Adversarial Search

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Games, man vs. algorithm

- Deep Blue
- Alpha Go
- Deep Stack
- Why Games, actually?

Games are interesting for AI *because* they are hard (to solve).

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More: Adversarial Learning

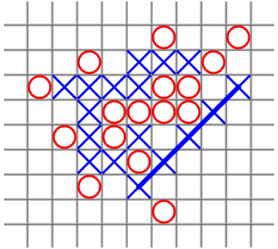
Video: Adversing visual segmentation

Vision for Robotics and Autonomous Systems, http://cyber.felk.cvut.cz/vras

Elements of the game

- s_0: The initial state
- PLAYER(s). Which player has to move in s.
- ACTIONS(s). What are the legal moves?
- RESULT(s, a). Transition, result of a move.
- TERMINAL-TEST(s). Game over?
- TERMINAL-UTILITY(s, p). What is prize?

Examples for some games ...



https://commons.wikimedia.org/wiki/File:Tic-tac-toe_5.png

Considering the notation, we are making slight transition from [1] to [2].

- Players: $P = \{1, 2, \ldots, N\}$ (often just $N = 2$)
- Transition functions: $S \times A \to S$.
- Terminal utilities: $S \times P \to R$. (R - as a Reward)

What are we loking for? A strategy/policy $S \to A$

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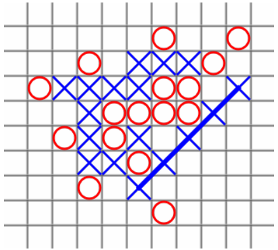

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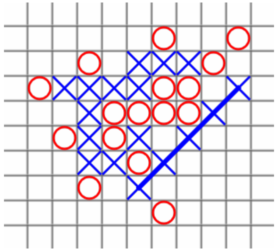

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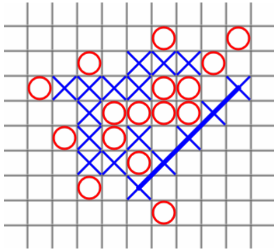

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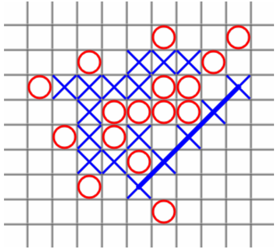

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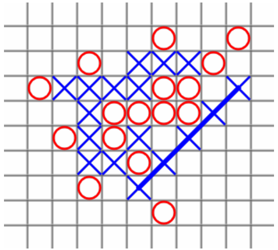

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Terminal utilitity: Zero-Sum and General games

- Zero-sum: players have opposite utilities (values)
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- General game: cooperations, competition, ...

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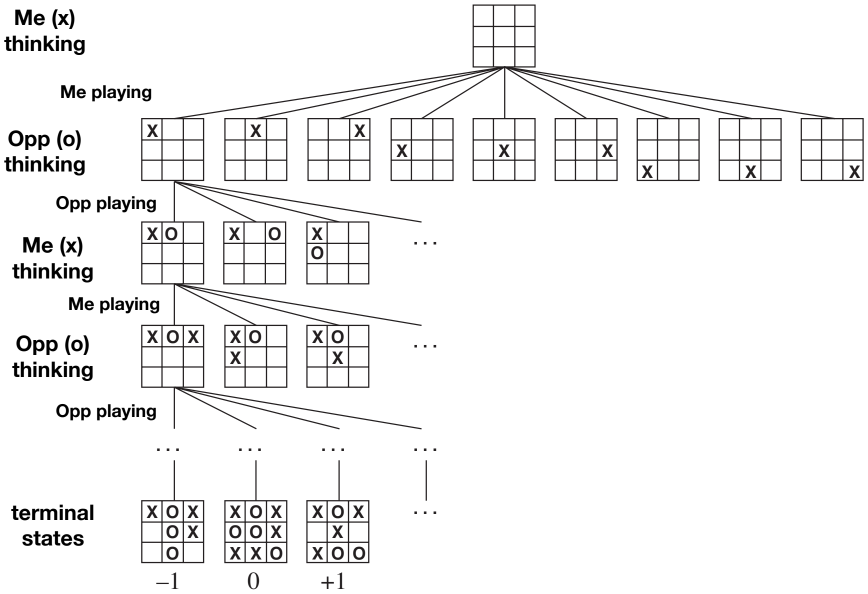
Game Tree(s)



TERMINAL-UTILITY$(s, \mathbf{x})$

Init state, ACTIONS function, and RESULT function defines game tree

State Value $V(s)$

$V(s)$ – value V of a state s : The best utility achievable from this state.

$$V(s) = \max_{s' \in \text{children}(s)} V(s')$$

Think about the State Value. It is a theoretical construct, definition. Depending on the problem, there may be various computational algorithms. In a game, what State Values are known? Usually, only terminal states.

Think, for a moment, you are the only player. You can control every step.

How would you compute the $V(s)$ for a given state s?

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How would you compute the $V(s)$ for a given state s?

Two-ply game: **max** for me, **min** for the opponent.

$$a_1 = \operatorname*{arg\,max}_{a \in \textsc{Actions}(A)} \textsc{Result}(A, a)$$

I'm player that starts (state A) and want to decide what to play, actions/plays/moves a_1, a_2, a_3 are the options. B, C, D are the possible outcomes of my moves. Now the opponent is about to play. The numbers in terminal states denote *my* profit/utility.

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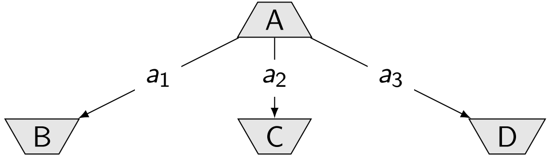

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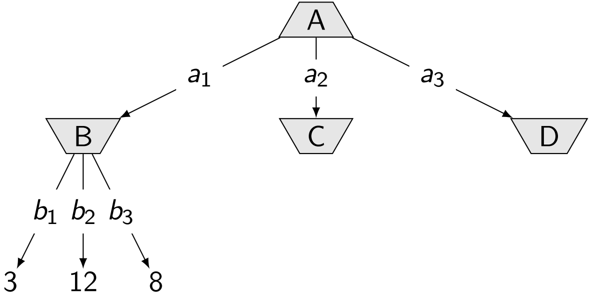

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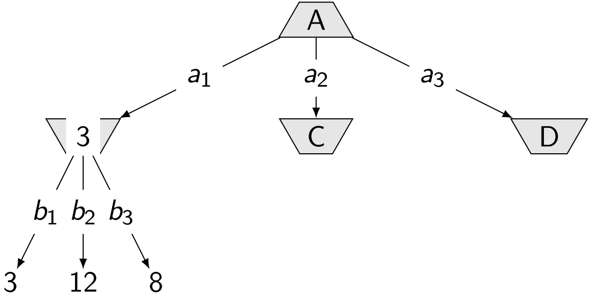

