Adversarial Search

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March 6, 2019

Games, man vs. algorithm

- Deep Blue
- Alpha Go
- Deep Stack
- Why Games, actually?

Games are interesting for AI *because* they are hard (to solve).

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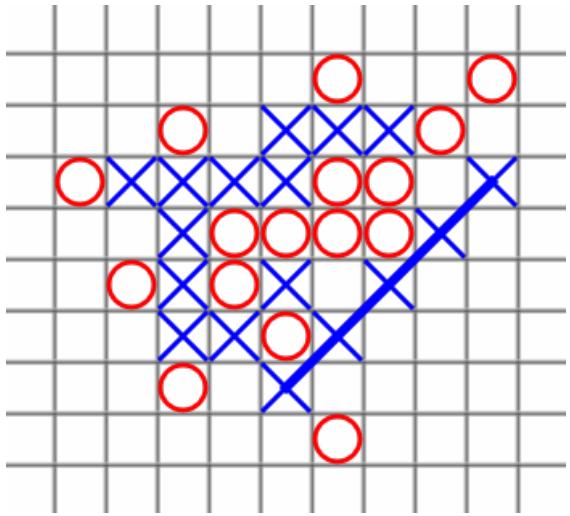
More: Adversarial Learning

Video: Adversing visual segmentation

Elements of the game

- s_0: The initial state
- PLAYER(s). Which player has to move in s.
- ACTIONS(s). What are the legal moves?
- RESULT(s, a). Transition, result of a move.
- TERMINAL-TEST(s). Game over?
- TERMINAL-UTILITY(s, p). What is prize?

Examples for some games ...



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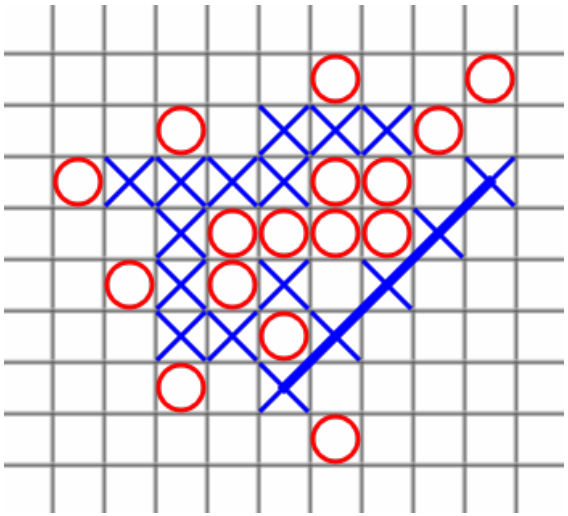

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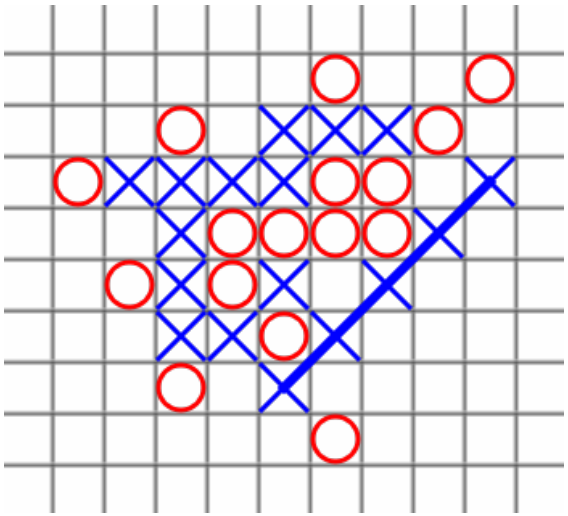

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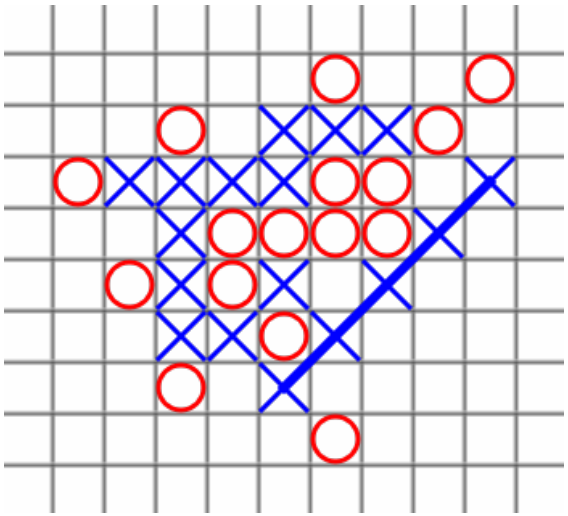

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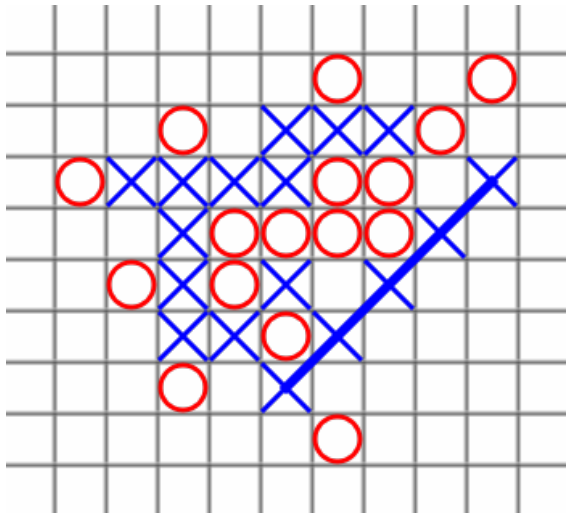

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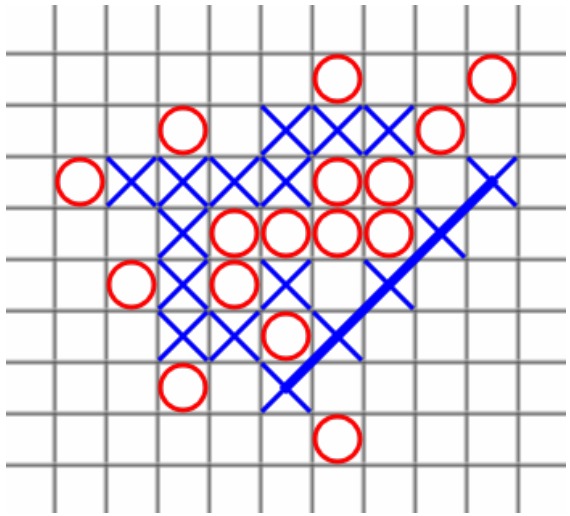

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Terminal utilitity: Zero-Sum and General games

- Zero-sum: players have opposite utilities (values)
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- General game: independent utilities
- General game: cooperations, competition, ...

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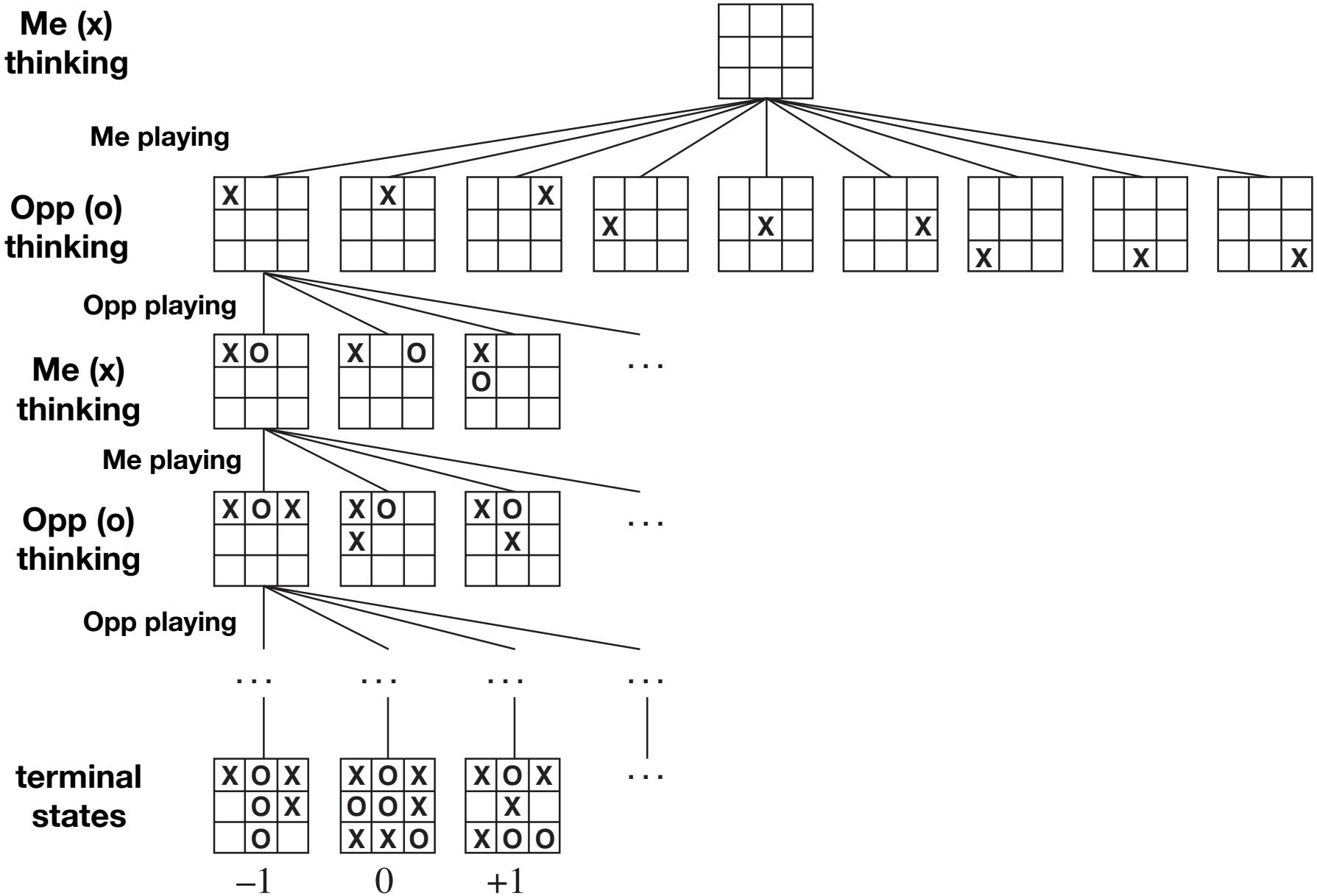
Game Tree(s)



TERMINAL-UTILITY$(s, \mathbf{x})$

State Value $V(s)$

$V(s)$ – value V of a state s : The best utility achievable from this state.

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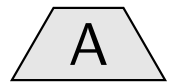

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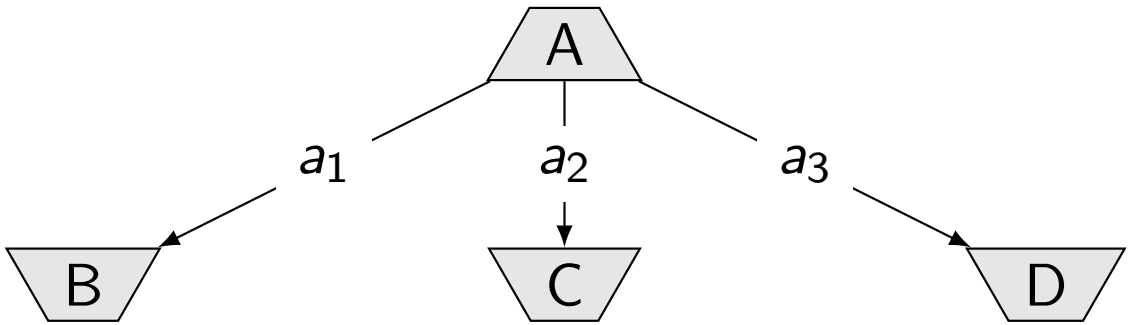

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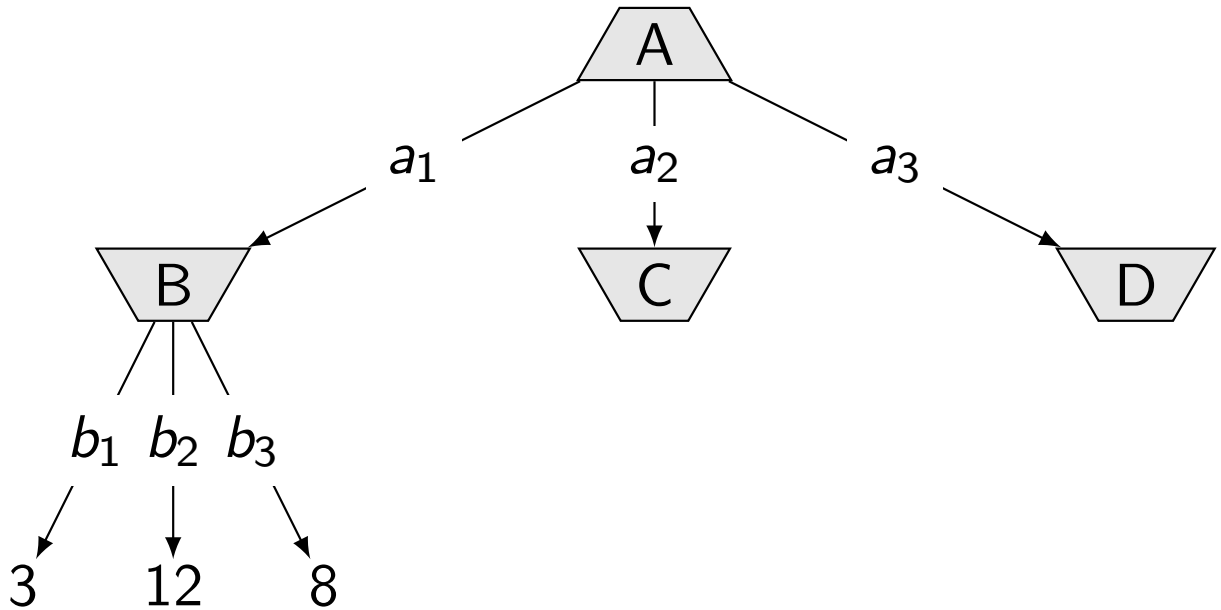

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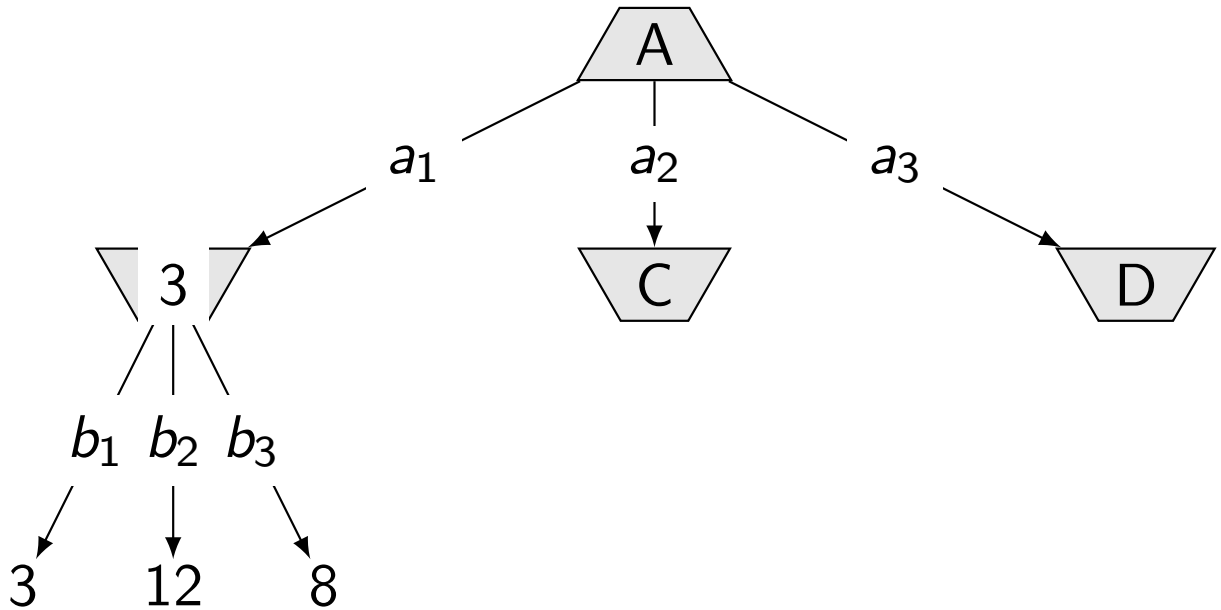

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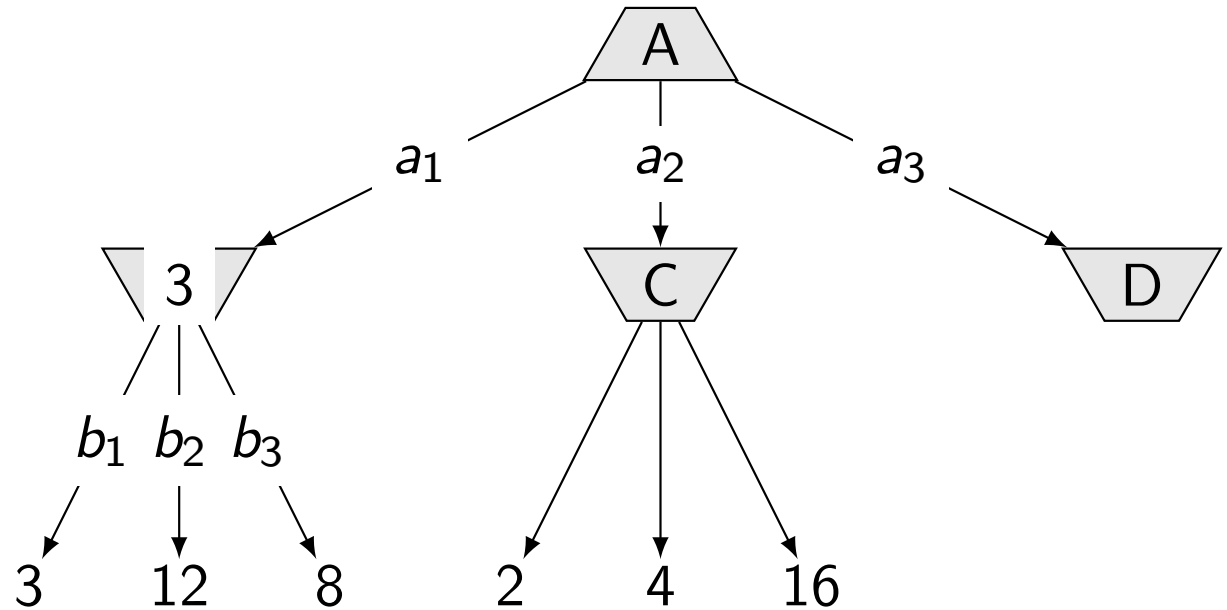

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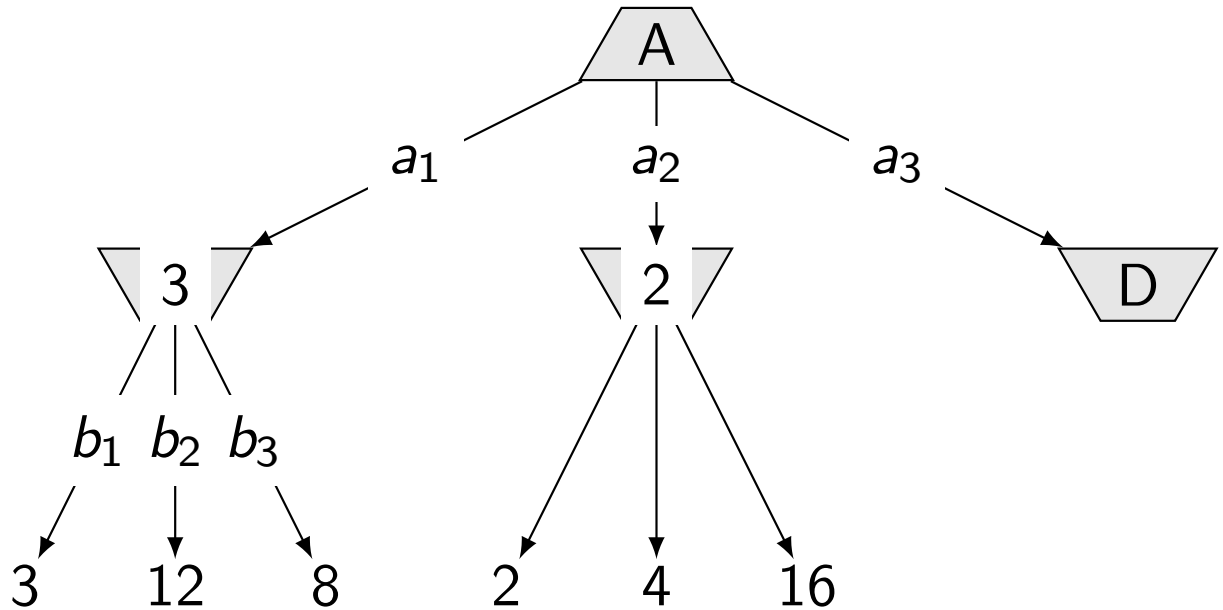

