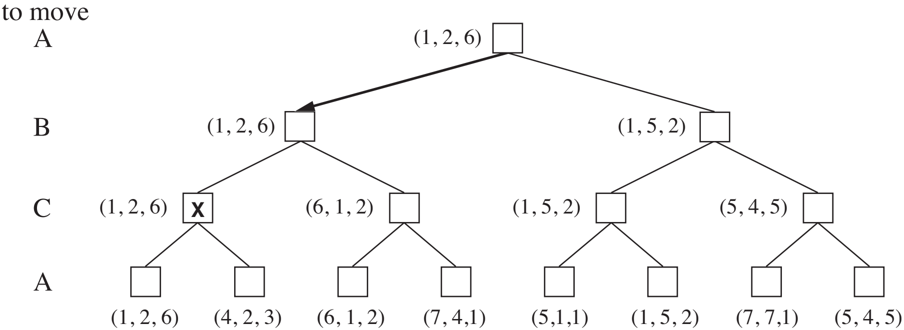

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- Uncertain outcome of an action.
- Robot/Agent may not know the current state!

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- Robot/Agent may not know the current state!

Uncertain, partially observable environment

- Currents state s may be unknown, observations $\mathbf{e}$
- Uncertain outcome, random variable $\textsc{Result}(a)$
- Probability of outcome s' given $\mathbf{e}$ is

$$P(\textsc{Result}(a) = s' | a, \mathbf{e})$$

- Utility function $U(s)$ corresponds to agent preferences.
- Expected utility of an action a given $\mathbf{e}$:

$$EU(a|\mathbf{e}) = \sum_{s'} P(\textsc{Result}(a) = s' | a, \mathbf{e}) U(s')$$

Amatrice, Italy, 2016.

Rational agent

Agent's expected utility of an action a given $\mathbf{e}$:

$$EU(a|\mathbf{e}) = \sum_{s'} P(\textsc{Result}(a) = s'|a, \mathbf{e})U(s')$$

What should a rational agent do?

Is it then all solved?

- $P(\textsc{Result}(a) = s'|a, \mathbf{e})$
- $U(s')$

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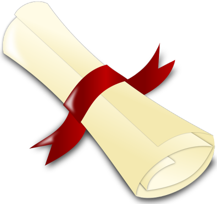
Utilities

- Where do utilities come frome?
- Does averging make sense?
- Do they exist?
- What if our preferences can't be described by utilities?

Agent/Robot Preferences

- Prizes A, B
- Lottery: uncertain prizes $L = [p, A; (1-p), B]$

Preference, indifference, ...

- Robot prefers A over B: $A \succ B$
- Robot has no preferences: $A \sim B$
- in between: $A \succsim B$

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Rational preferences

- Transitivity
- Orderability (Completeness)
- Continuity
- Substituability
- Monotonocity
- Decomposability (Reduction)

Axioms of utility theory.

Rational preferences

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- Orderability (Completeness)
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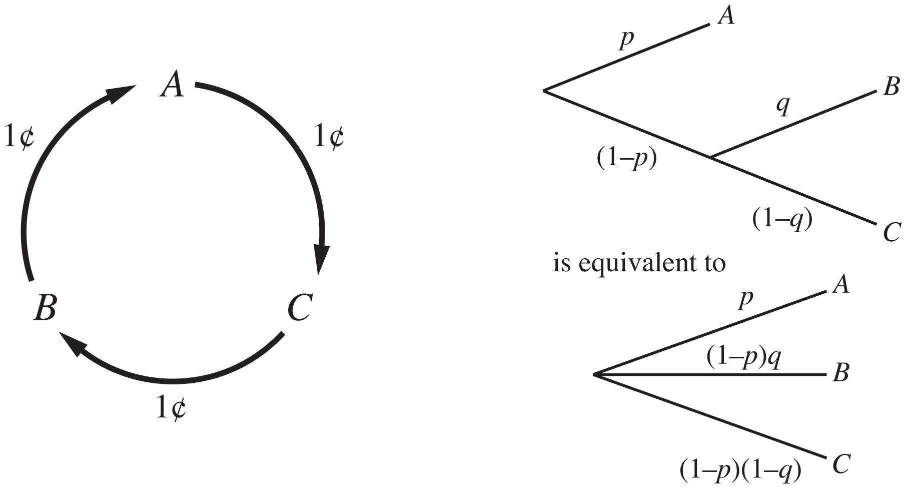
Axioms of utility theory.

Transitivity and decomposability

Goods A, B, C and agent (nontransitive) preferences $A \succ B \succ C \succ A$.



(a) (b)

Maximum expected utility principle

Given the rational preferences (contraints), there exists a real valued function U such that:

$$U(A) > U(B) \quad \Leftrightarrow \quad A \succ B$$
$$U(A) = U(B) \quad \Leftrightarrow \quad A \sim B$$

Expected utility of a Lotery L:

$$L([p_1, S_1; \cdots ; p_n, S_n]) = \sum_i p_i U(S_i)$$

Proof in [3].
Is a utility U unique?

Maximum expected utility principle

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$$
\begin{aligned}
U(A) > U(B) &\Leftrightarrow A \succ B \\
U(A) = U(B) &\Leftrightarrow A \sim B
\end{aligned}
$$

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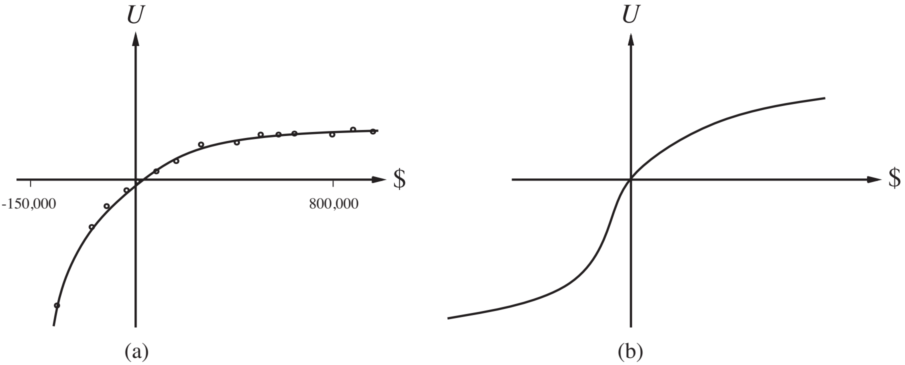
Human utilities

Utility of money

You triumphed in a TV show!

a) Take $\$1,000,000$ ... or

b) Flip a coin and loose all or win $\$2,500,000$



The (human) Utility of money.

References

Some figures from [2], Chapters 5, 16. Human utilities are discussed in [1]. This lecture has been also greatly inspired by the 7th lecture of CS 188 at `http://ai.berkeley.edu` as it convenietly bridges the world of deterministic search and sequential decisions in uncertain worlds.

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