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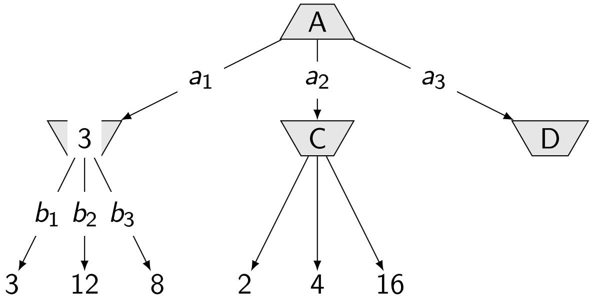

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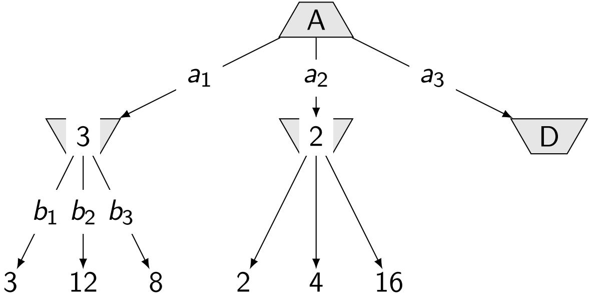

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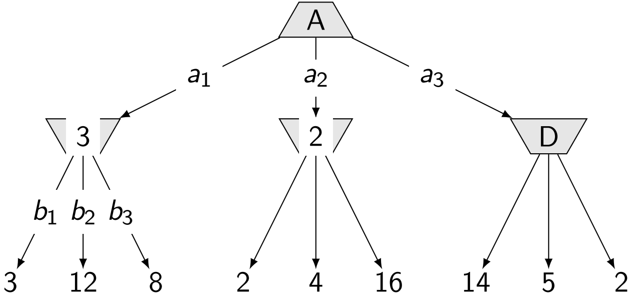

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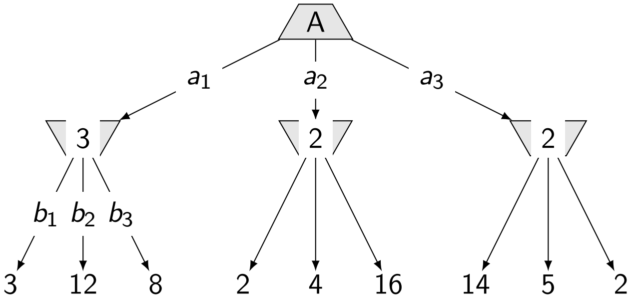

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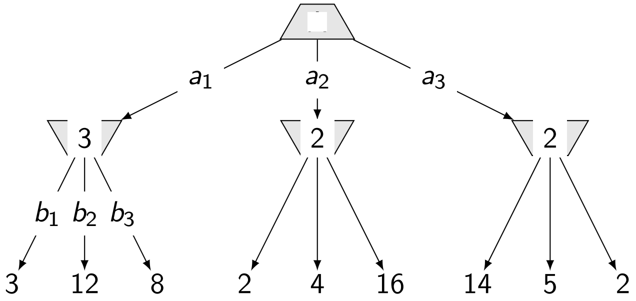

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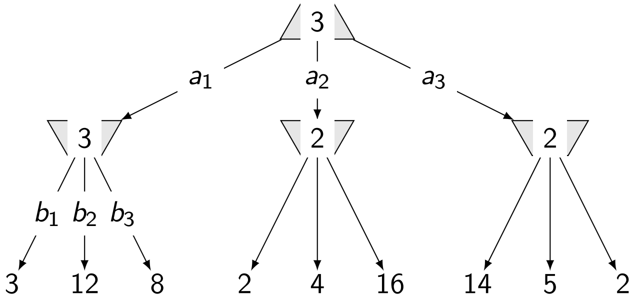

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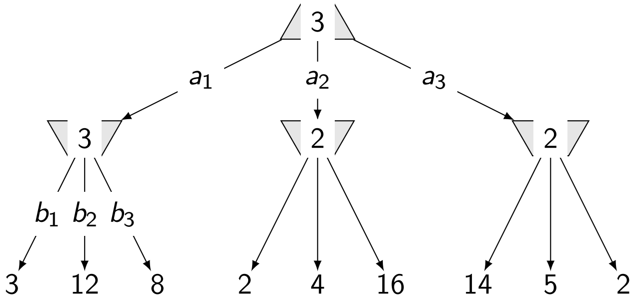

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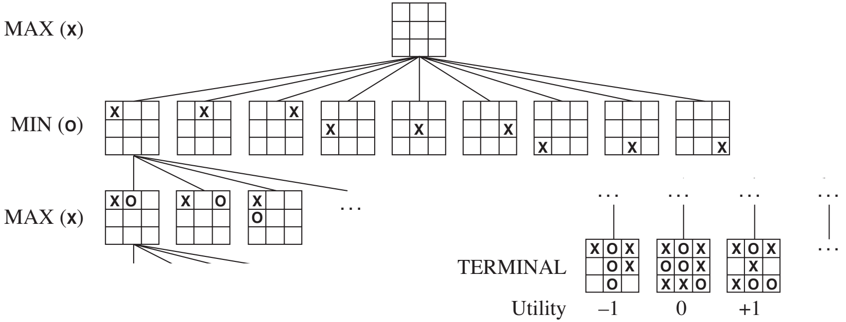
Zero-Sum game: **max** for me, **min** for the opponent.



$$
\text{MINIMAX}(s) =
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\max_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s,a)) & \text{if} \quad \text{PLAYER}(s) = \text{MAX} \\
\min_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s,a)) & \text{if} \quad \text{PLAYER}(s) = \text{MIN}
\end{cases}
$$

Max step: I want to maximize my outcome.

Min step: I want to minimize the outcome of the opponent.

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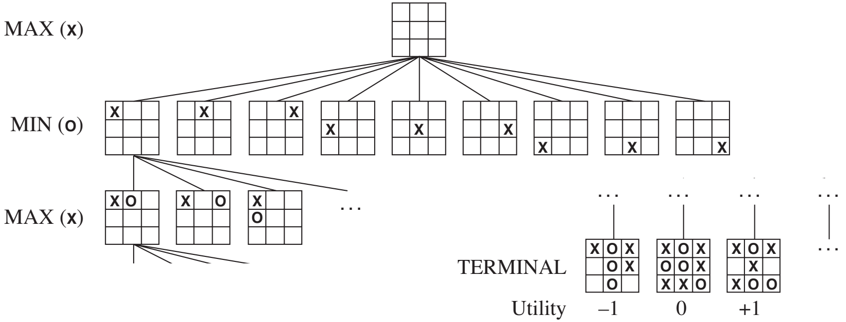

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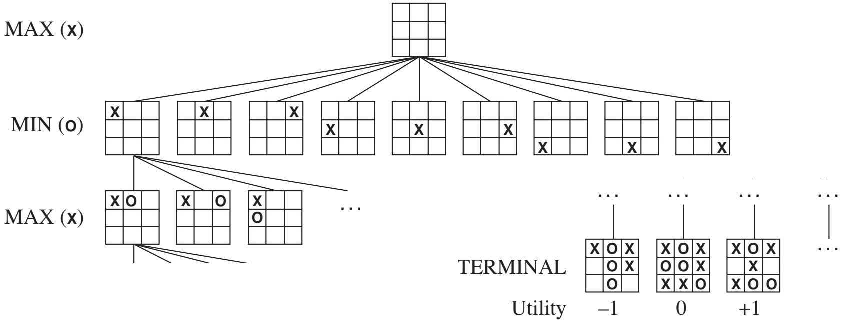