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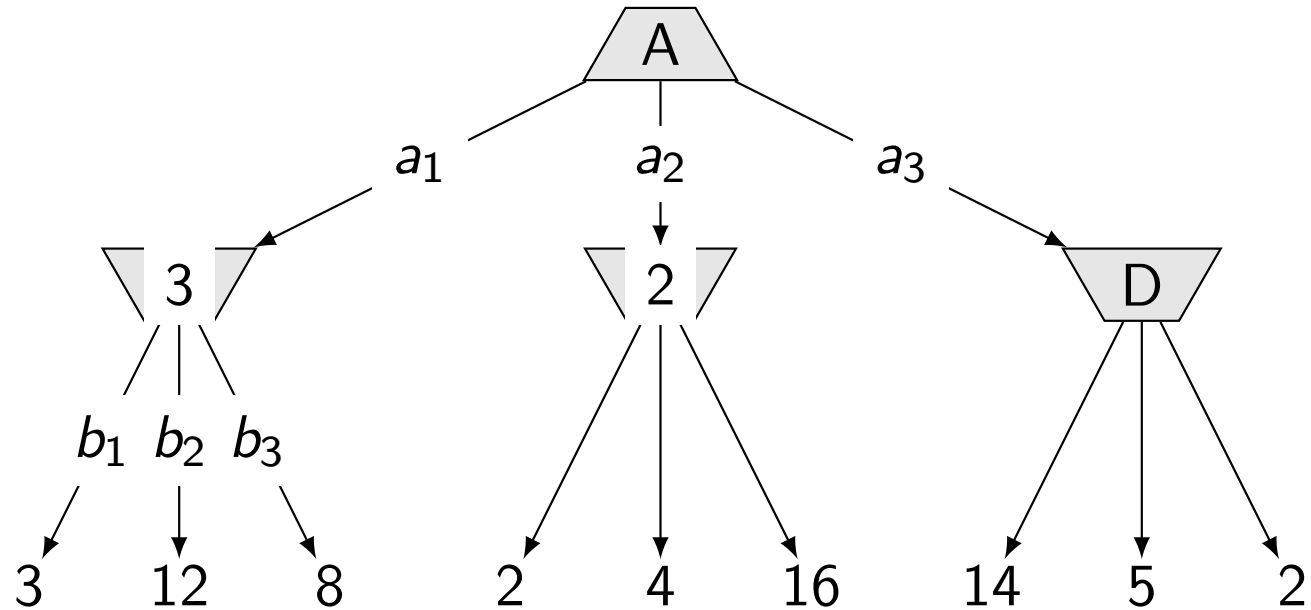

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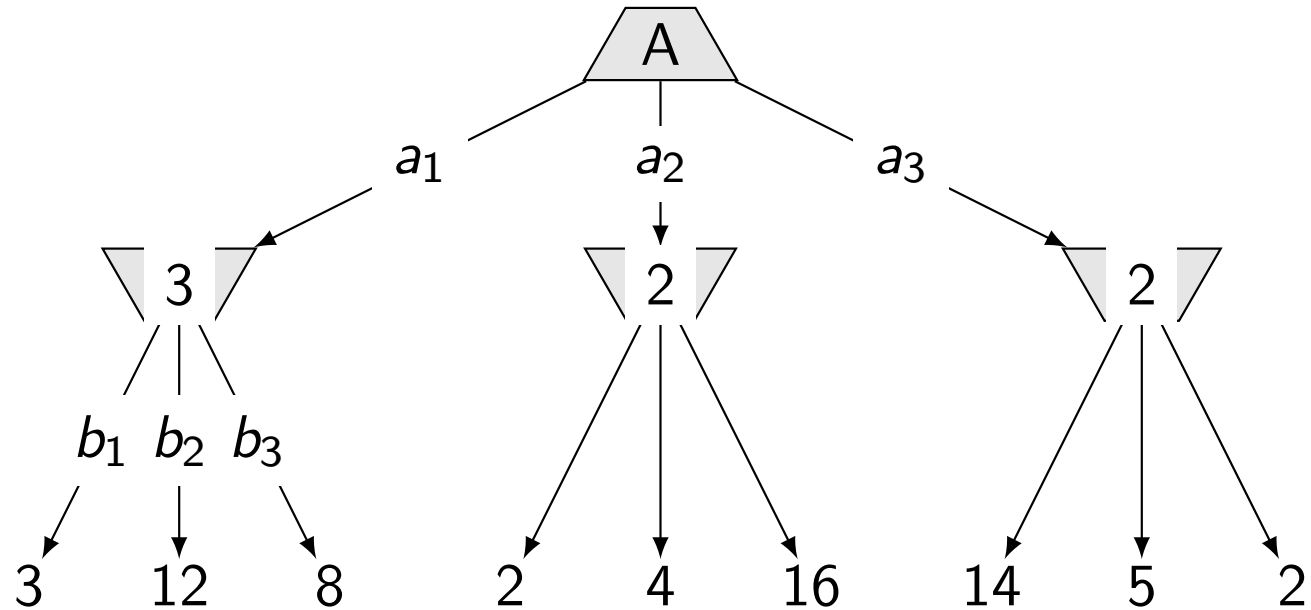

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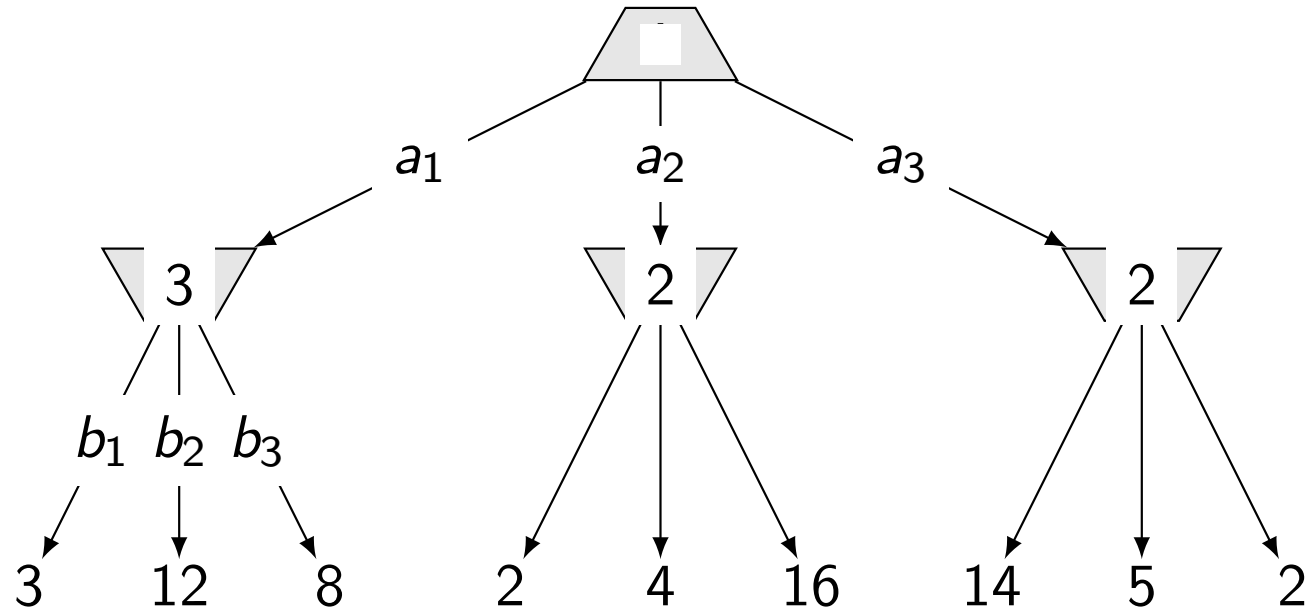

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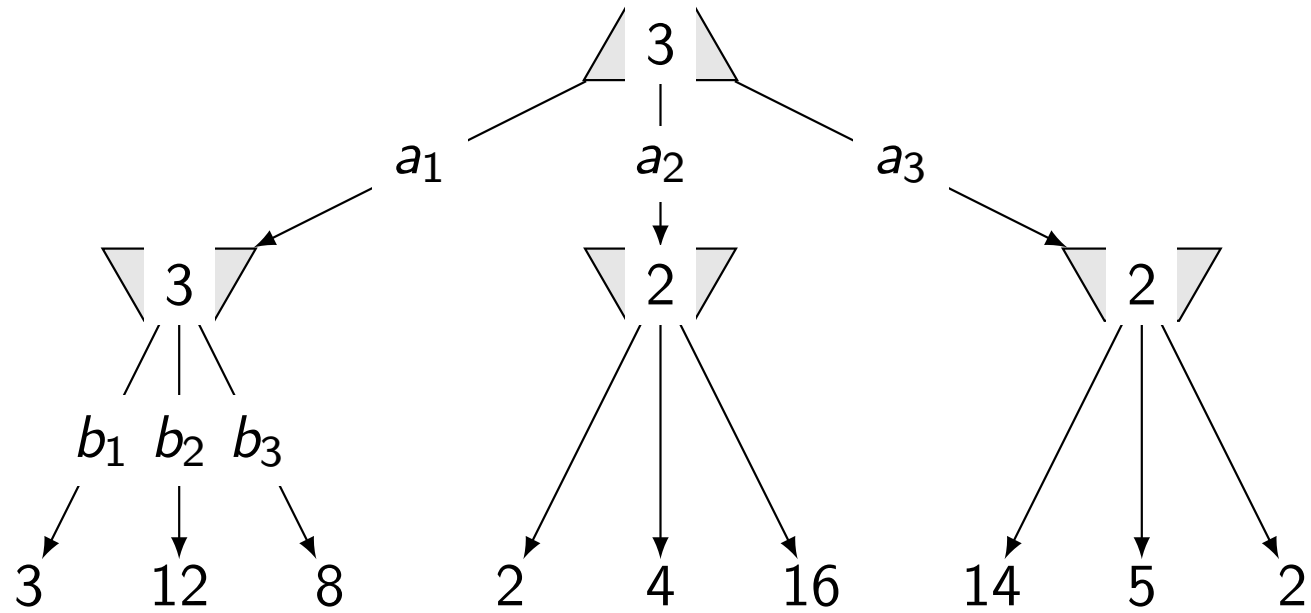

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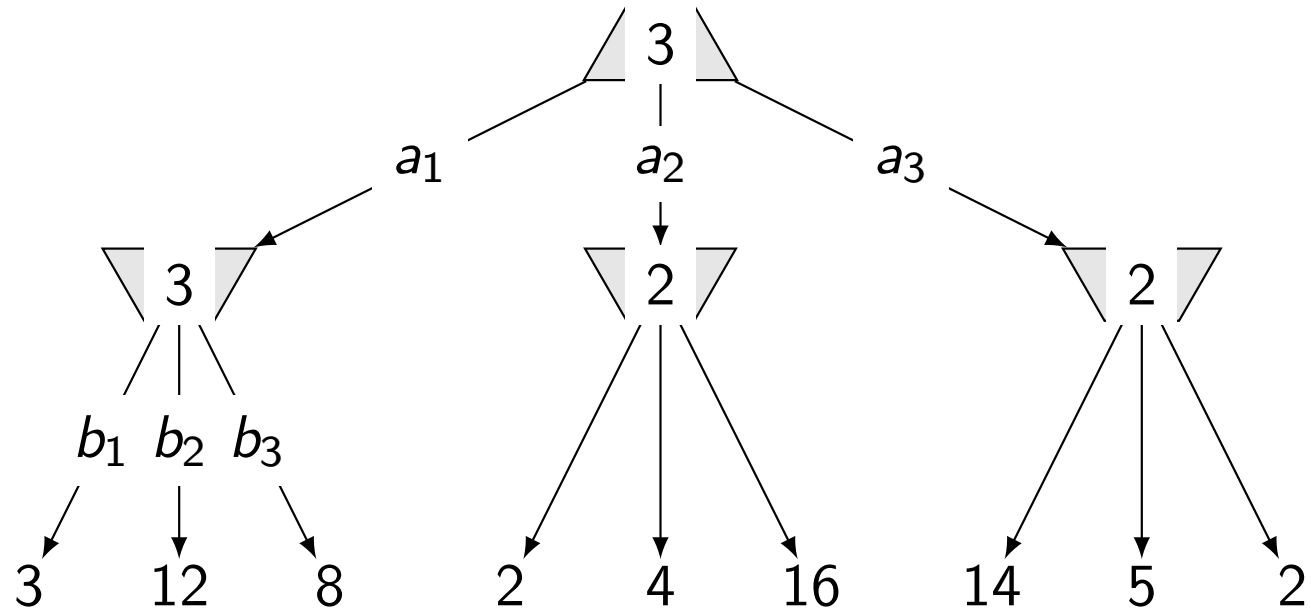

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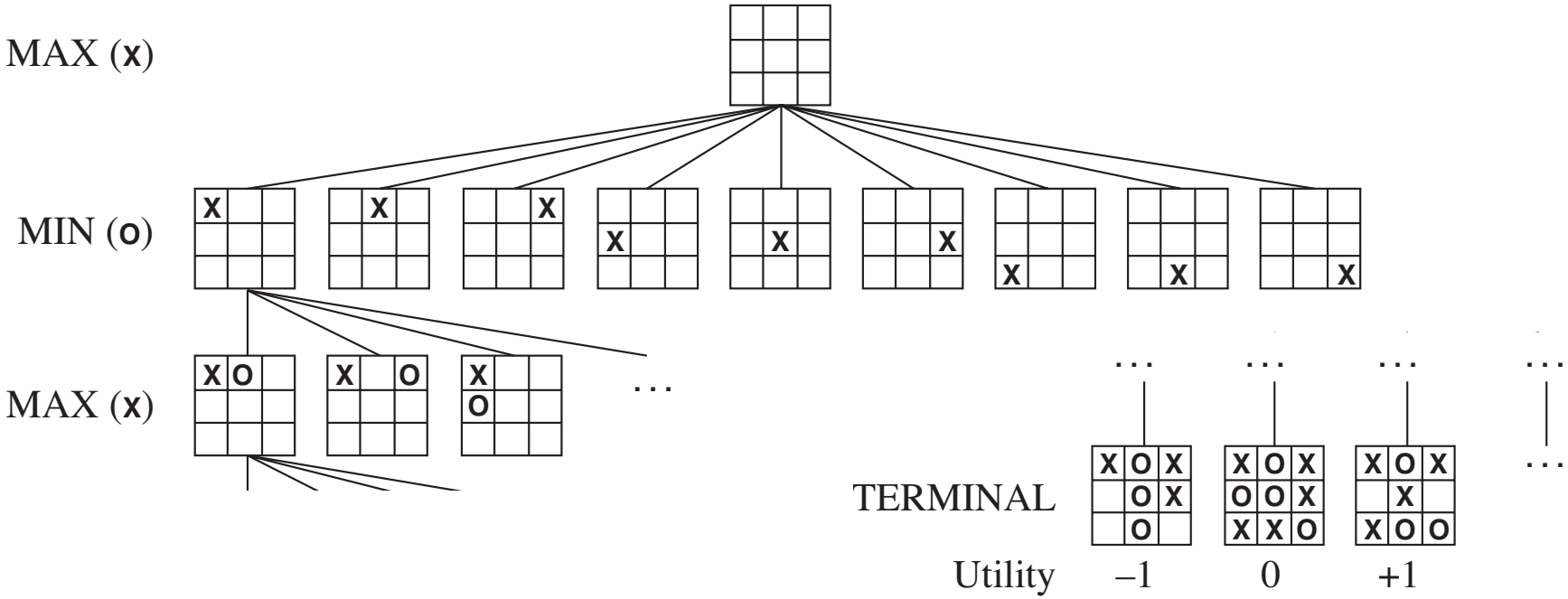

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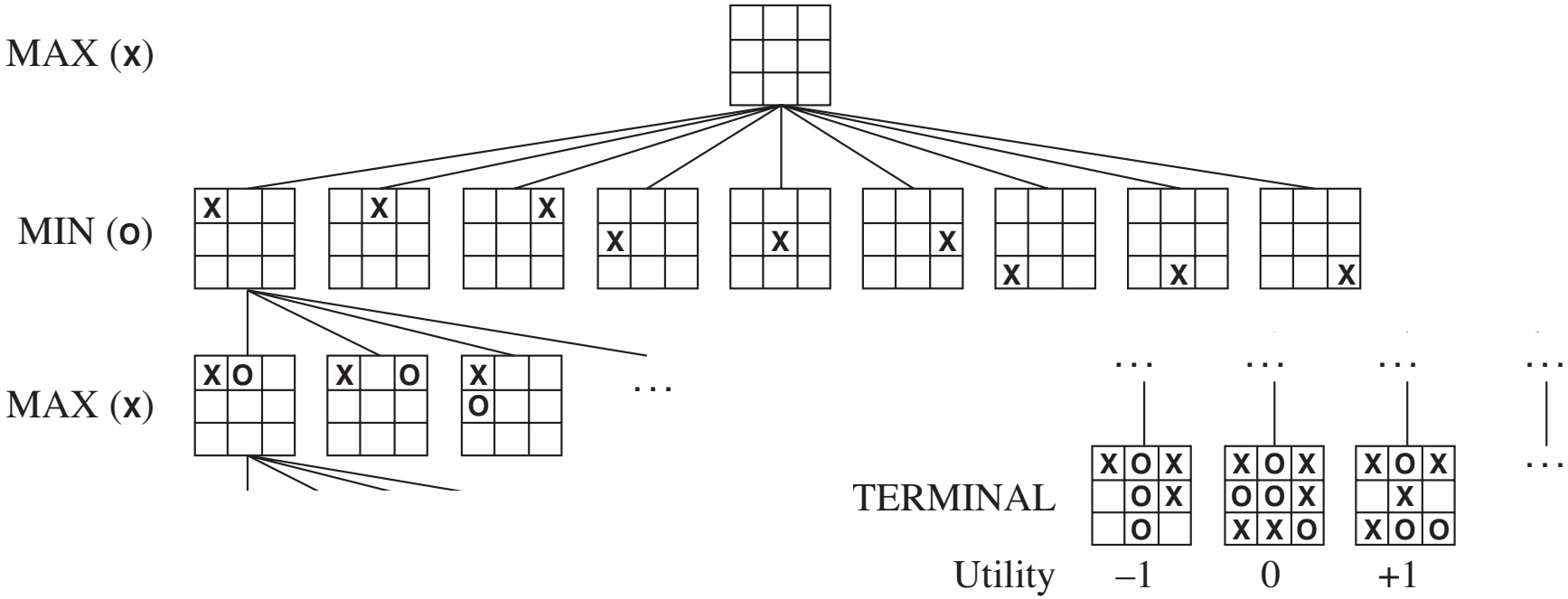

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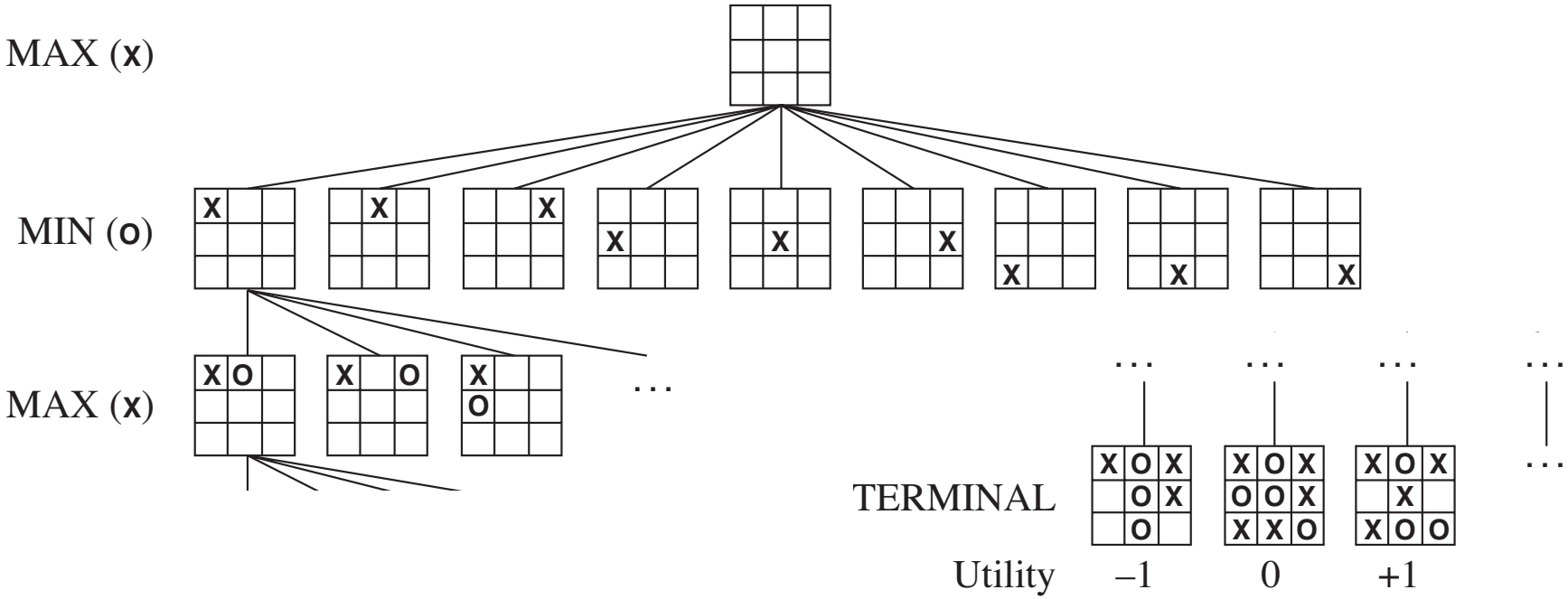

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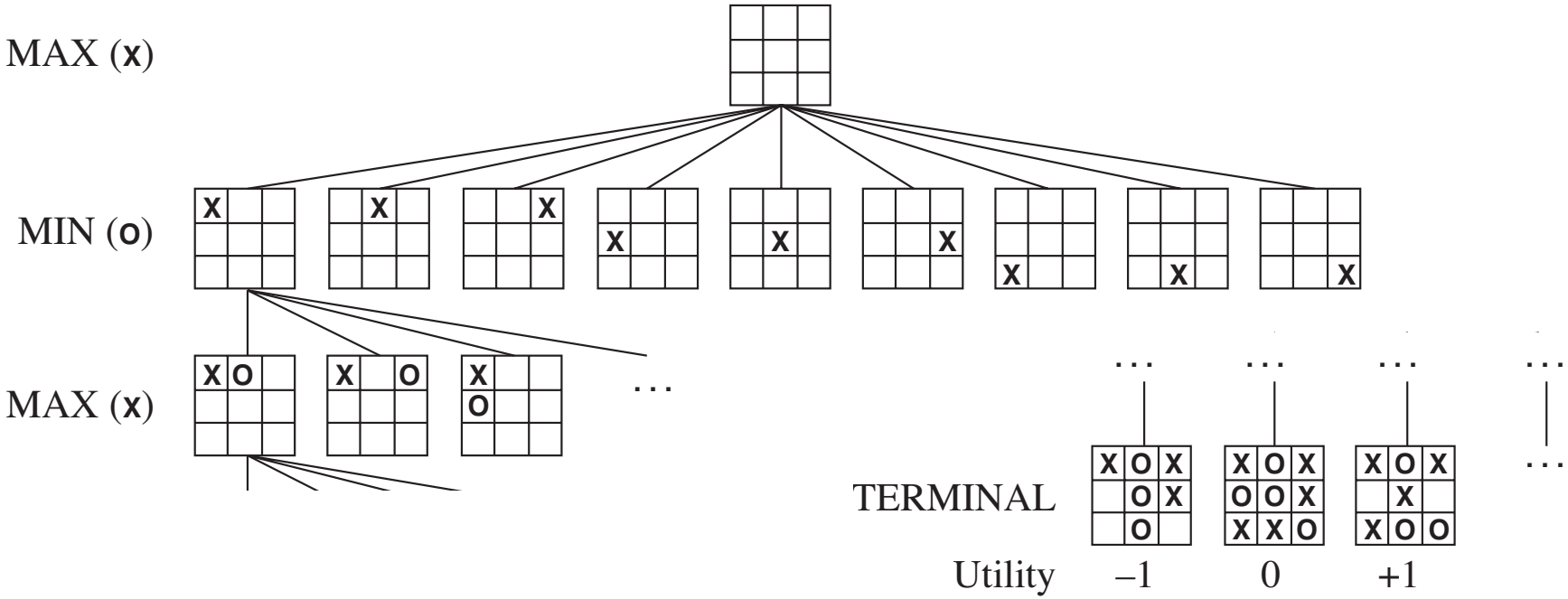

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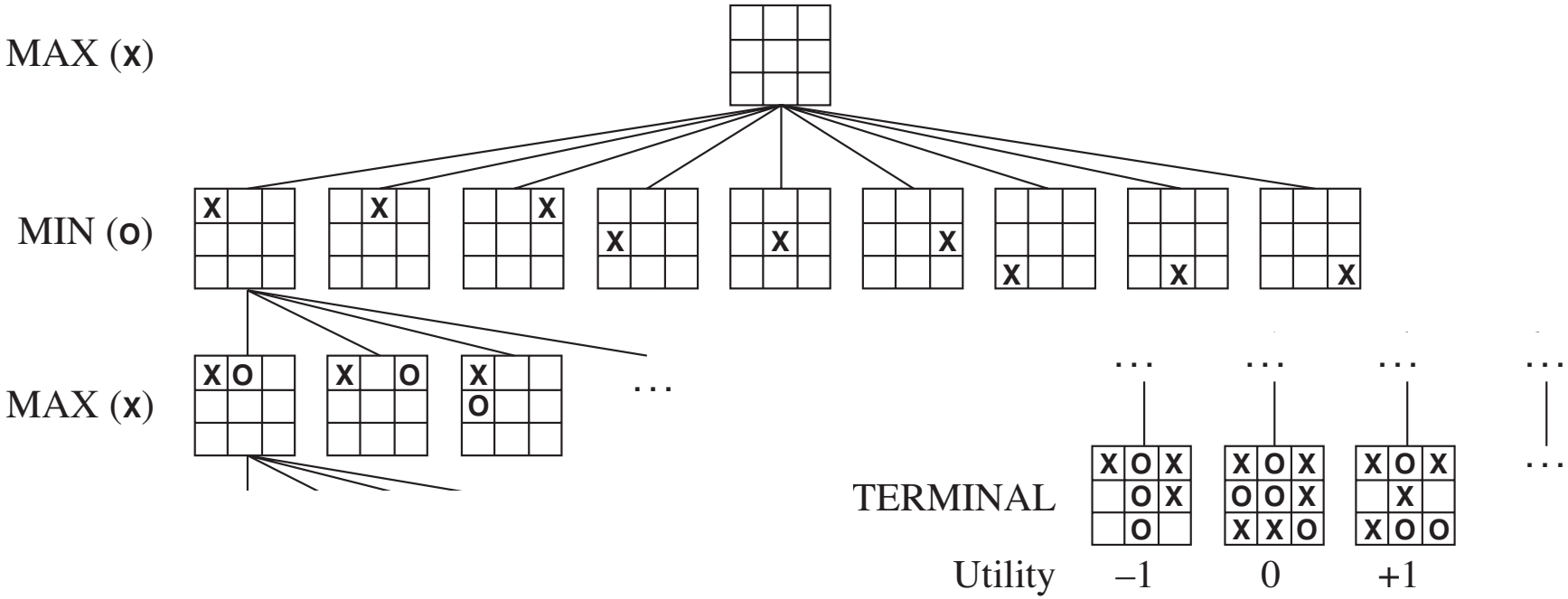

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Minimax algorithm

function MINIMAX(state) **returns** an action

    return $\underset{a \in \text{Actions}(s)}{\text{argmax}}$ MIN-VALUE(RESULT(state, a))