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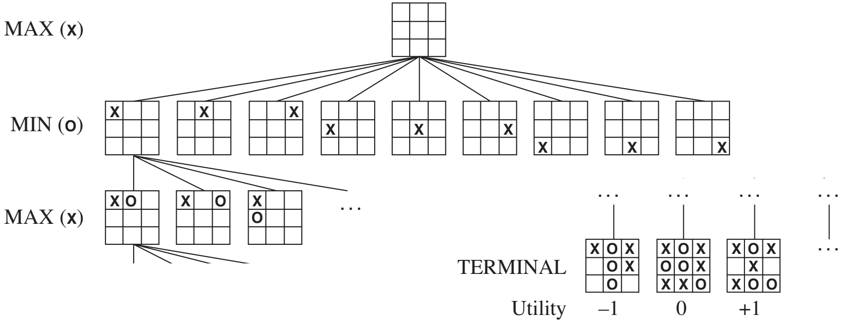

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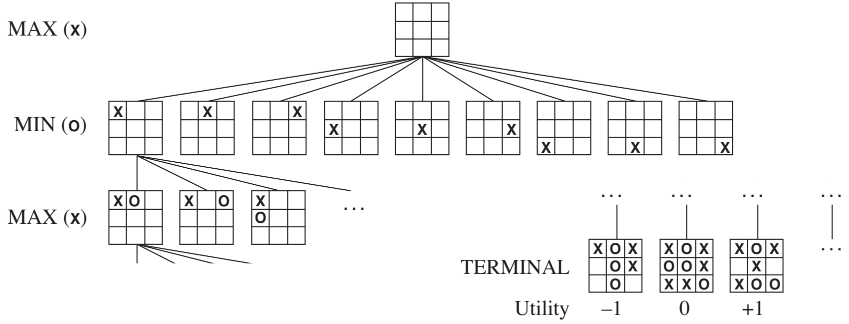

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Minimax algorithm

function MINIMAX(state) **returns** an action

    return $\arg\max_{a \in \text{Actions}(s)}$ MIN-VALUE(RESULT(state, a))

end function

function MIN-VALUE(state) **returns** a utility value v

    if TERMINAL-TEST(state) **then return** UTILITY(state)

    end if

    $v \leftarrow \infty$

    for all ACTIONS(state) **do**

        $v \leftarrow \min(v,$ MAX-VALUE(RESULT(state,a)))

    end for

end function

function MAX-VALUE(state) **returns** a utility value v

    if TERMINAL-TEST(state) **then return** UTILITY(state)