end function

function MIN-VALUE(state) **returns** a utility value v

    if TERMINAL-TEST(state) **then return** UTILITY(state)
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    $v \leftarrow \infty$
    for all ACTIONS(state) **do**
        $v \leftarrow$ **min**(v, MAX-VALUE(RESULT(state,a)))
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A two ply game, down to terminal and back again . . .

```
function MINIMAX(s) returns a                                   MAX
    argmax_{a∈Actions(s)} MINVAL(RES(s, a))
end function
-------------------------------------------------------------   MIN
function MINVAL(s) returns v
    if TERMINAL(s) then UTIL(s)
    end if
    v ← ∞
    for all ACTIONS(s) do
        v ← min(v, MAXVAL(RES(s, a)))
    end for
end function
-------------------------------------------------------------
function MAXVAL(s) returns v
    if TERMINAL(s) then UTIL(s)
    end if
    v ← −∞
    for all ACTIONS(s) do
        v ← max(v, MINVAL(RES(s, a)))
    end for
end function
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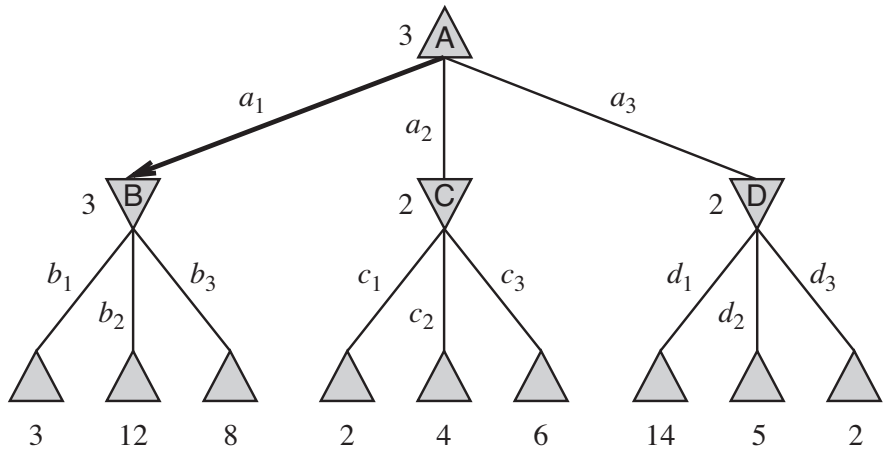
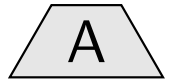


A two ply game, recursive run



What is the complexity? How many nodes to visit?

Can we do better? How?

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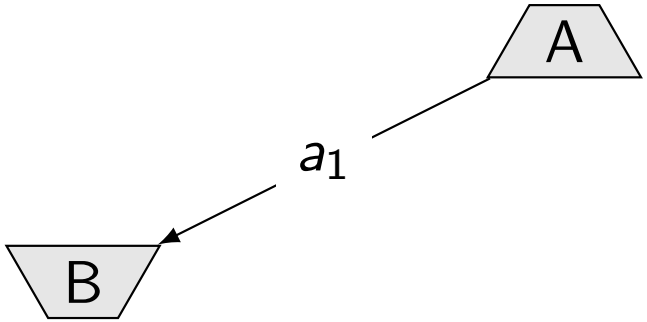

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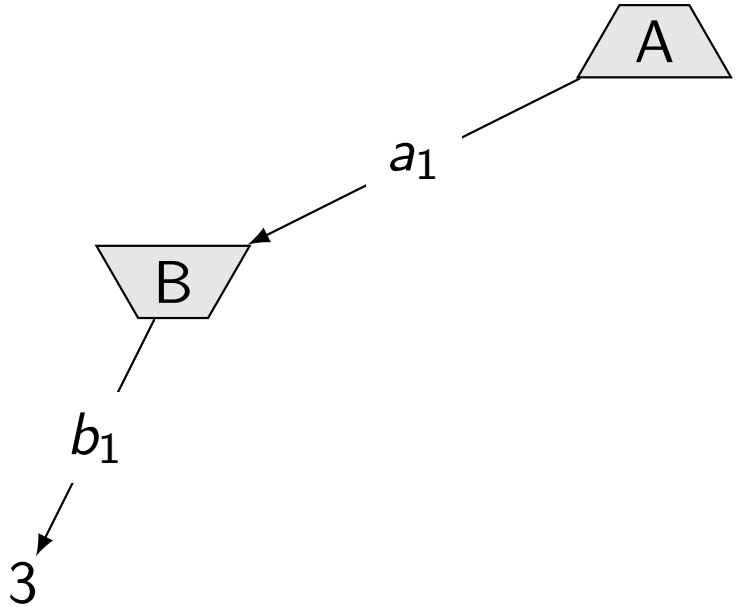

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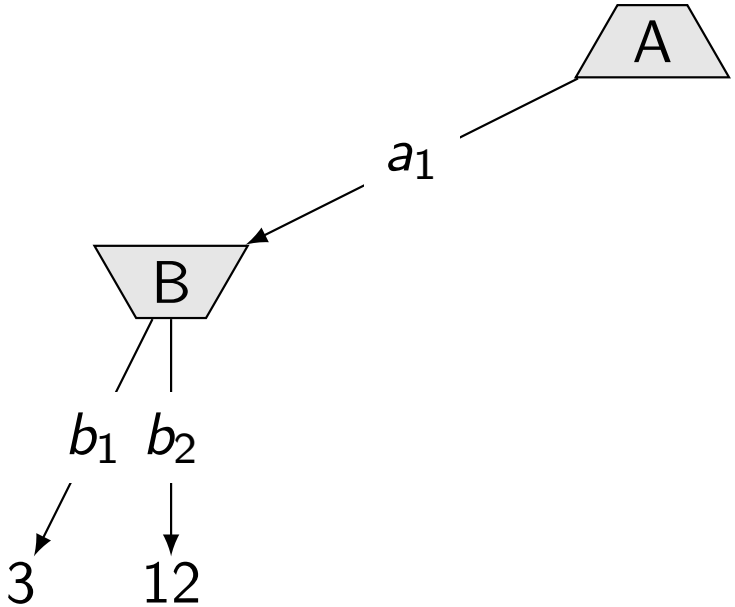

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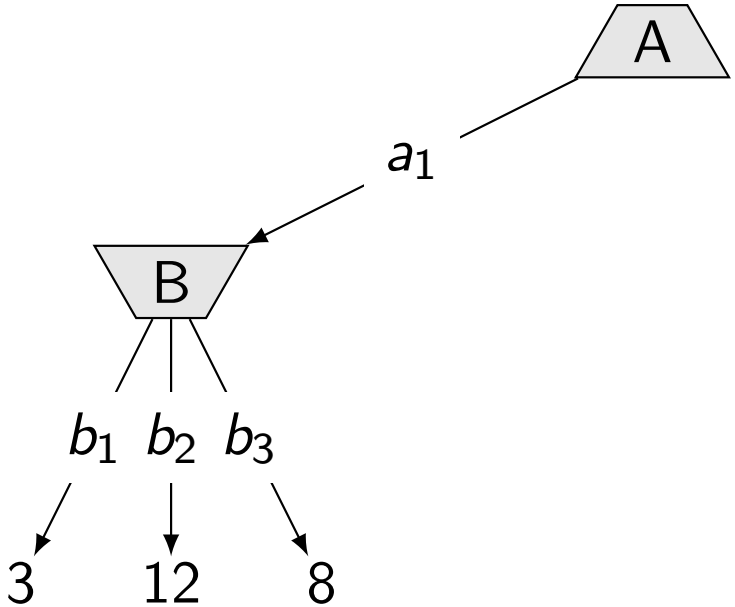

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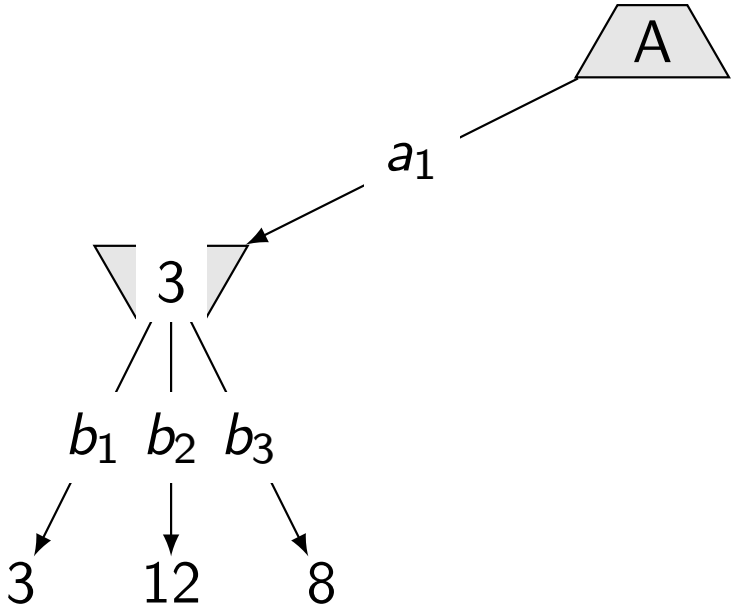

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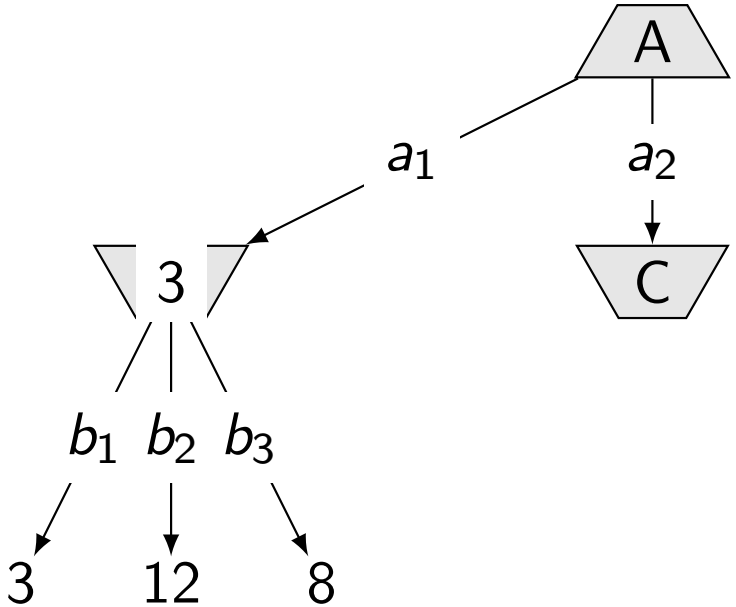