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    end for

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    end for
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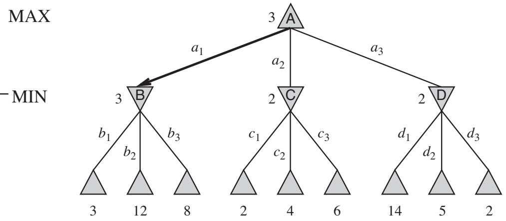
    end if
    v ← −∞
    for all ACTIONS(state) do
        v ← max(v, MIN-VALUE(RESULT(state,a)))
    end for
end function
```

A two ply game, down to terminal and back again ...

function MINIMAX(s) **returns** a
  $\underset{a \in \text{Actions}(s)}{\text{arg}\mathbf{max}}$ MINVAL(RES(s, a))
end function

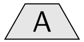
function MINVAL(s) **returns** v
  if TERMINAL(s) **then** UTIL(s)
  end if
  $v \leftarrow \infty$
  for all ACTIONS(s) **do**
    $v \leftarrow$ **min**(v, MAXVAL(RES(s, a)))
  end for
end function

function MAXVAL(s) **returns** v
  if TERMINAL(s) **then** UTIL(s)
  end if
  $v \leftarrow -\infty$
  for all ACTIONS(s) **do**
    $v \leftarrow$ **max**(v, MINVAL(RES(s, a)))
  end for
end function



Before going to the animation on the next slide, try to follow the algorithm by a pencil and paper.

A two ply game, recursive run



What is the complexity? How many nodes to visit?

Can we do better? How?

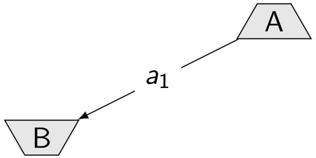
Efficiency/complexity:

- Exhaustive DFS
- Time $O(b^m)$
- Space $O(bm)$

Chess $b \approx 35, m \approx 100 \dots$

- We cannot go(dive) to the end
- Can we save something?

A two ply game, recursive run



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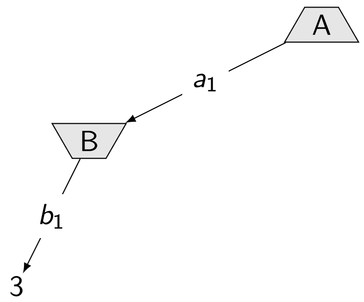

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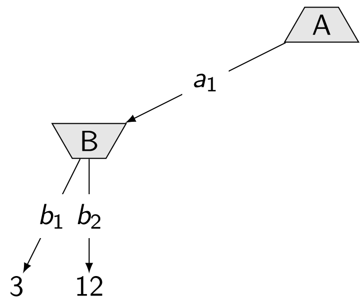

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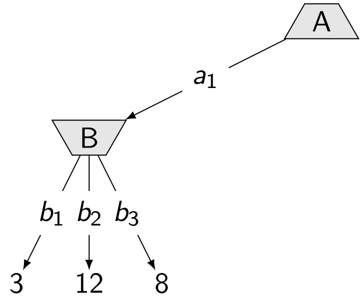

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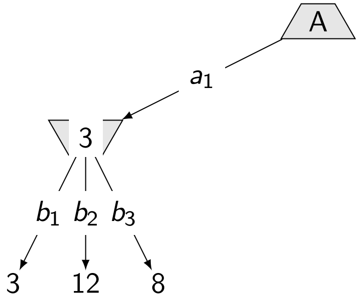

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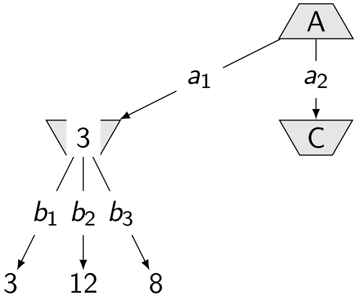

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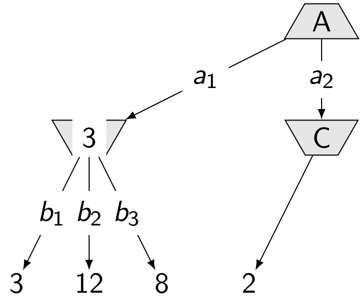

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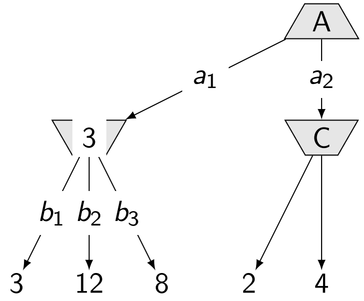

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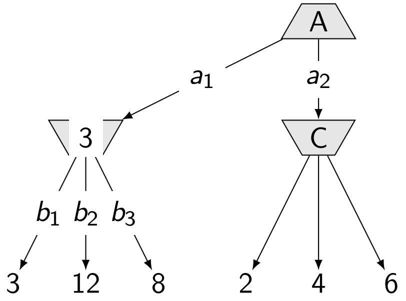

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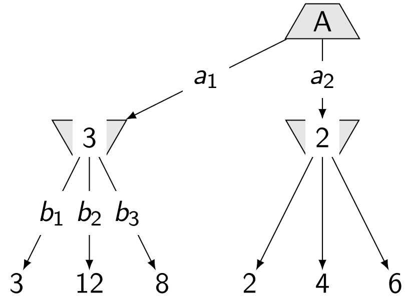

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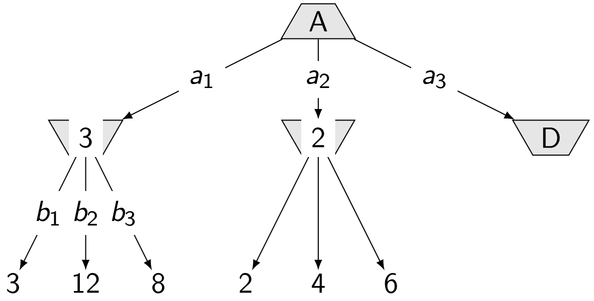

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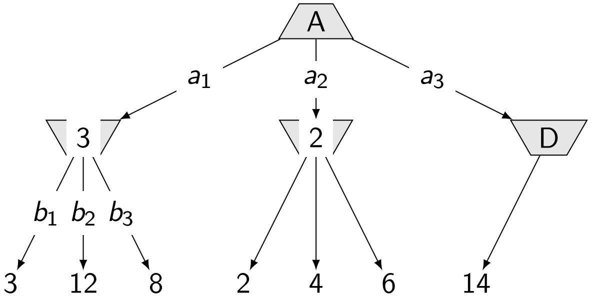

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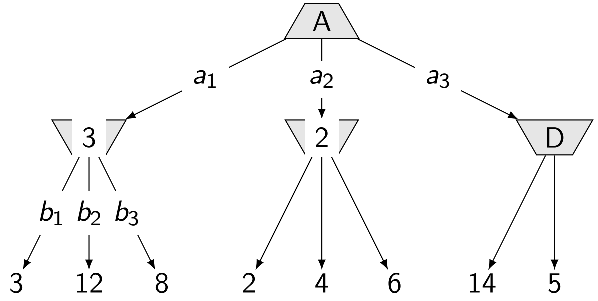

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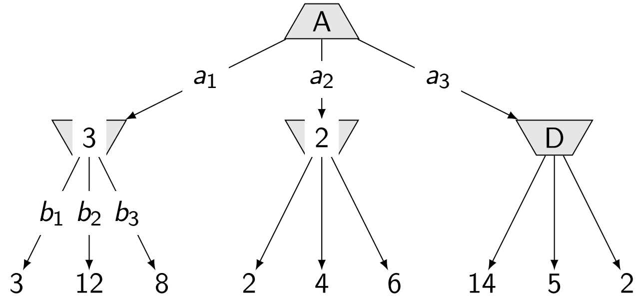

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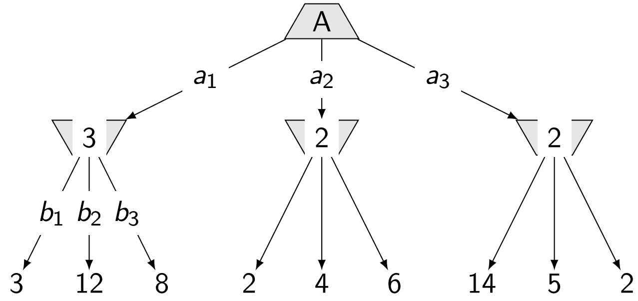

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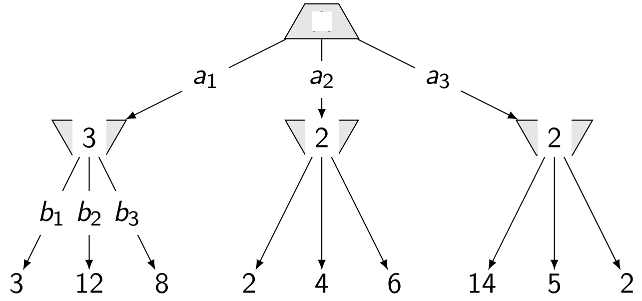

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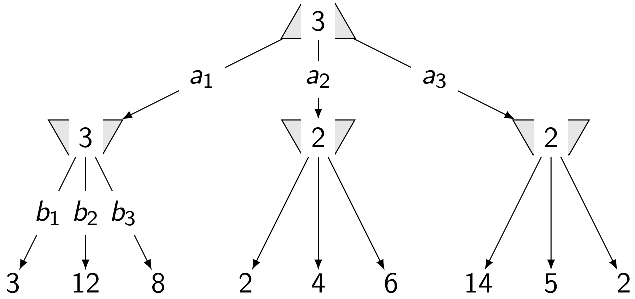

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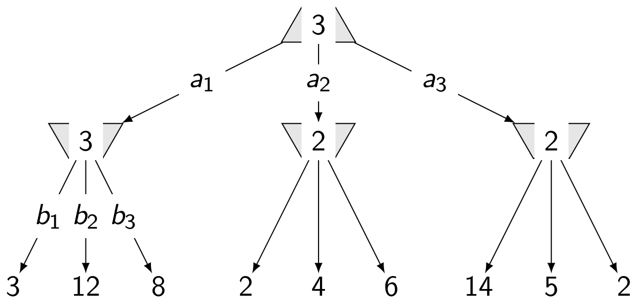

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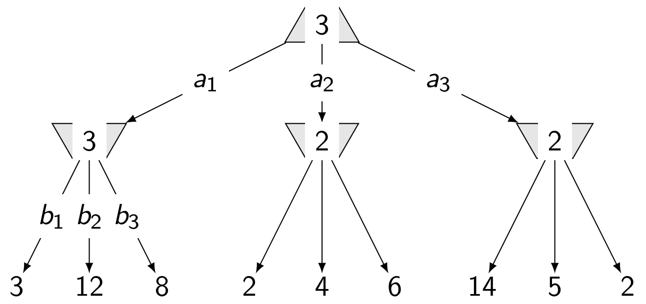

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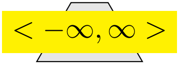

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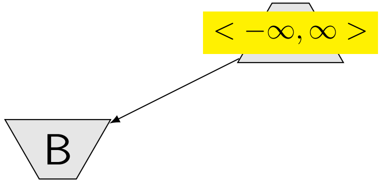

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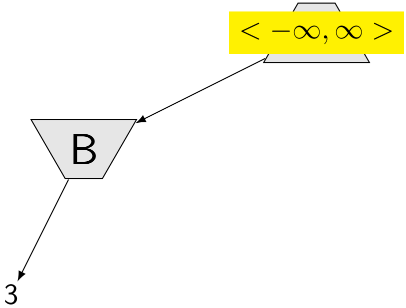

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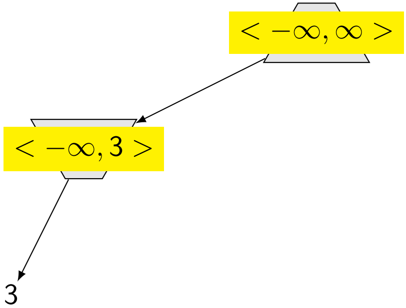

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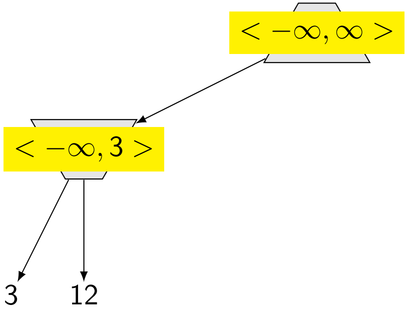

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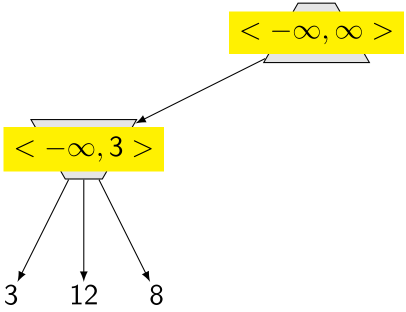

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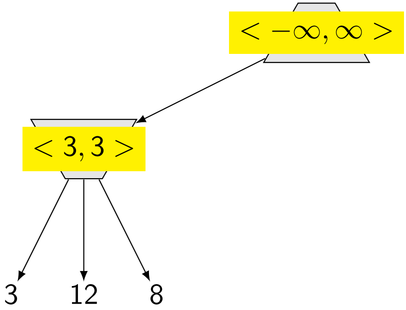

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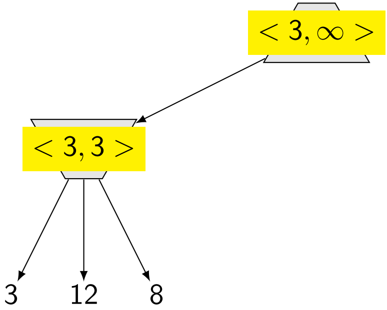

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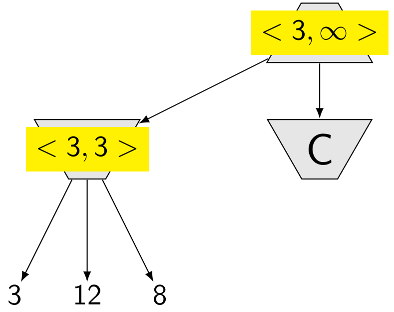

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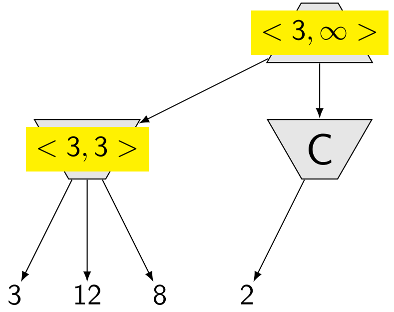

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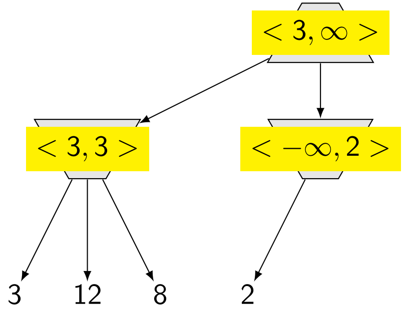

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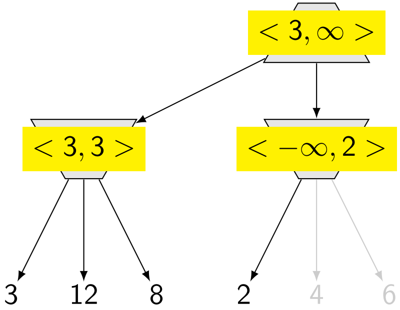