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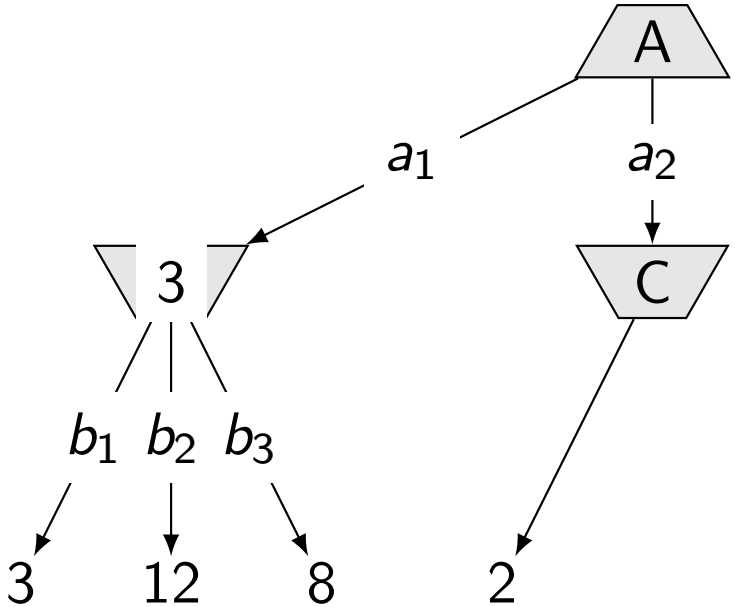

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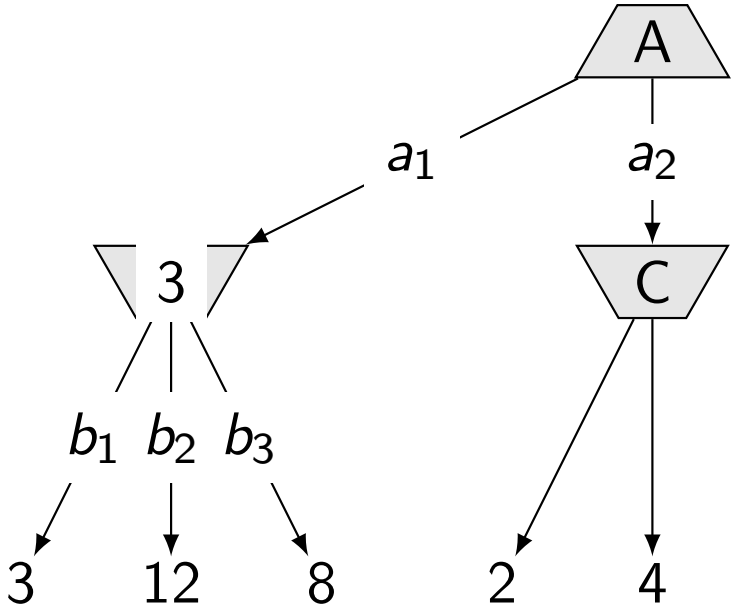
A two ply game, recursive run



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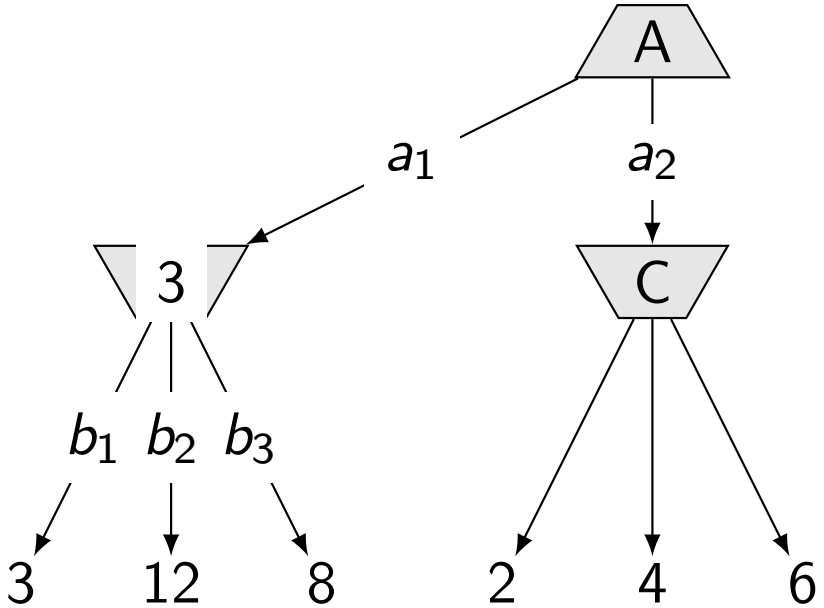

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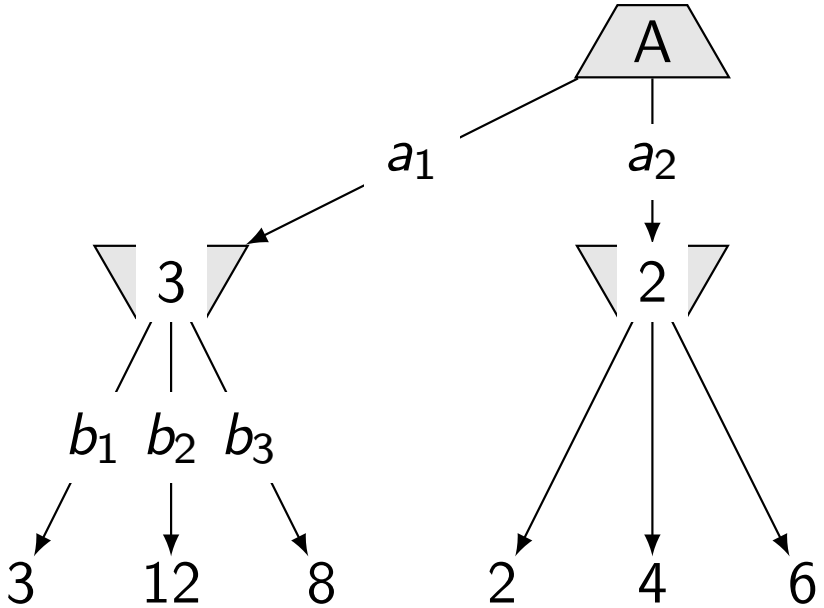

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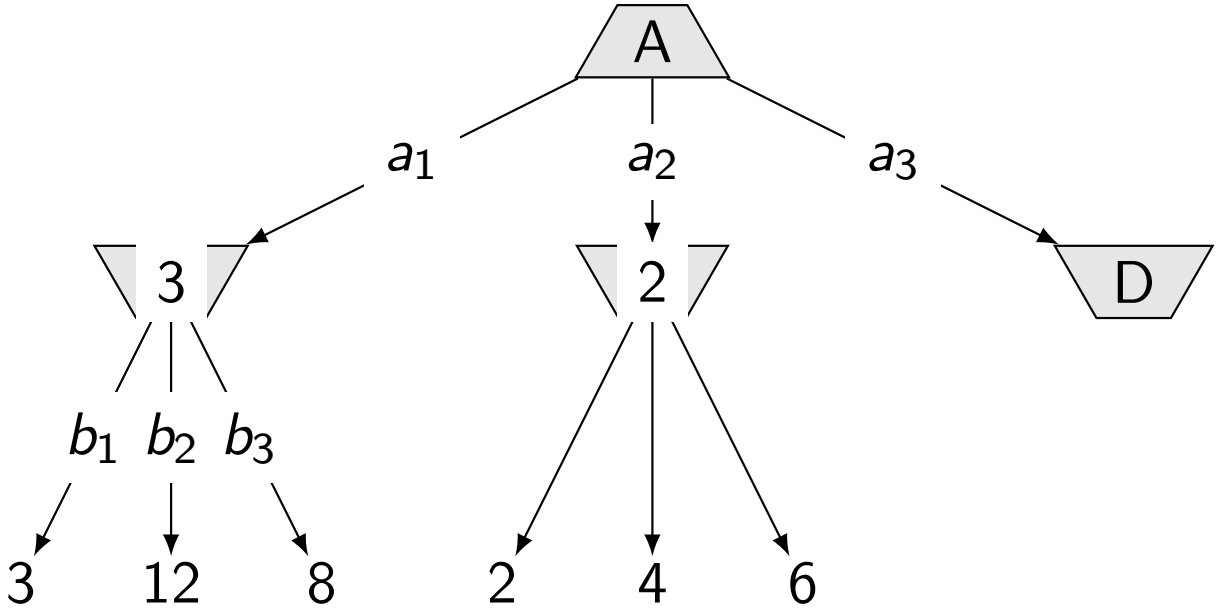

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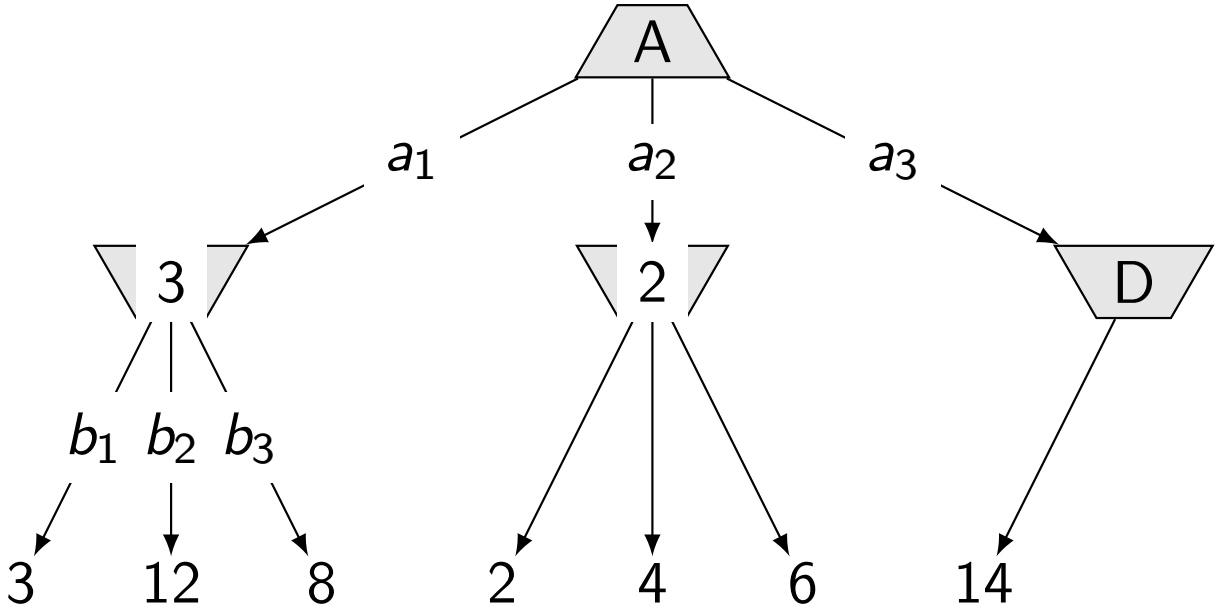

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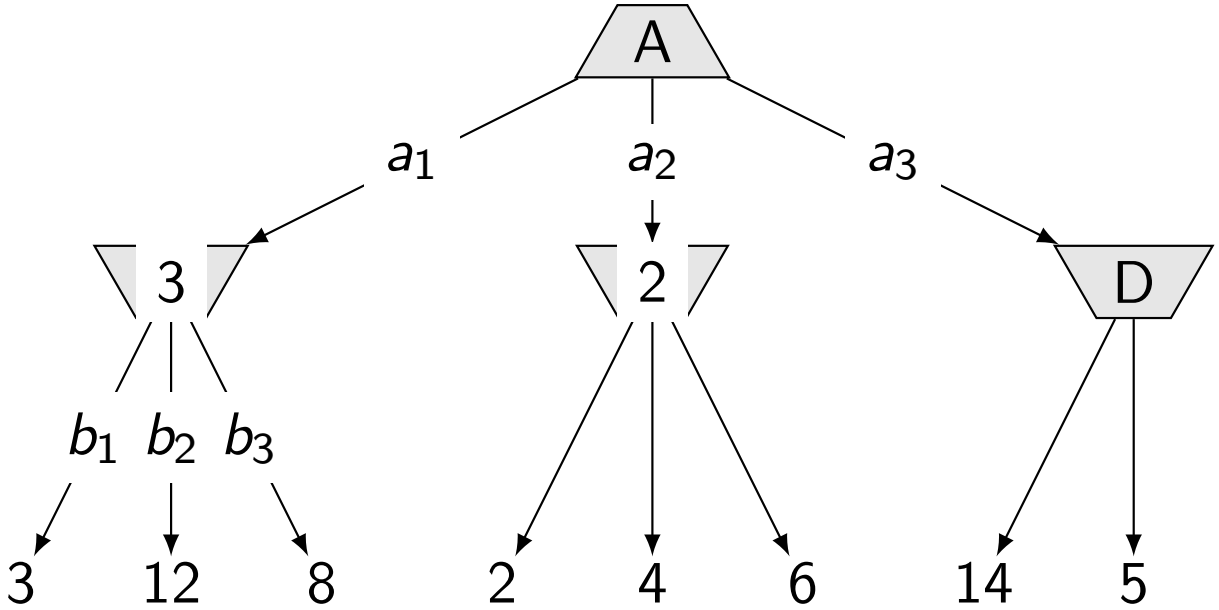

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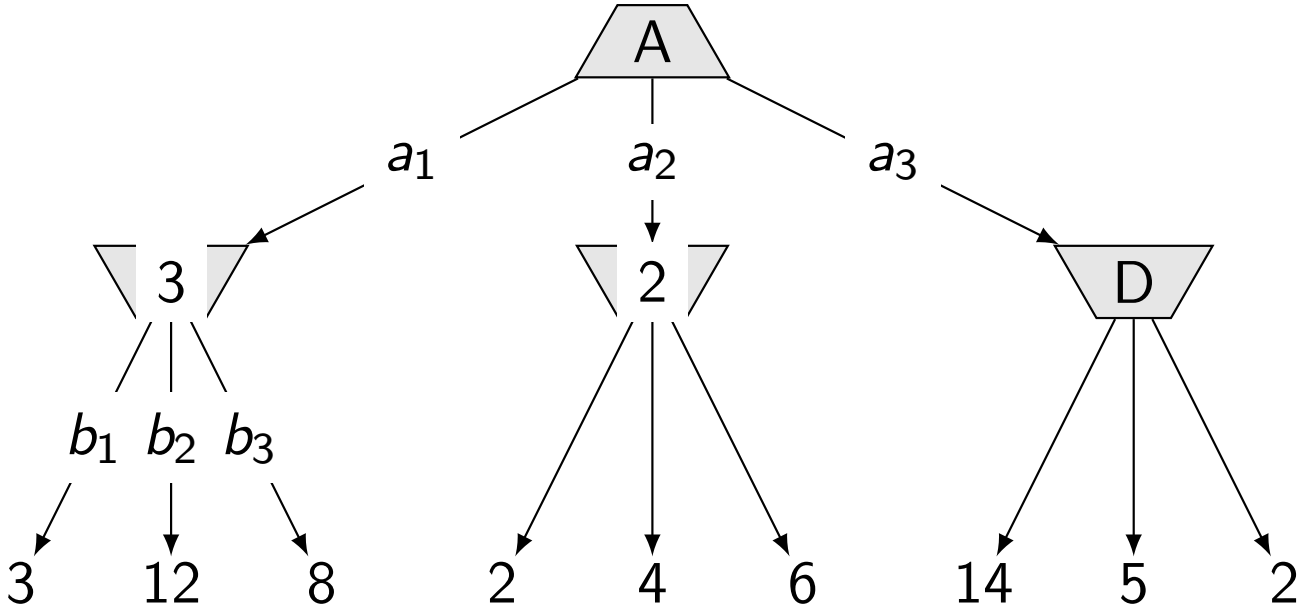

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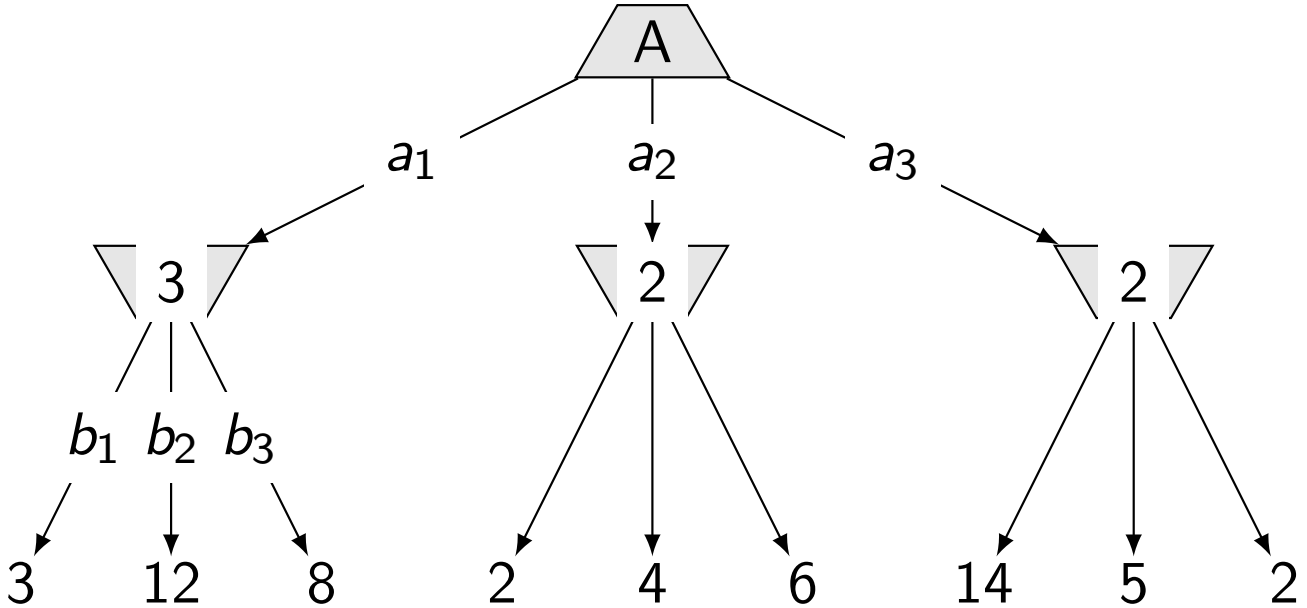

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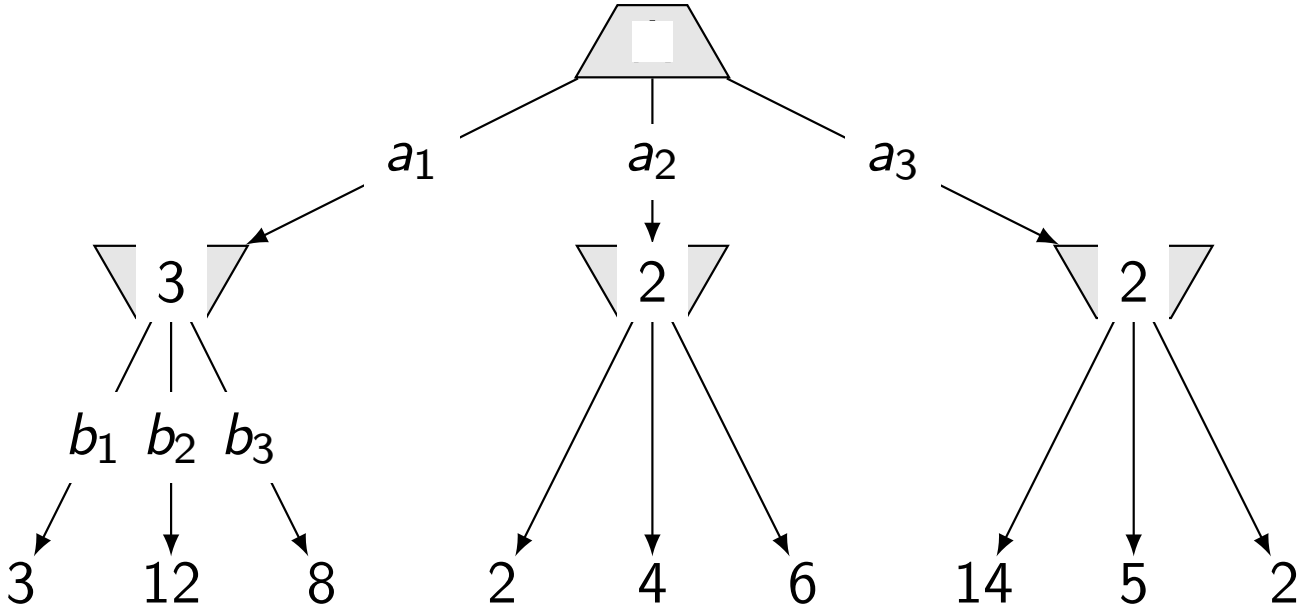

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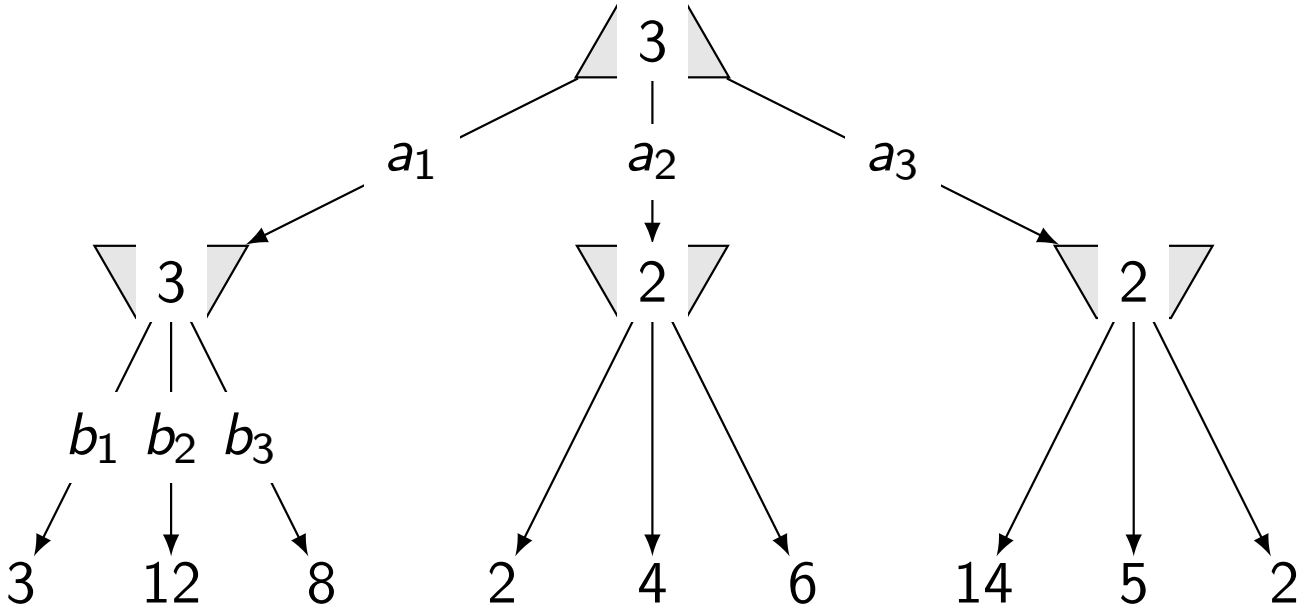

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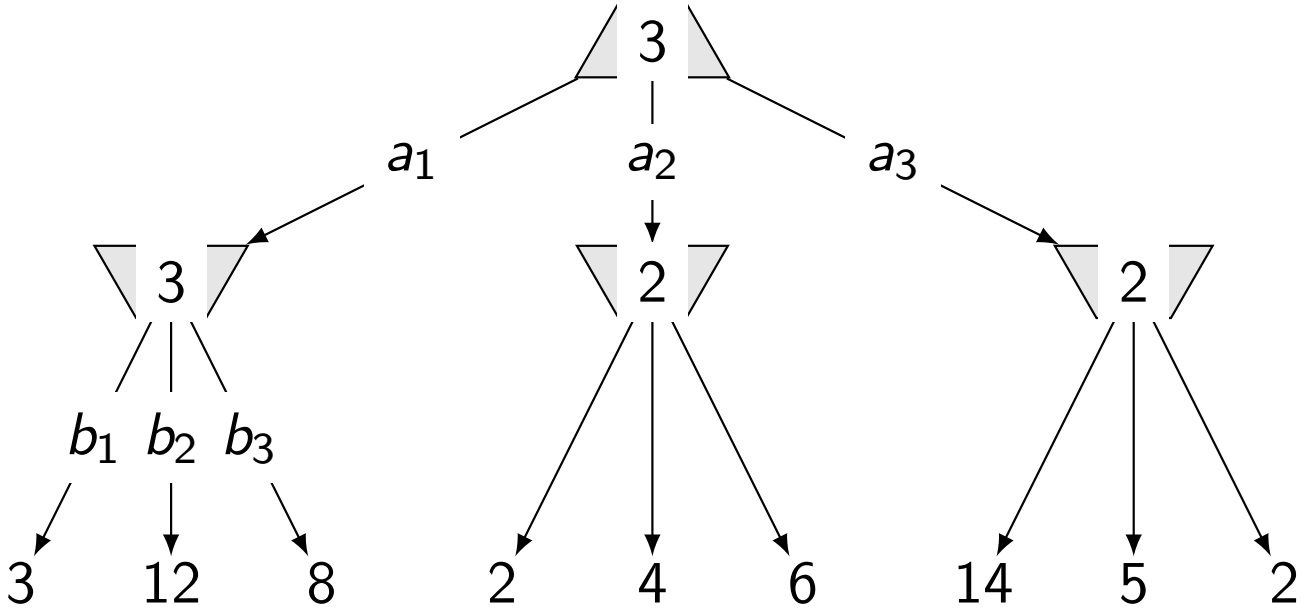

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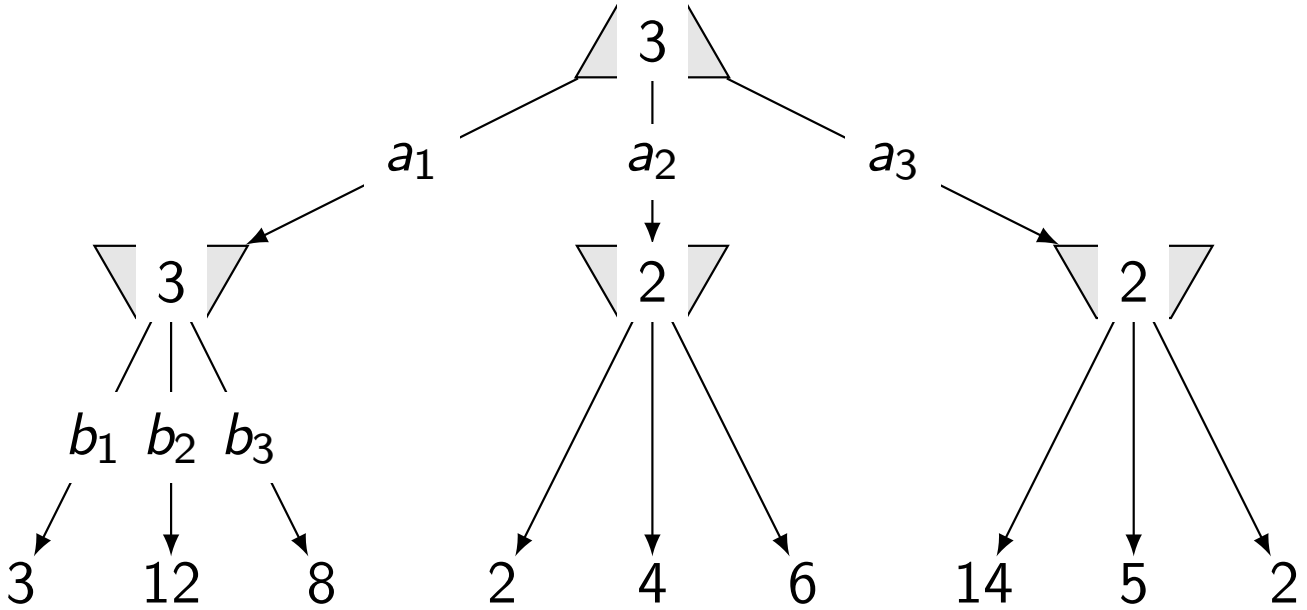

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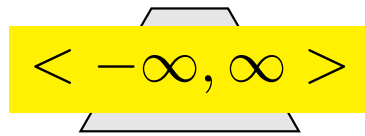

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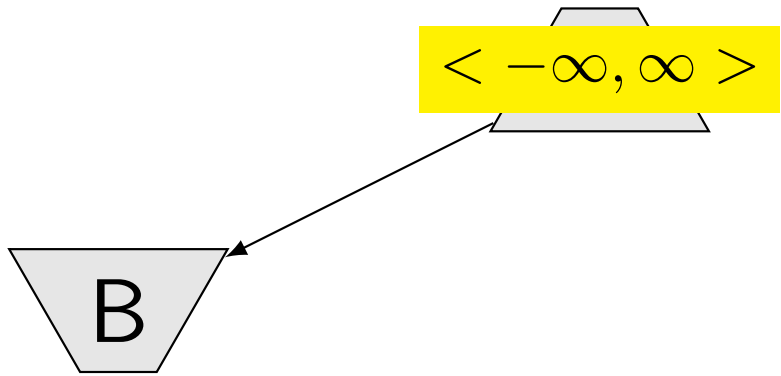

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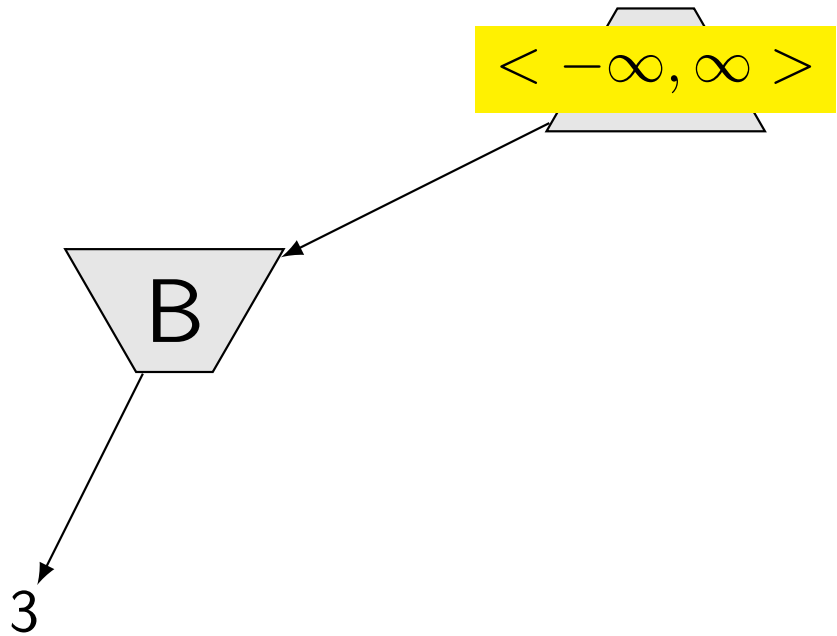

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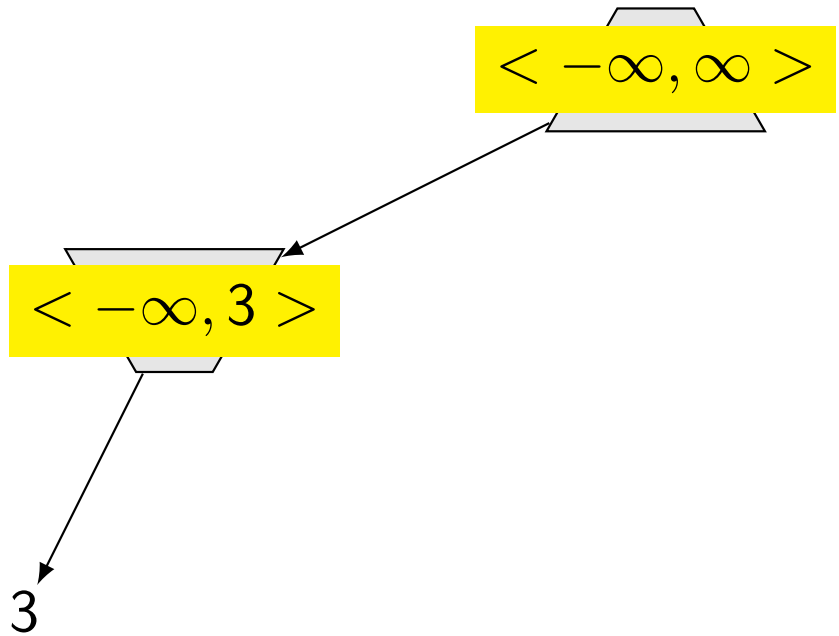

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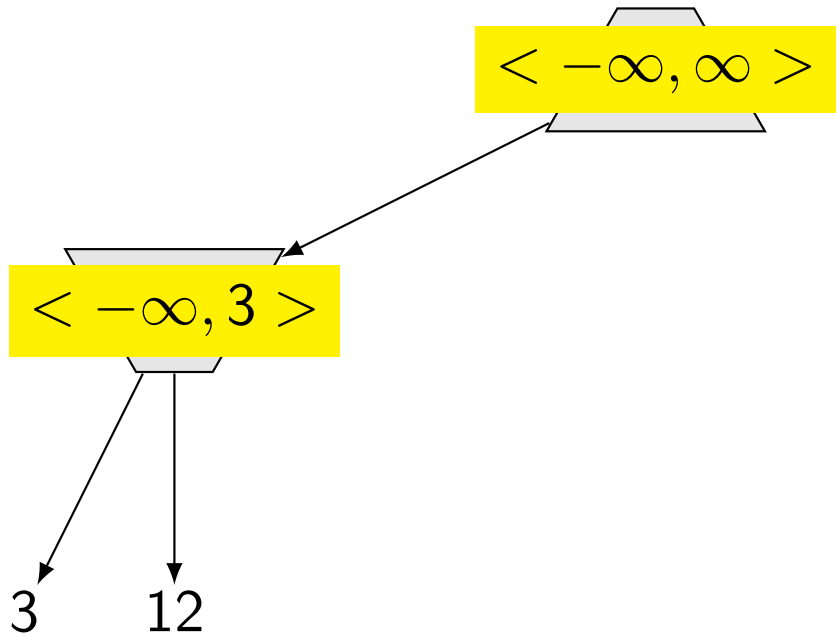