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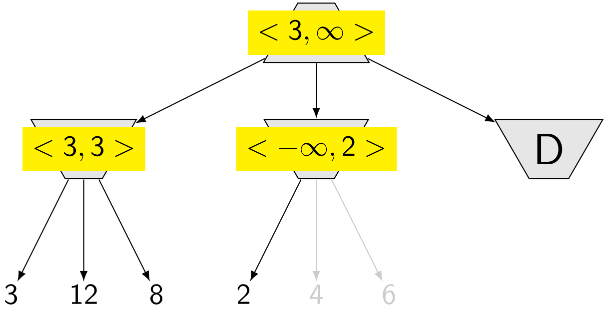

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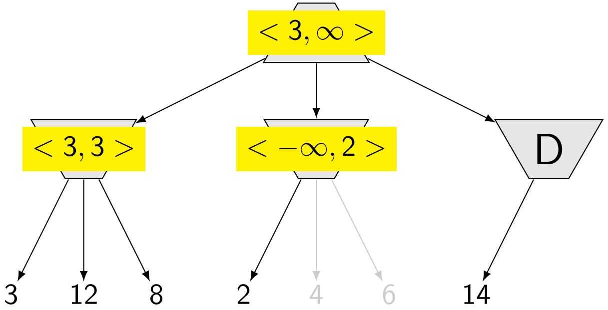

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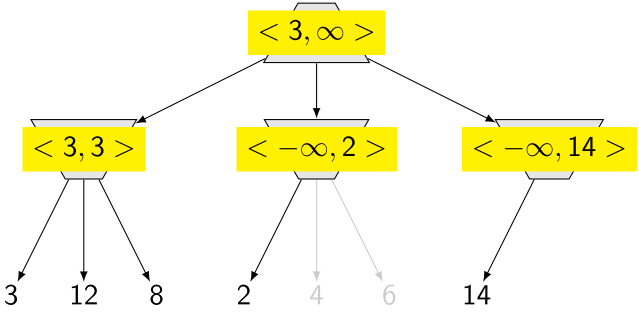

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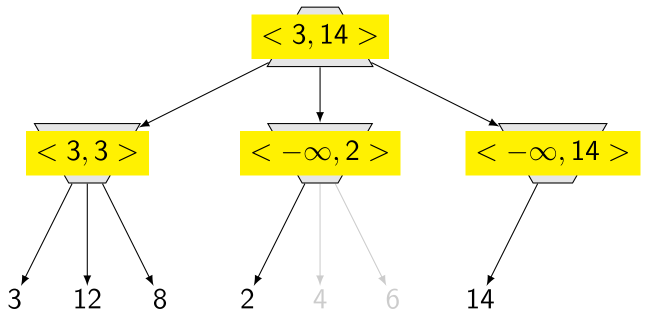

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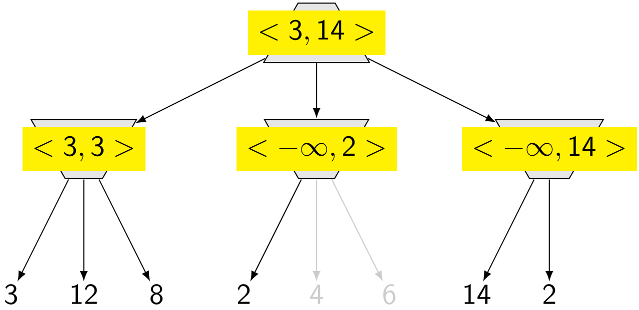

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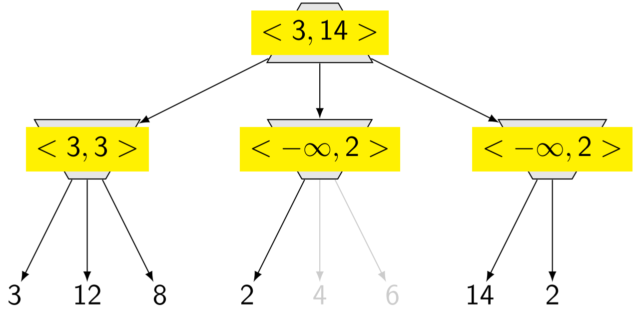

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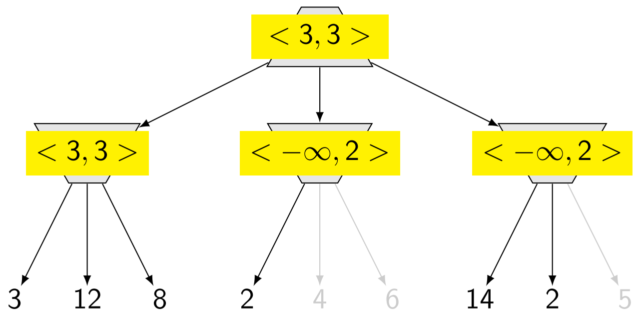
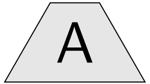


α-β prunning

α highest (best) value choice found so far for any choice along MAX
β lowest (best) value choice found so far for any choice along MIN
v value of the state



In MIN-VAL: $v \leftarrow 2$
$v \leq \alpha$ then: return v!

Functions scope: MAX-VALUE MIN-VALUE
In MAX nodes α is changing and β is stopping, in MIN nodes β is changing and α is stopping.

It is clear that ordering of child nodes matters. Draw tree of α-β search in case of perferct ordering. Effective branching factor becomes $\sqrt{b}$ instead of b which effectively doubles the depth can be searched.

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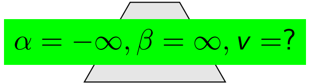

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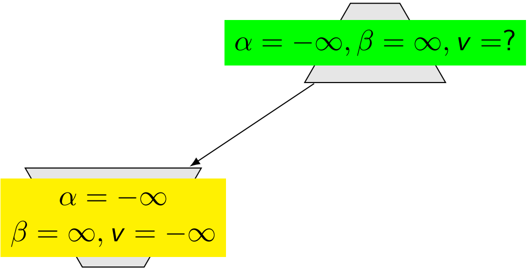

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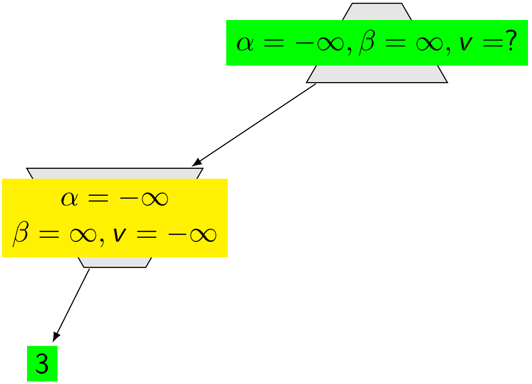

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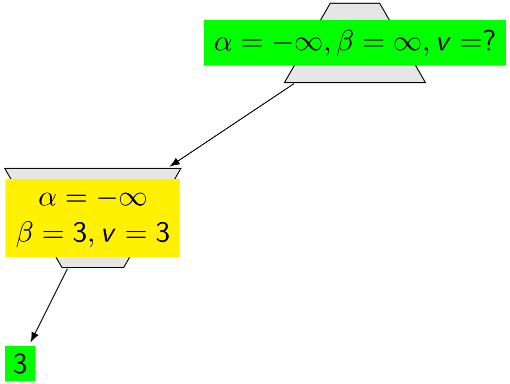

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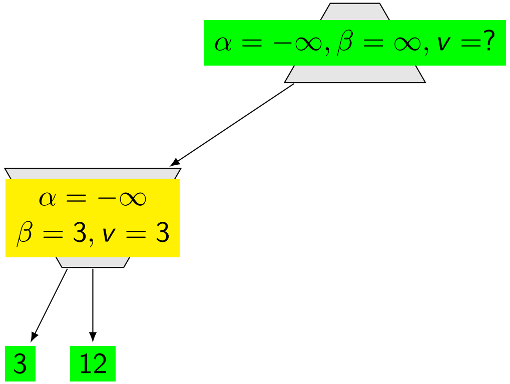

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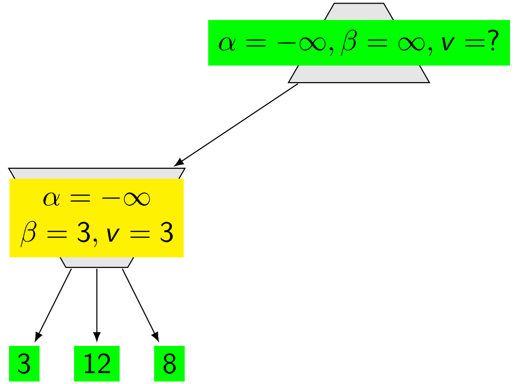

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It is clear that ordering of child nodes matters. Draw tree of α-β search in case of perferct ordering. Effective branching factor becomes $\sqrt{b}$ instead of b which effectively doubles the depth can be searched.

α-β prunning

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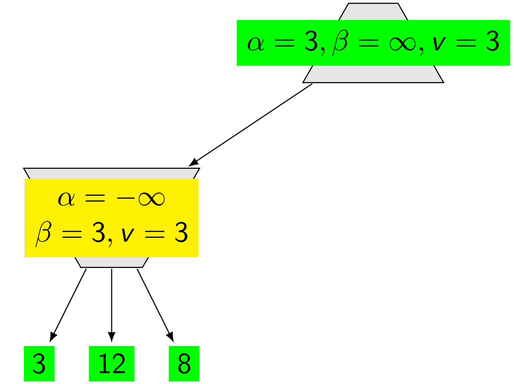
v value of the state



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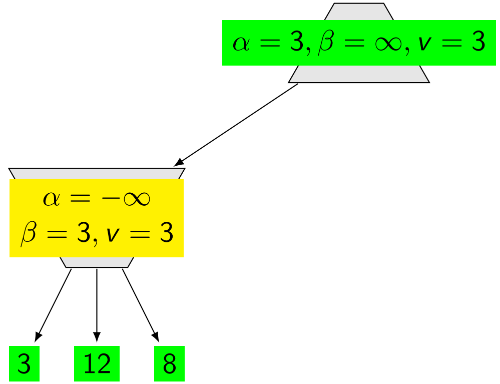

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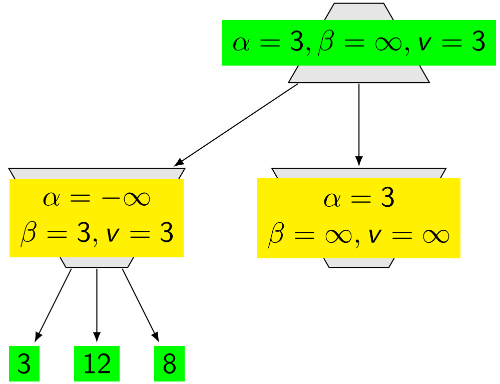

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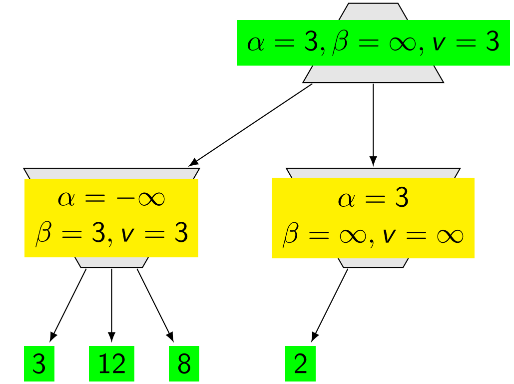

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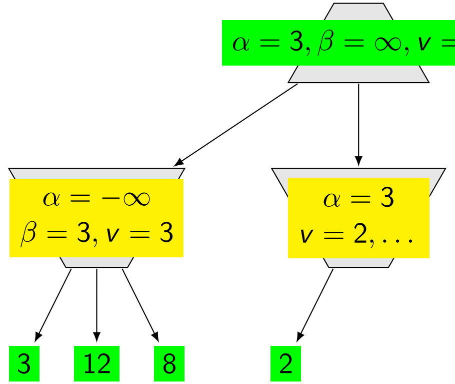

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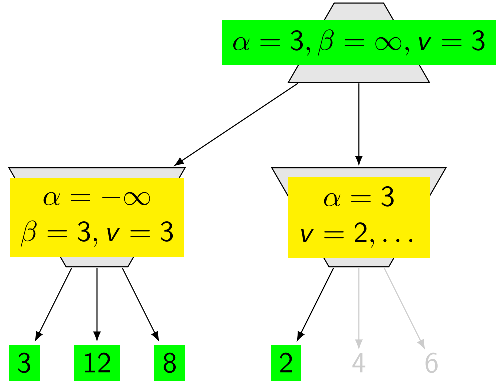

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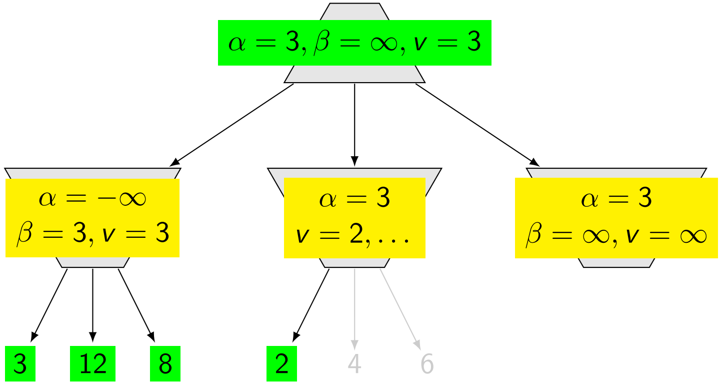

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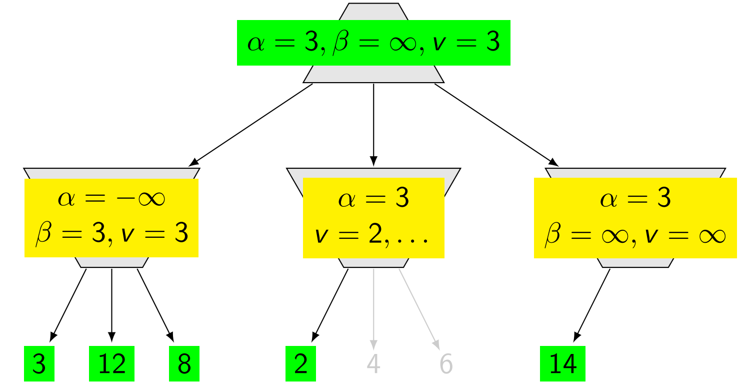

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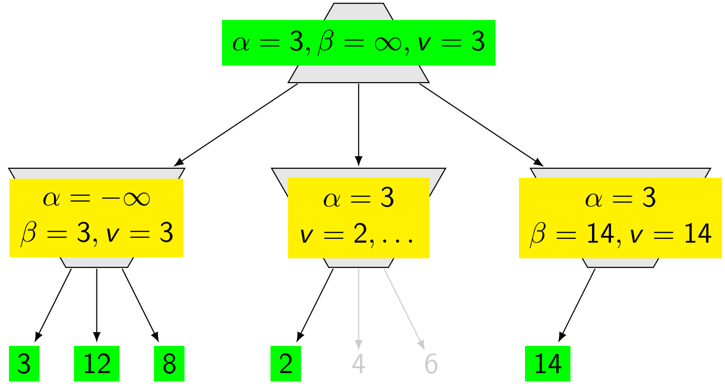

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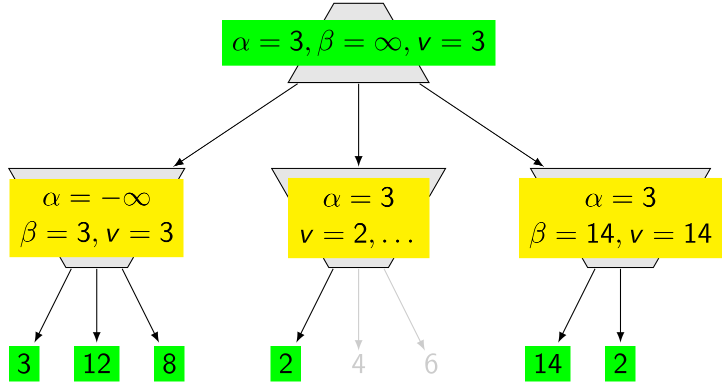

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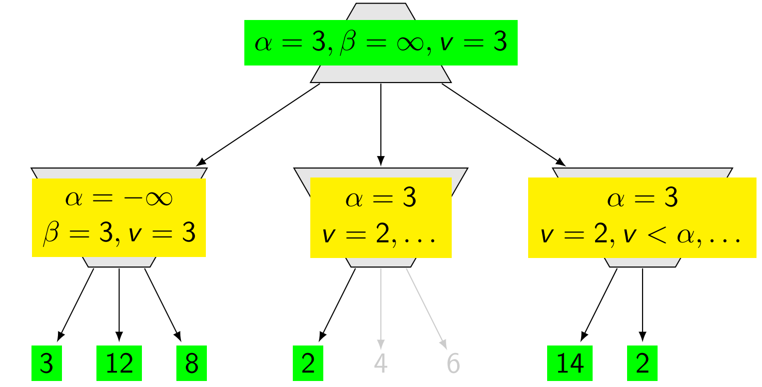

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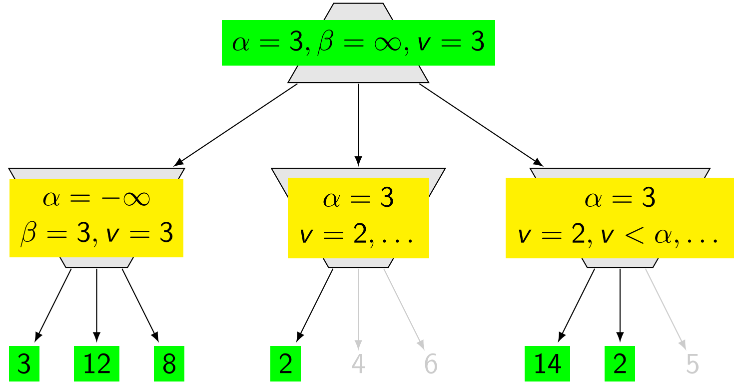

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function ALPHA-BETA-SEARCH(state) **returns** an action
    $v \leftarrow$ MAX-VALUE(state, $\alpha = -\infty$, $\beta = \infty$)
    return action corresponding to v
end function

---

function MAX-VALUE(state,α, β) **returns** a utility value v
    if TERMINAL-TEST(state) **return** UTILITY(state)
    $v \leftarrow -\infty$
    for all ACTIONS(state) **do**
        $v \leftarrow$ **max**(v, MIN-VALUE(RESULT(state,a),α, β))
        if $v \geq \beta$ **return** v
        $\alpha \leftarrow \max(\alpha, v)$
    end for
end function

---

function MIN-VALUE(state, α, β) **returns** a utility value v
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end function

Take the tree from the previous slide and try to go step-by-step, watch α, β and v

function ALPHA-BETA-SEARCH(state) **returns** an action
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  end for
end function

Take the tree from the previous slide and try to go step-by-step, watch α, β and v