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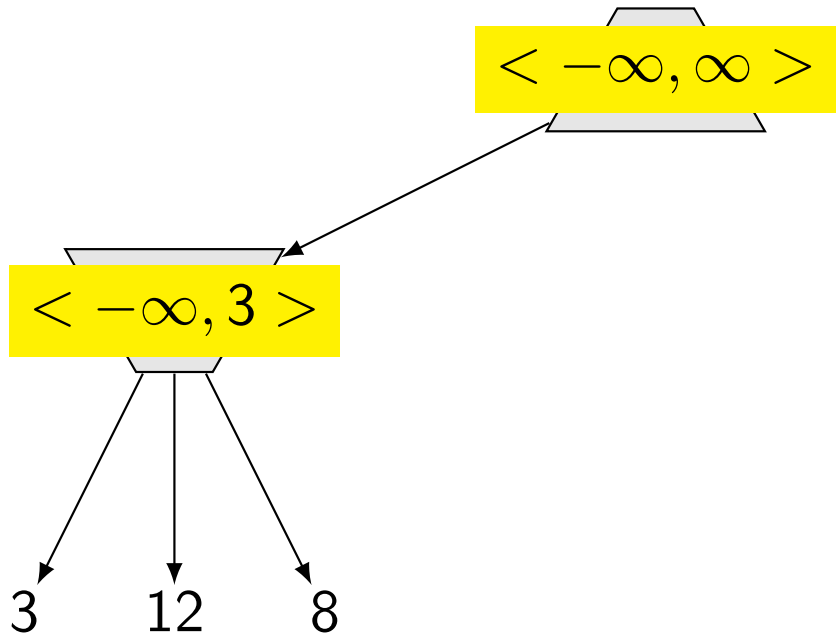

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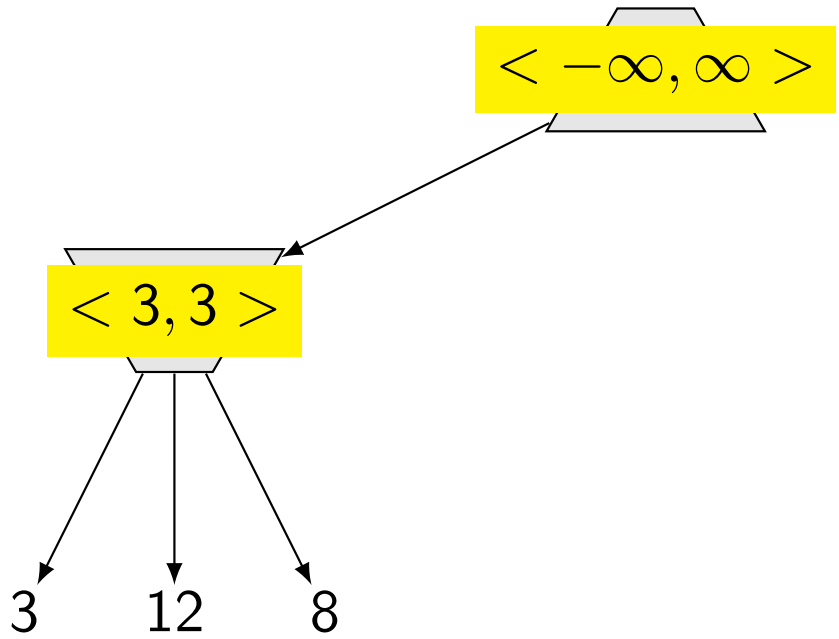

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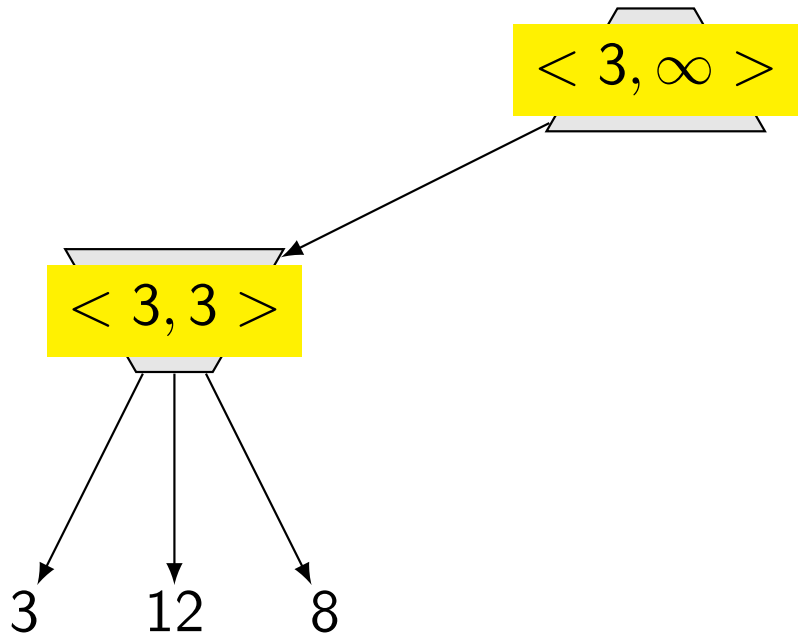

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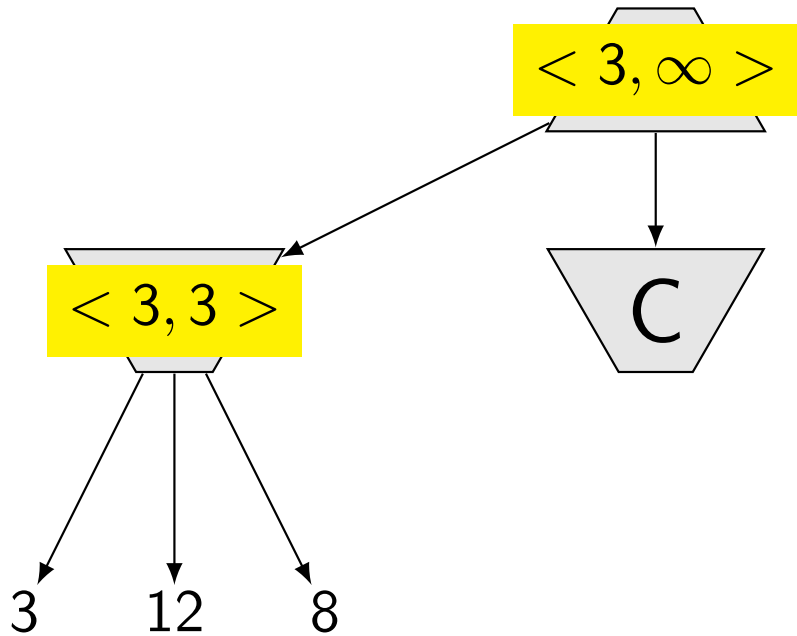

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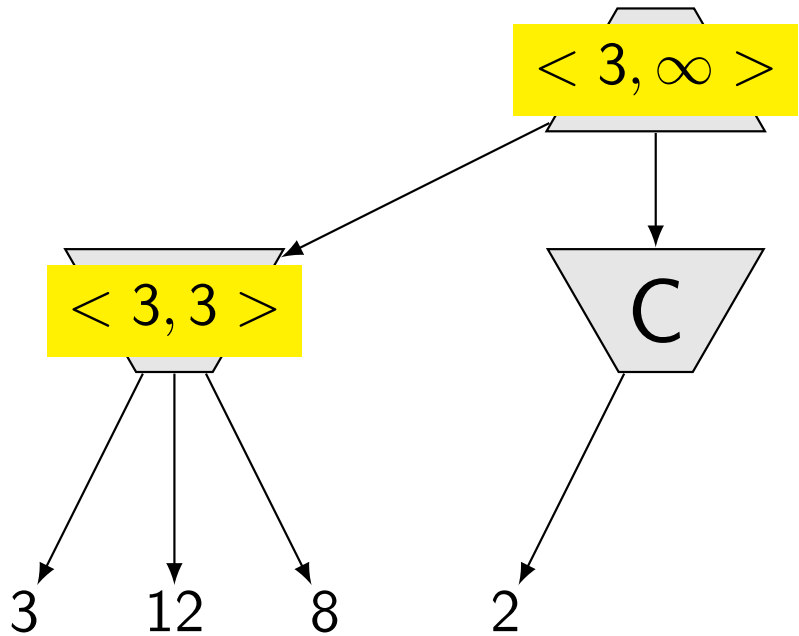

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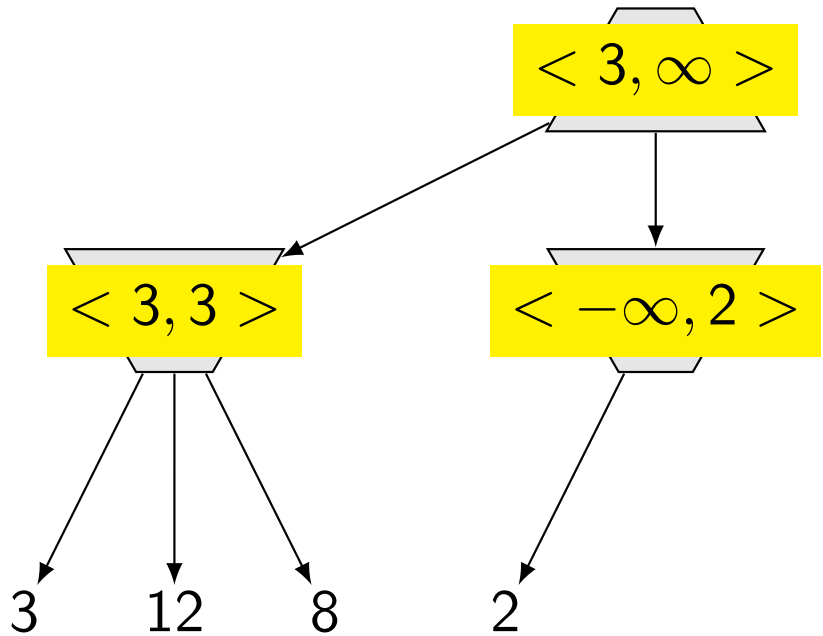

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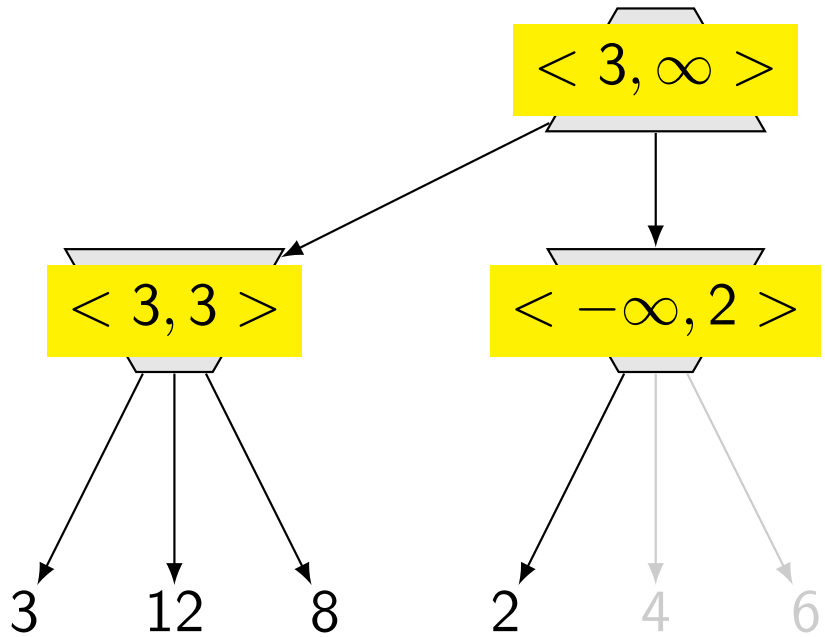

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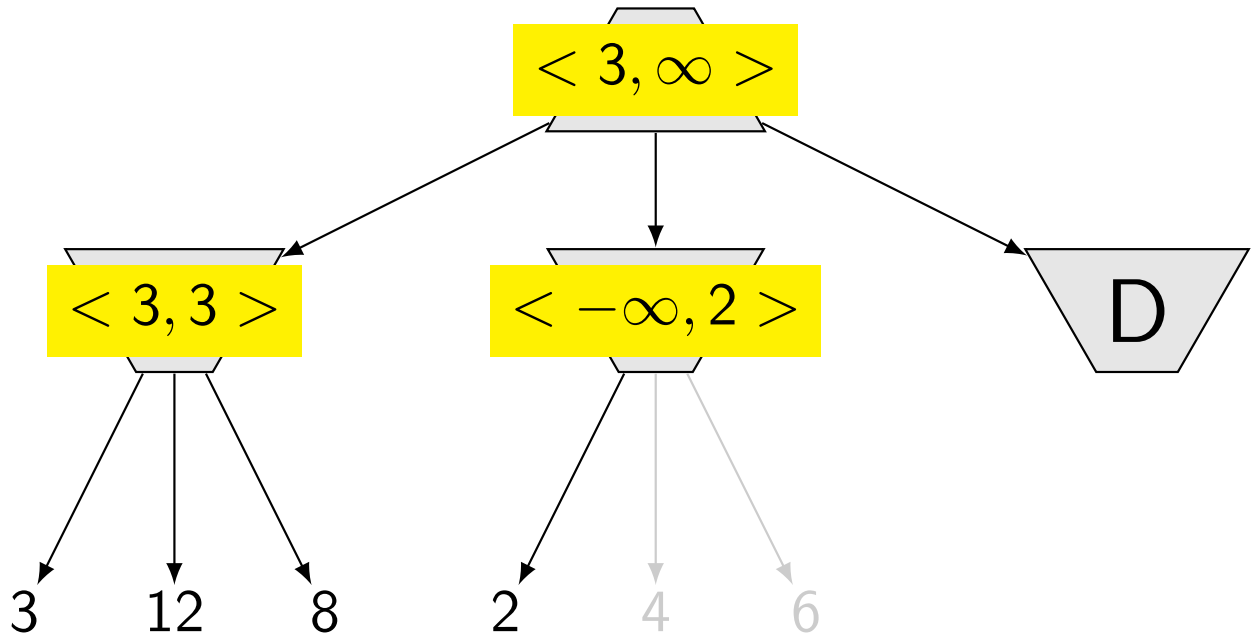

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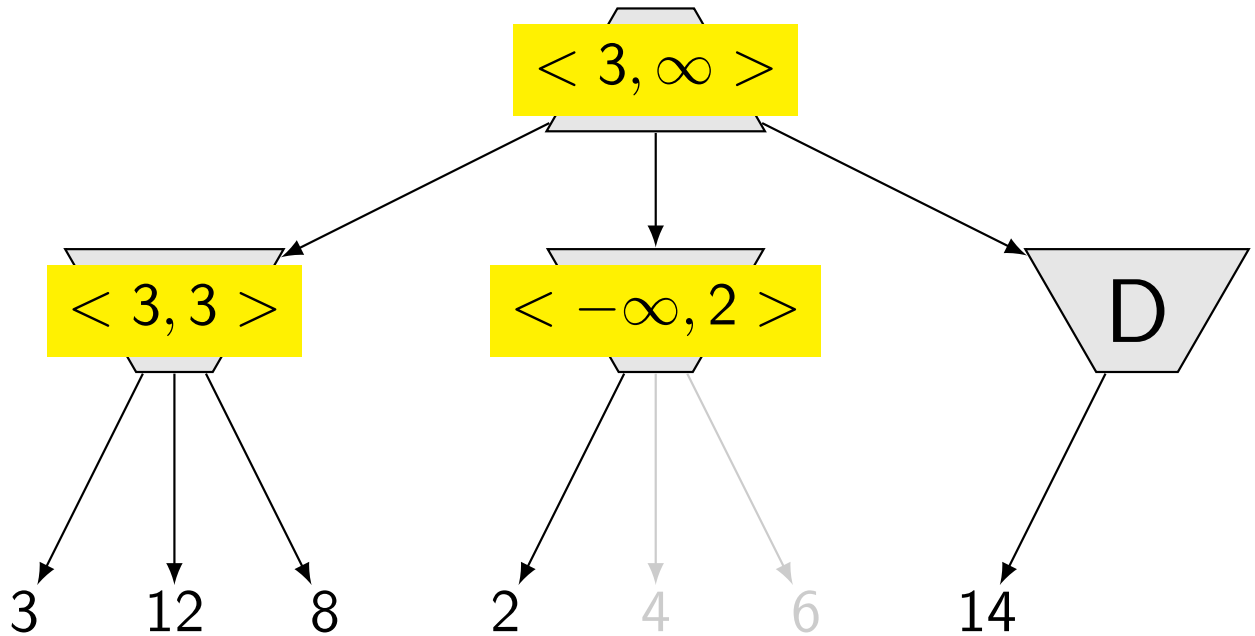

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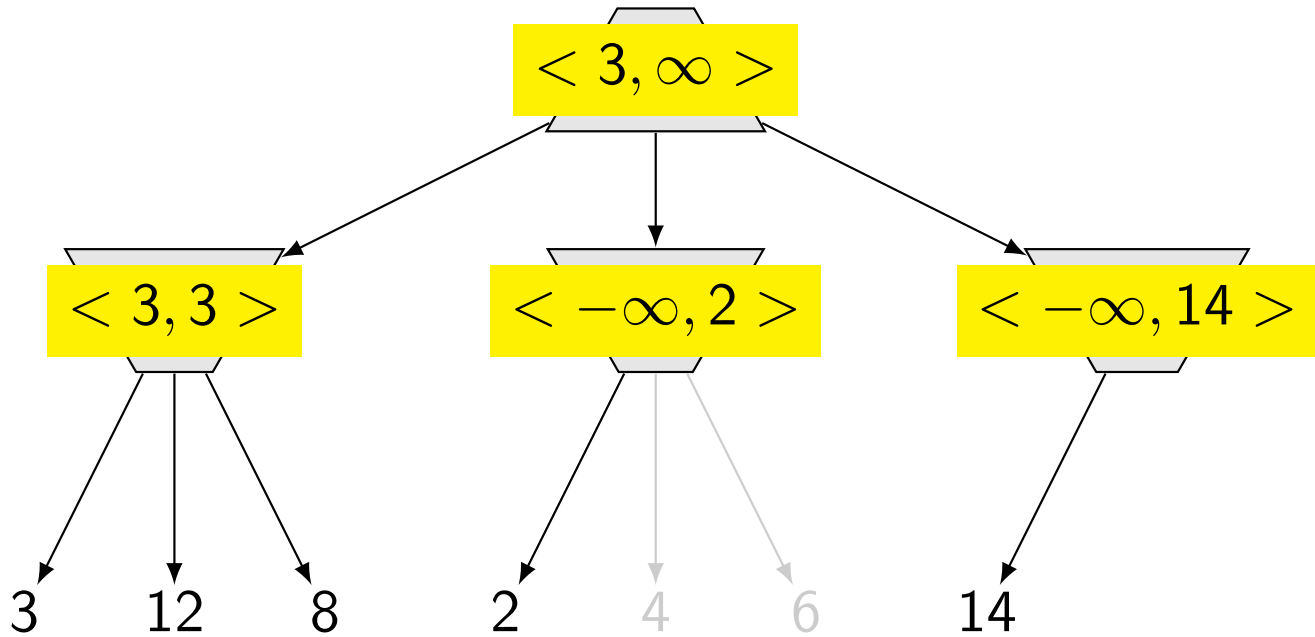

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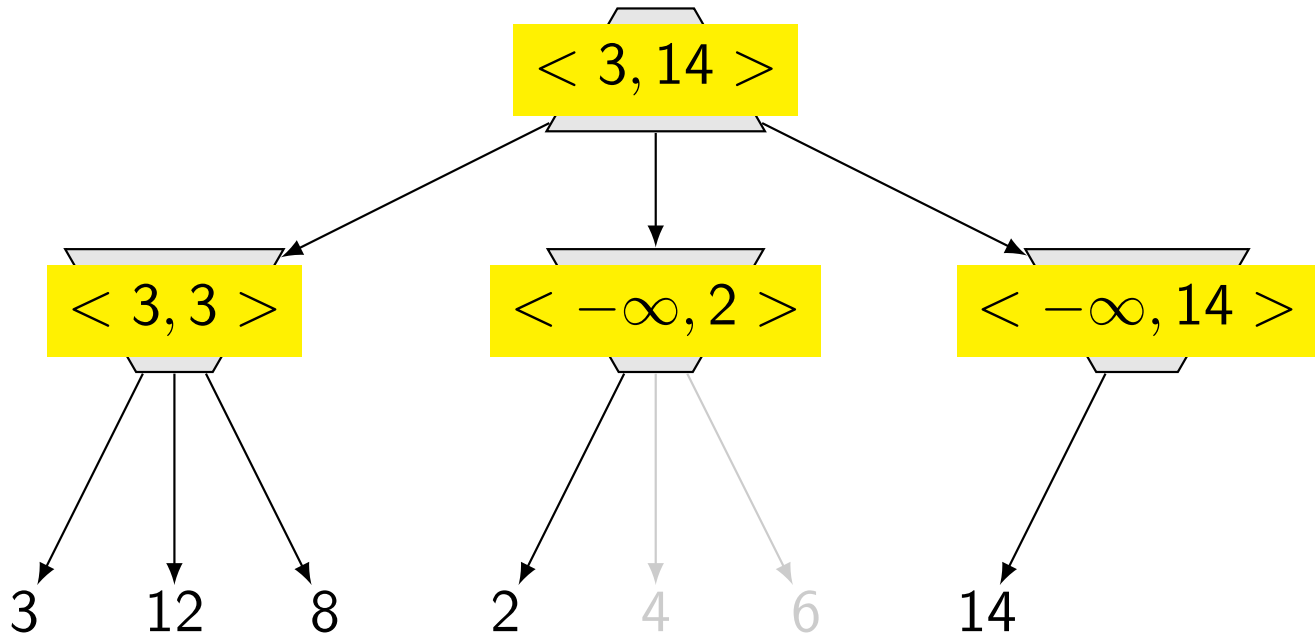

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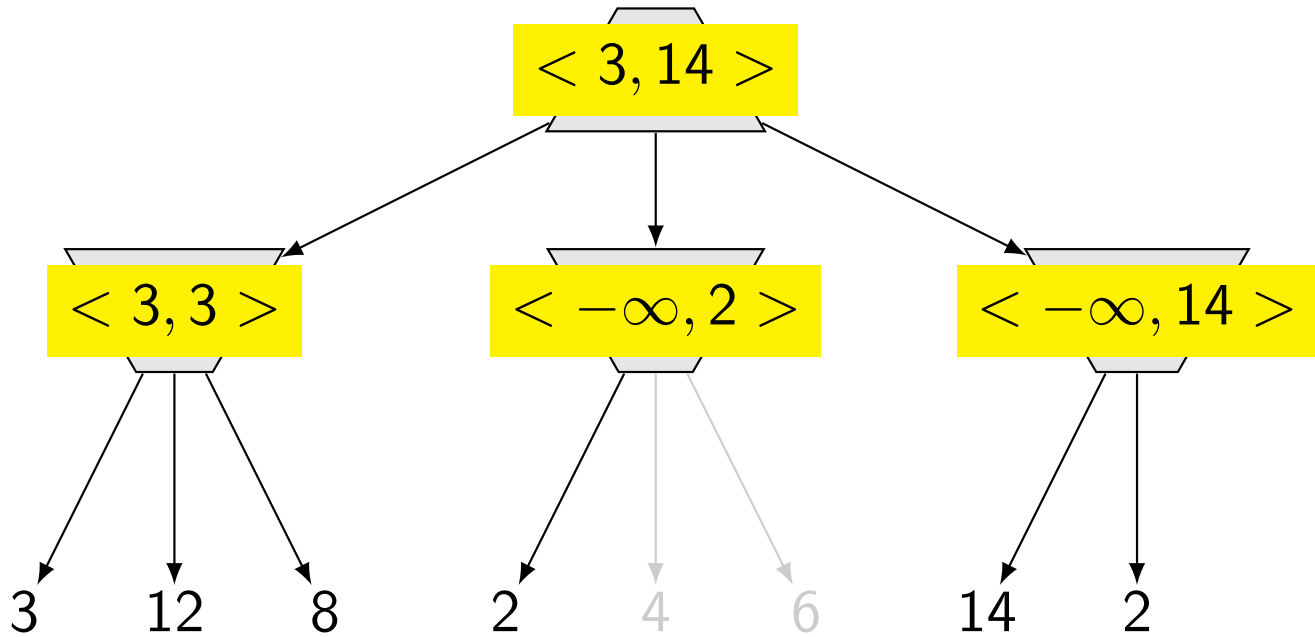

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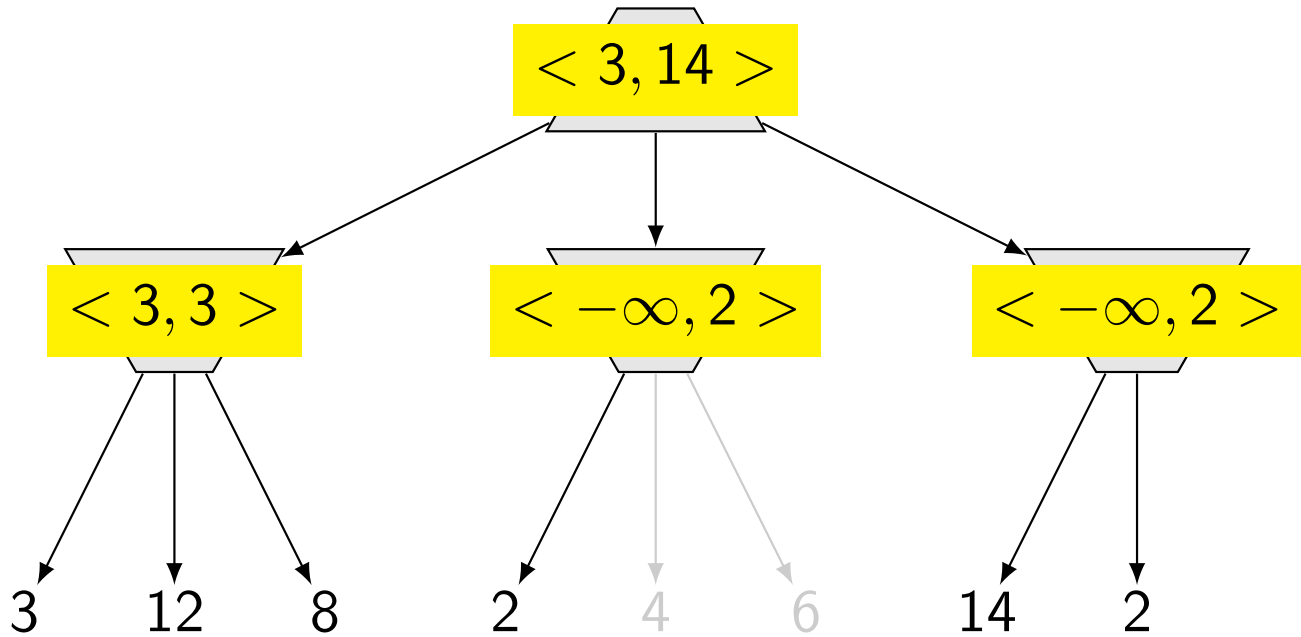

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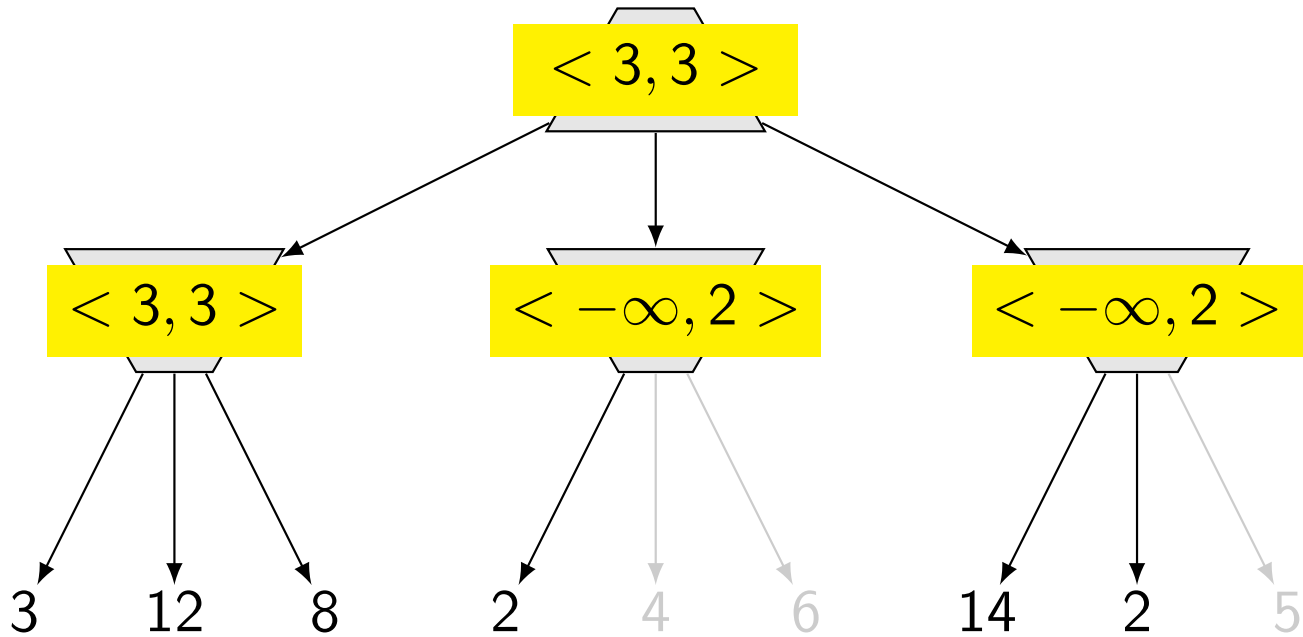

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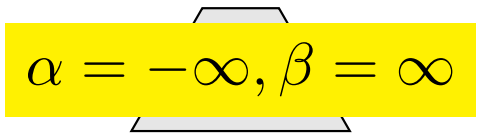

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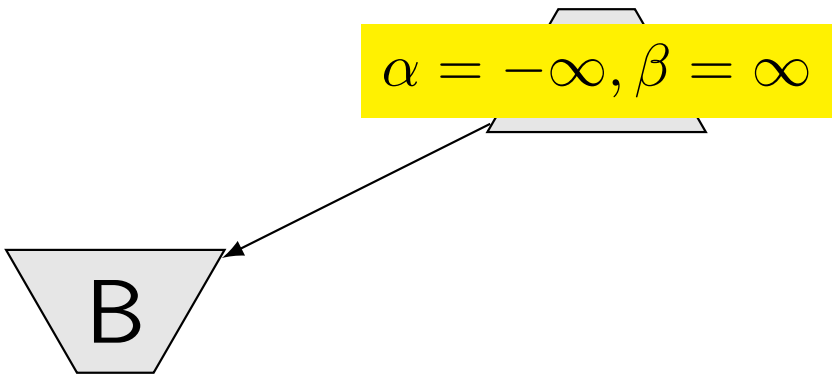

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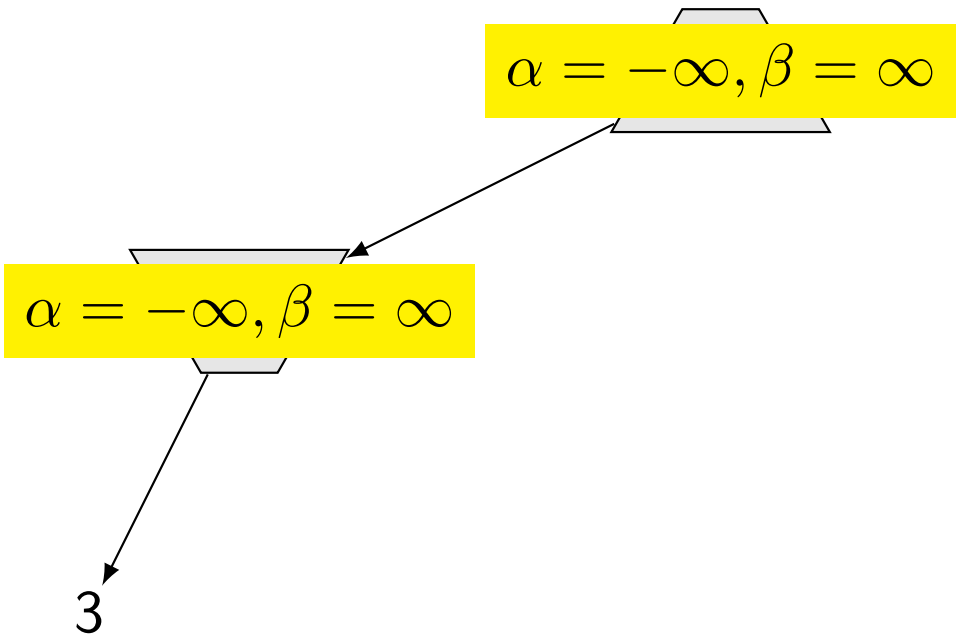

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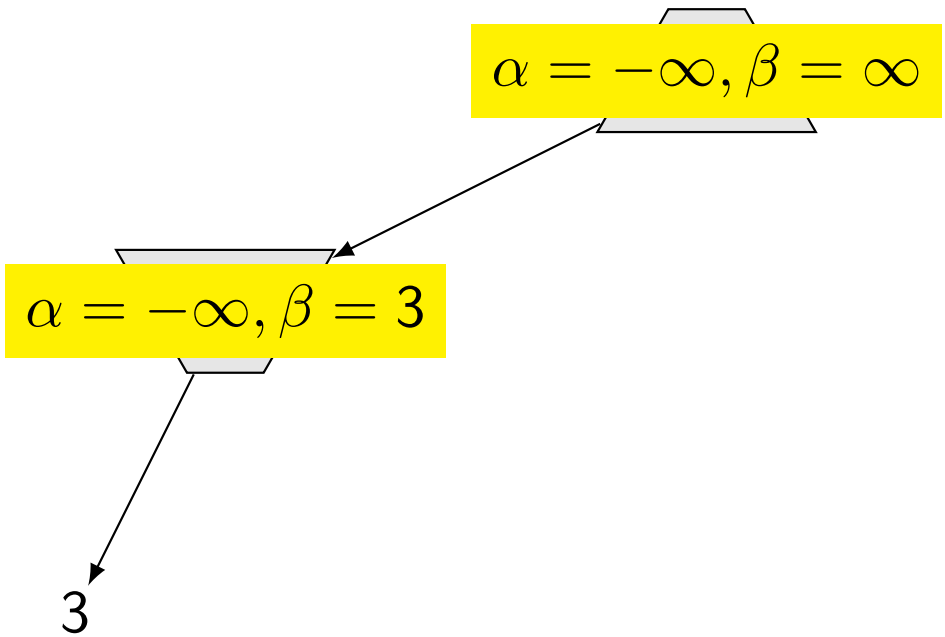

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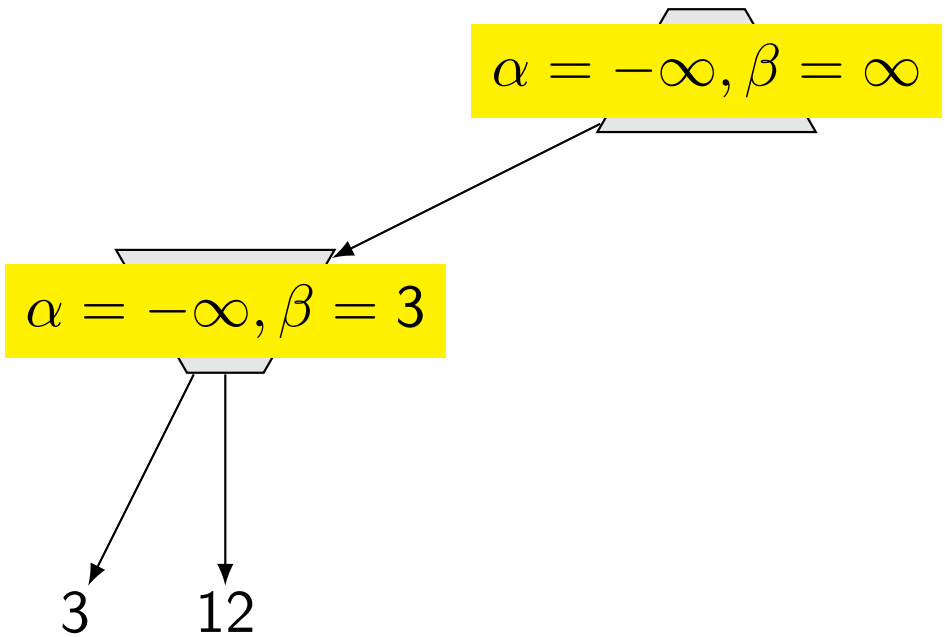

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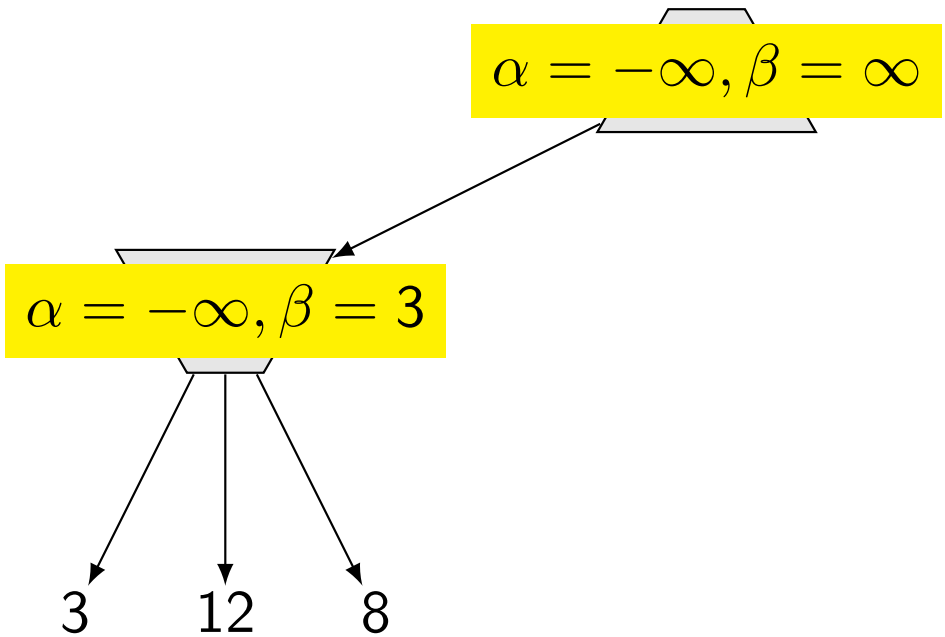

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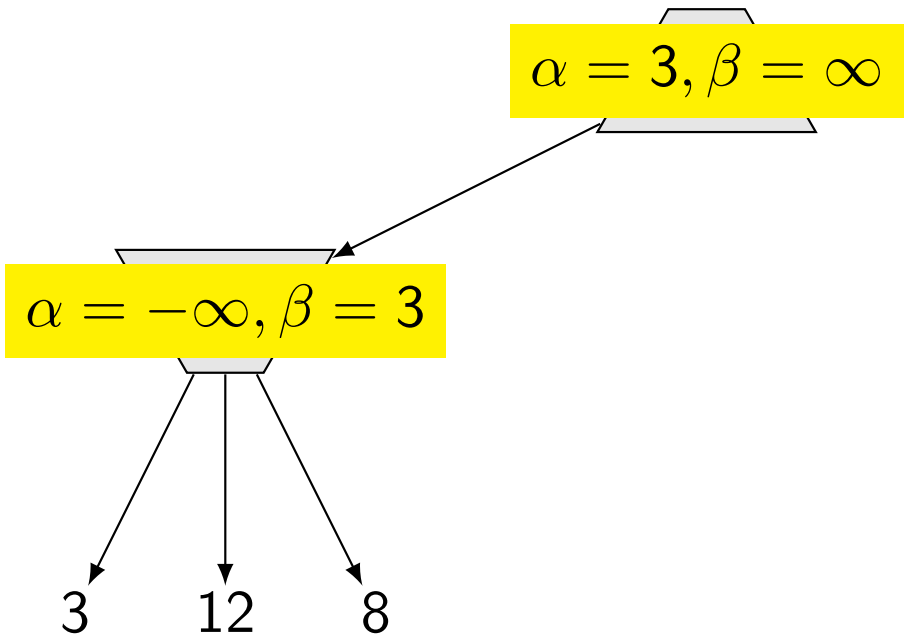

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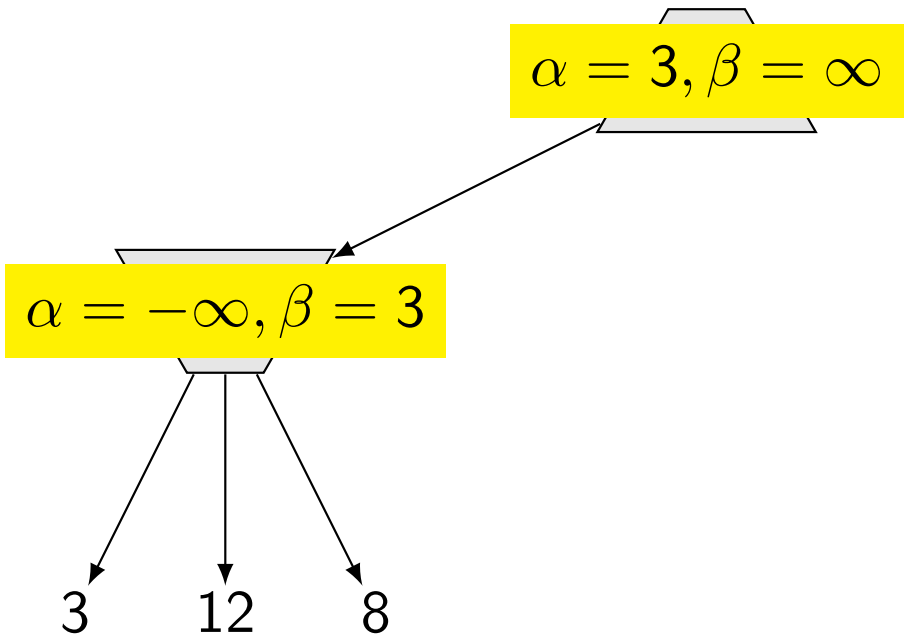

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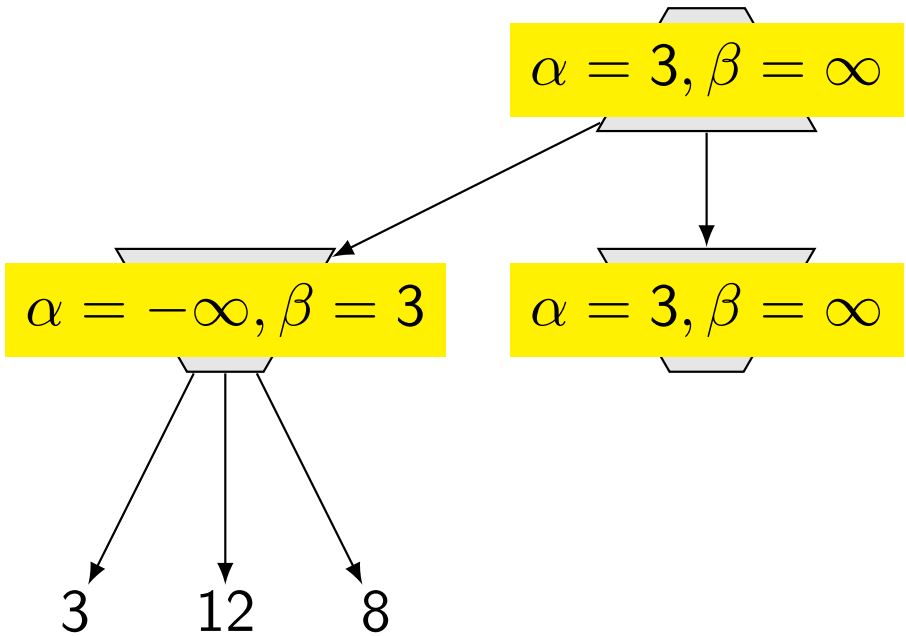

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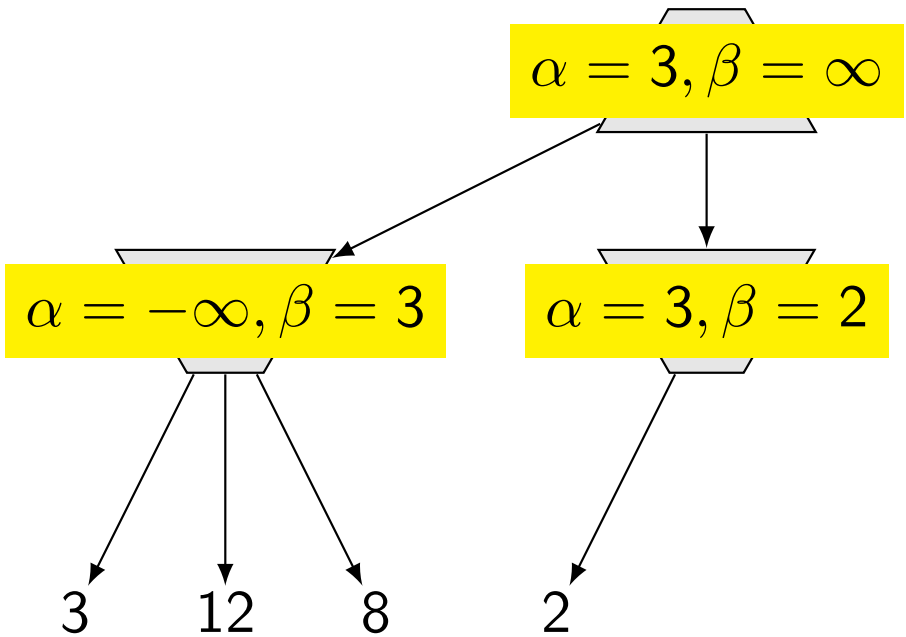

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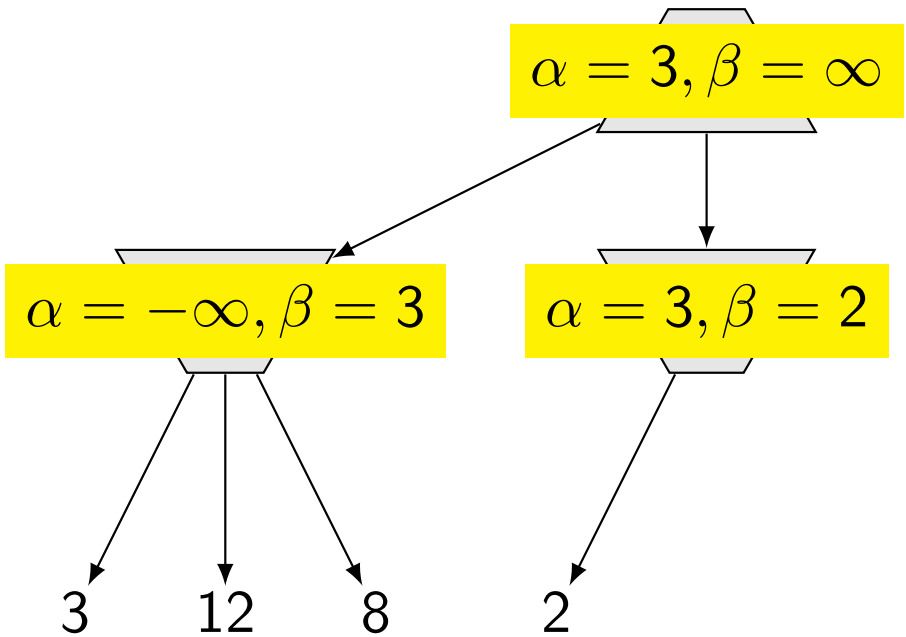

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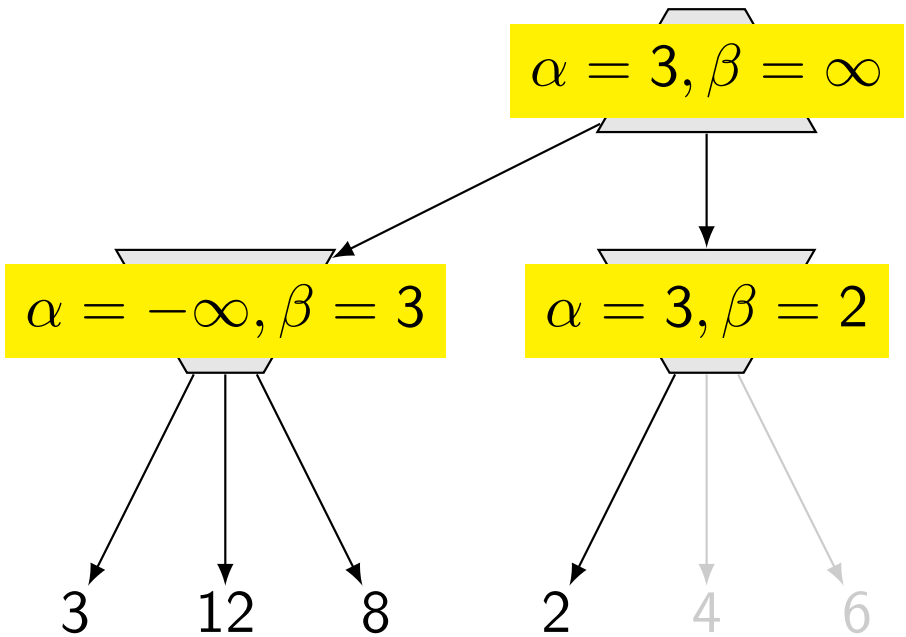

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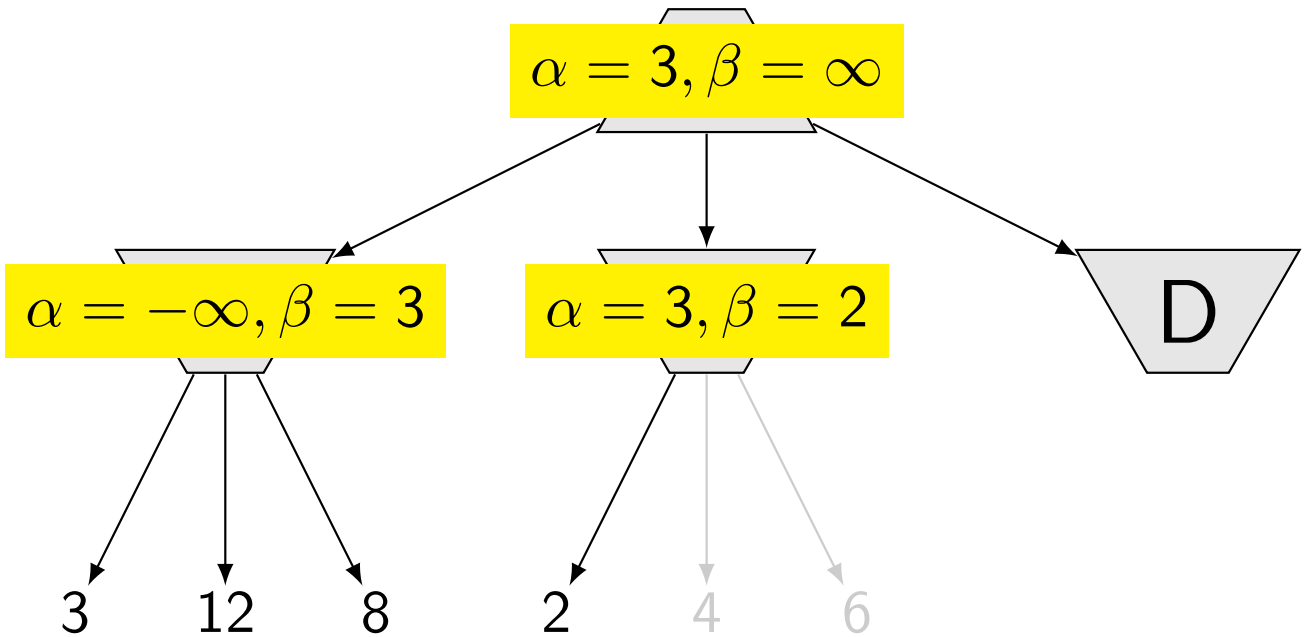

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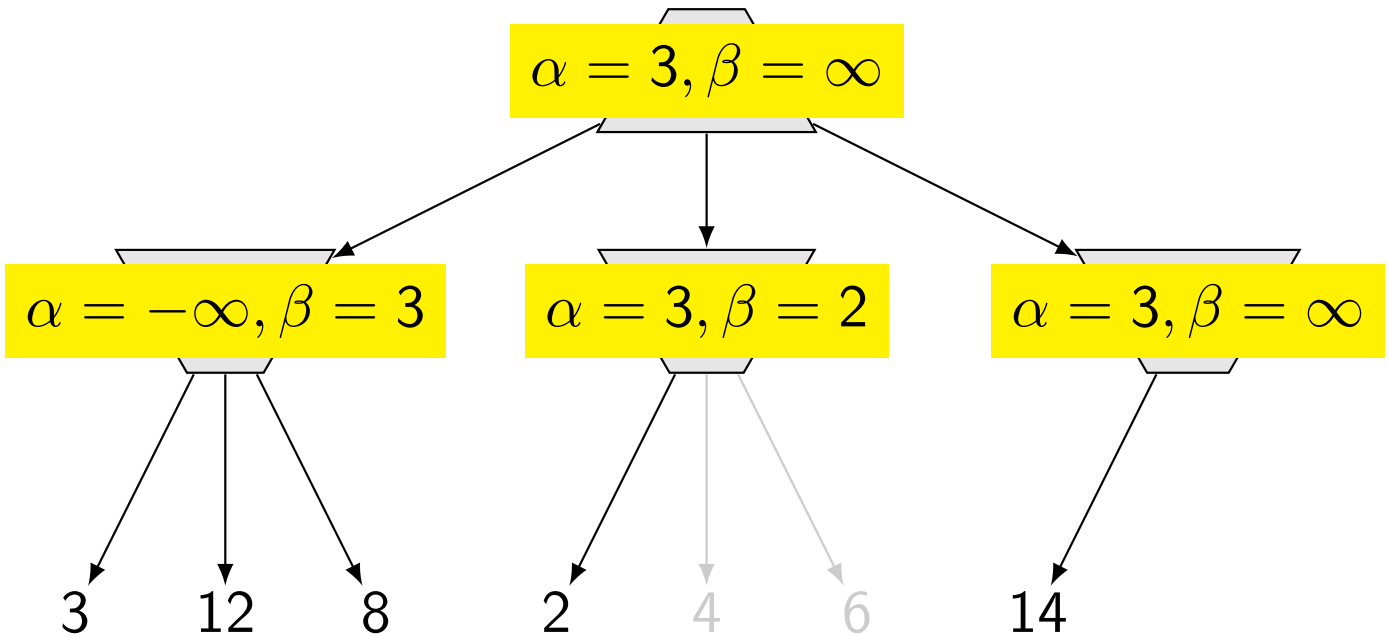

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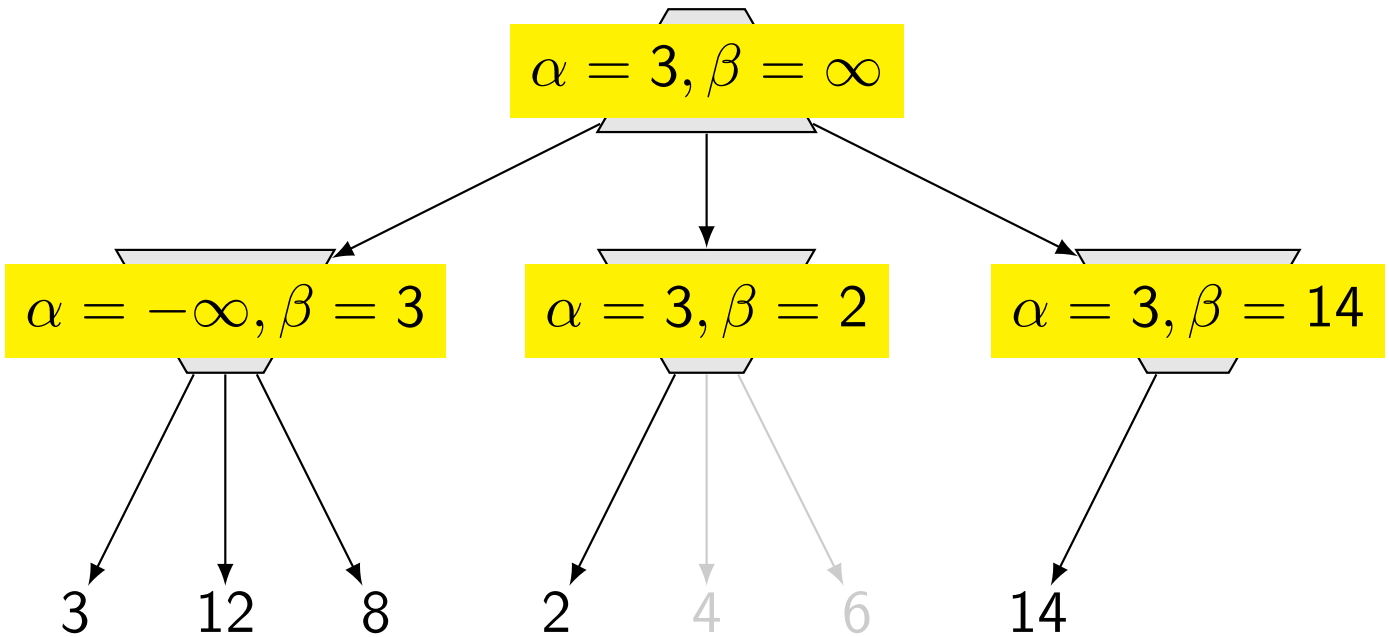

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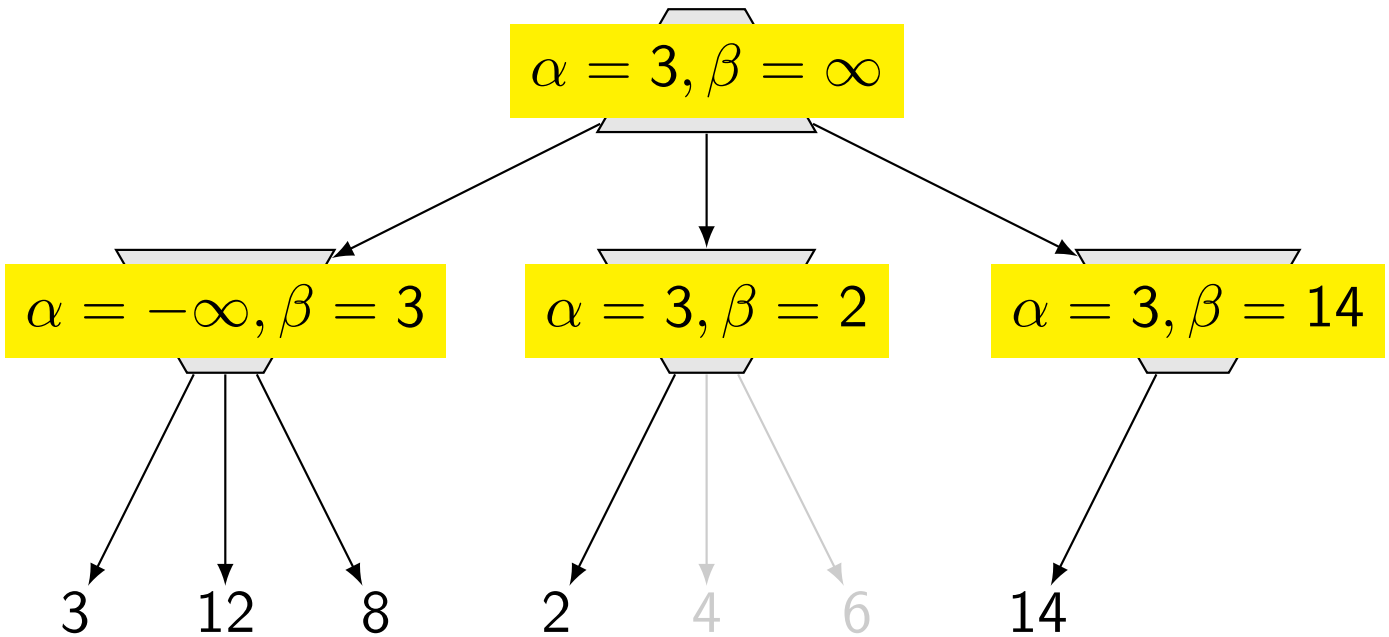

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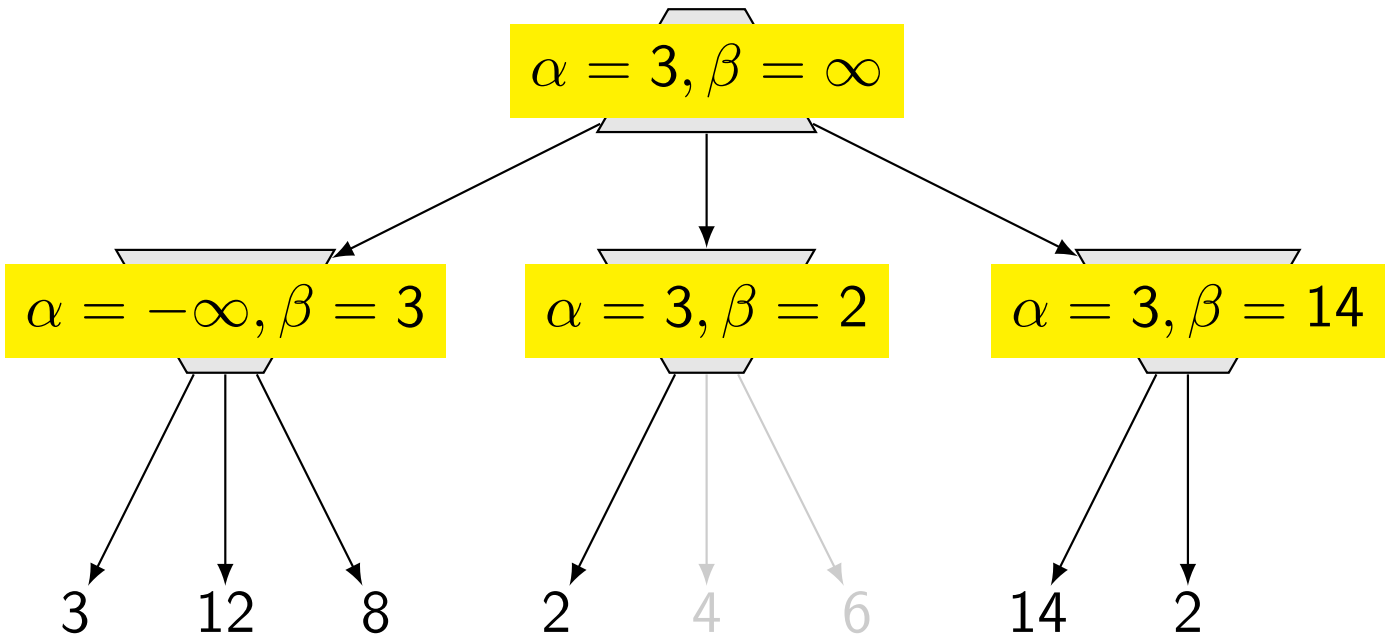

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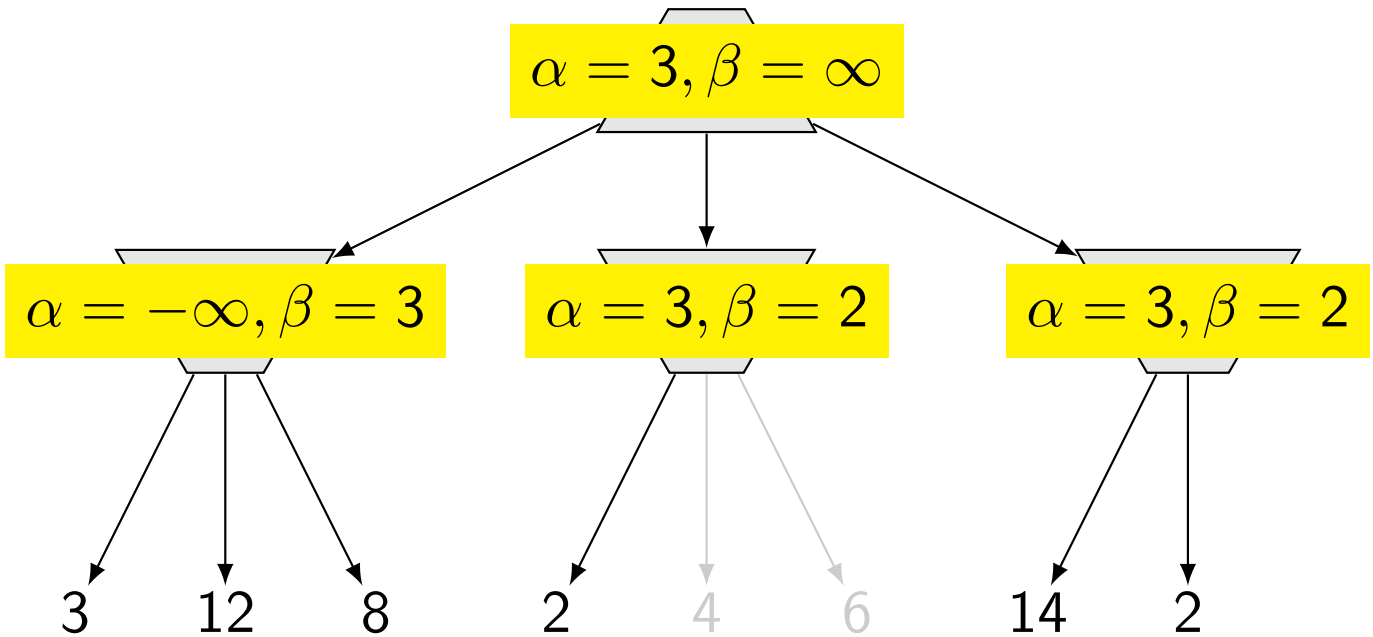

In MIN-VAL: $v \leftarrow 2$
$v \leq \alpha$ then: return v!

α-β prunning

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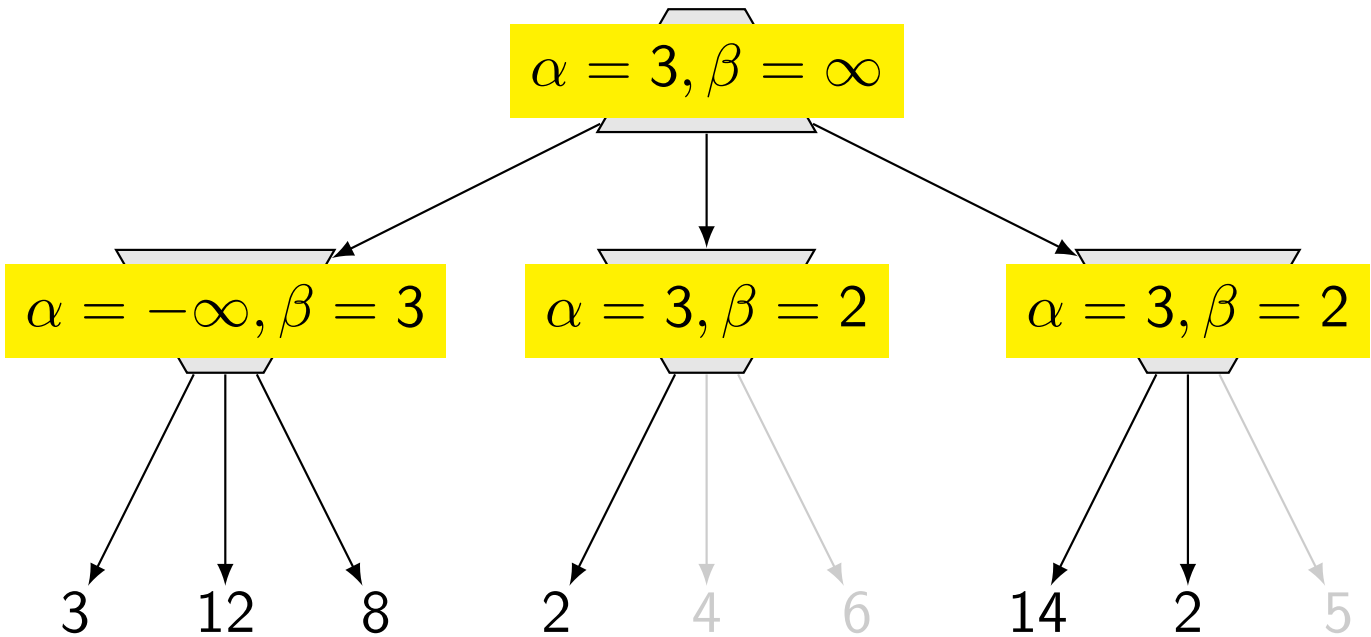

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function ALPHA-BETA-SEARCH(state) **returns** an action
  $v \leftarrow$ MAX-VALUE(state, $-\infty, \infty$)
  return the action in ACTIONS(state) with value v
end function

---

function MAX-VALUE(state, α, β) **returns** a utility value v
  if TERMINAL-TEST(state) **return** UTILITY(state)