```
function ALPHA-BETA-SEARCH(state) returns an action
    v ← MAX-VALUE(state, α = −∞, β = ∞)
    return action corresponding to v
end function
```

```
function MAX-VALUE(state,α, β) returns a utility value v
    if TERMINAL-TEST(state) return UTILITY(state)
    v ← −∞
    for all ACTIONS(state) do
        v ← max(v, MIN-VALUE(RESULT(state,a),α, β))
        if v ≥ β return v
        α ← max(α, v)
    end for
end function
```

```
function MIN-VALUE(state, α, β) returns a utility value v
    if TERMINAL-TEST(state) return UTILITY(state)
    v ← ∞
    for all ACTIONS(state) do
        v ← min(v, MAX-VALUE(RESULT(state,a),α, β))
        if v ≤ α return v
        β ← min(β, v)
    end for
end function
```


Take the tree from the previous slide and try to go step-by-step, watch α, β and v

Imperfect but real-time decisions - iterative deepening

$$\textsc{h-minimax}(s, d) =$$

$$\begin{cases} \textsc{eval}(s) & \text{if} \quad \textsc{cutoff-test}(s, d) \\ \max_{a \in \textsc{actions}(s)} \textsc{h-minimax}(\textsc{result}(s, a), d + 1) & \text{if} \quad \textsc{player}(s) = \textsc{max} \\ \min_{a \in \textsc{actions}(s)} \textsc{h-minimax}(\textsc{result}(s, a, d + 1)) & \text{if} \quad \textsc{player}(s) = \textsc{min} \end{cases}$$

Even with perfect ordering, α-β prunning does not save us.

Imperfect but real-time decisions - iterative deepening

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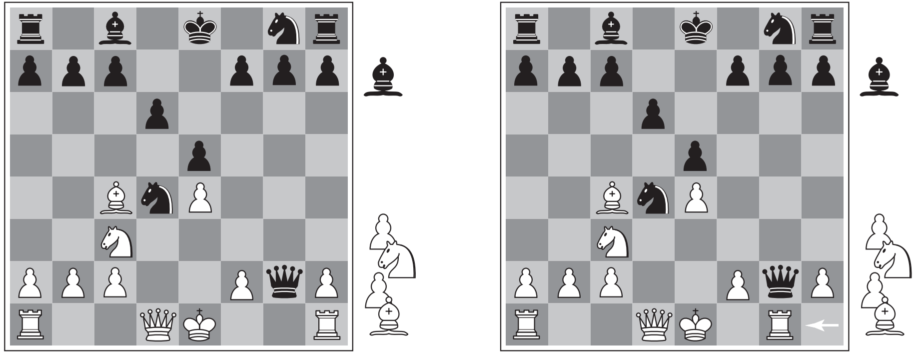
Cutting off search

Replace
if TERMINAL-TEST(s) **then return** UTILITY(s)
with:
if CUTOFF-TEST(s,d) **then return** EVAL(s)

Cutting depends on d only, why we need s as the input parameter?

EVAL(s) – Evaluation functions

(estimate of) State value for non-terminal states



(a) White to move (b) White to move

$$\textsc{Eval}(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots w_n f_n(s)$$

For many problems it is not so easy to find/construct proper function. We may try more functions and combine them conveniently.

$$f_1(s) = \text{number of white pawns} - \text{number of black pawns}$$

How to tune weights w_i?
or Deep Nets! Yeah!

How the get training data for supervised learning? More later.

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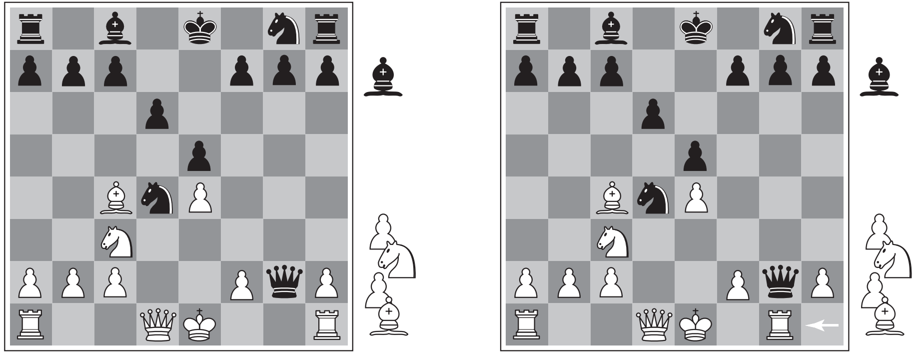

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References

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