  $v \leftarrow -\infty$
  for all ACTIONS(state) **do**
    $v \leftarrow$ **max**(v, MIN-VALUE(RESULT(state,a),α, β))
    if $v \geq \beta$ **return** v
    $\alpha \leftarrow \max(\alpha, v)$
  end for
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Imperfect but real-time decisions - iterative deepening

$$\text{H-MINIMAX}(s, d) =$$

$$\begin{cases} \text{EVAL}(s) & \text{if } \text{CUTOFF-TEST}(s, d) \\ \max_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d+1) & \text{if } \text{PLAYER}(s) = \text{MAX} \\ \min_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a, d+1)) & \text{if } \text{PLAYER}(s) = \text{MIN} \end{cases}$$

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Imperfect but real-time decisions - iterative deepening

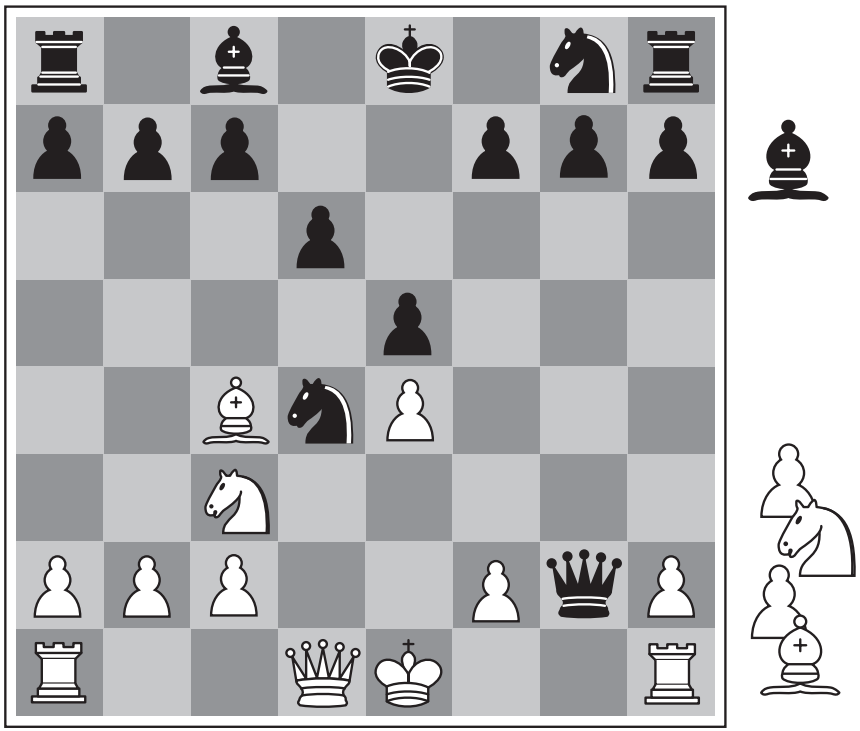
$$
\begin{aligned}
&\textsc{h-minimax}(s, d) = \\
&\qquad \textsc{eval}(s) \quad \text{if} \quad \textsc{cutoff-test}(s, d) \\
&\max_{a \in \textsc{actions}(s)} \textsc{h-minimax}(\textsc{result}(s, a), d + 1) \quad \text{if} \quad \textsc{player}(s) = \textsc{max} \\
&\min_{a \in \textsc{actions}(s)} \textsc{h-minimax}(\textsc{result}(s, a, d + 1)) \quad \text{if} \quad \textsc{player}(s) = \textsc{min}
\end{aligned}
$$

Cutting off search

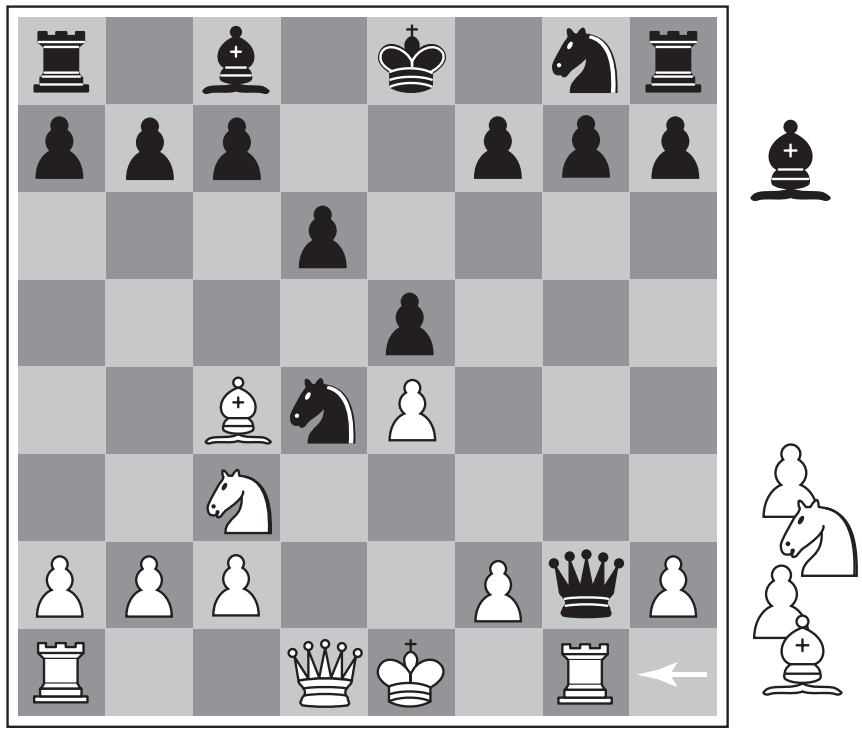
Replace
if TERMINAL-TEST(s) **then return** UTILITY(s)
with:
if CUTOFF-TEST(s,d) **then return** EVAL(s)

EVAL(s) – Evaluation functions

(estimate of) State value for non-terminal states



(a) White to move

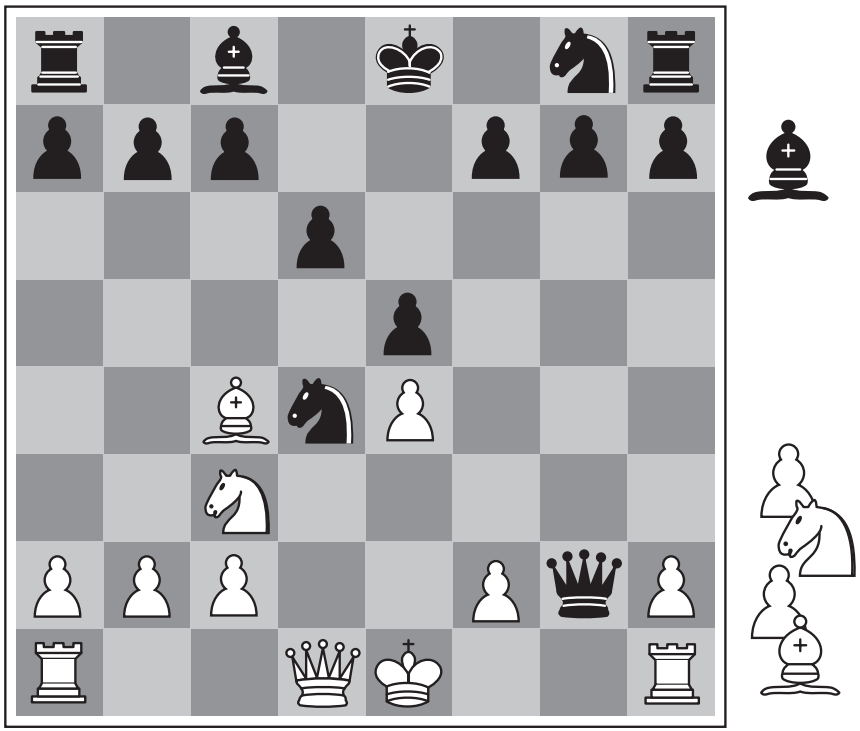


(b) White to move

$$\text{EVAL}(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots w_n f_n(s)$$

EVAL(s) – Evaluation functions

(estimate of) State value for non-terminal states



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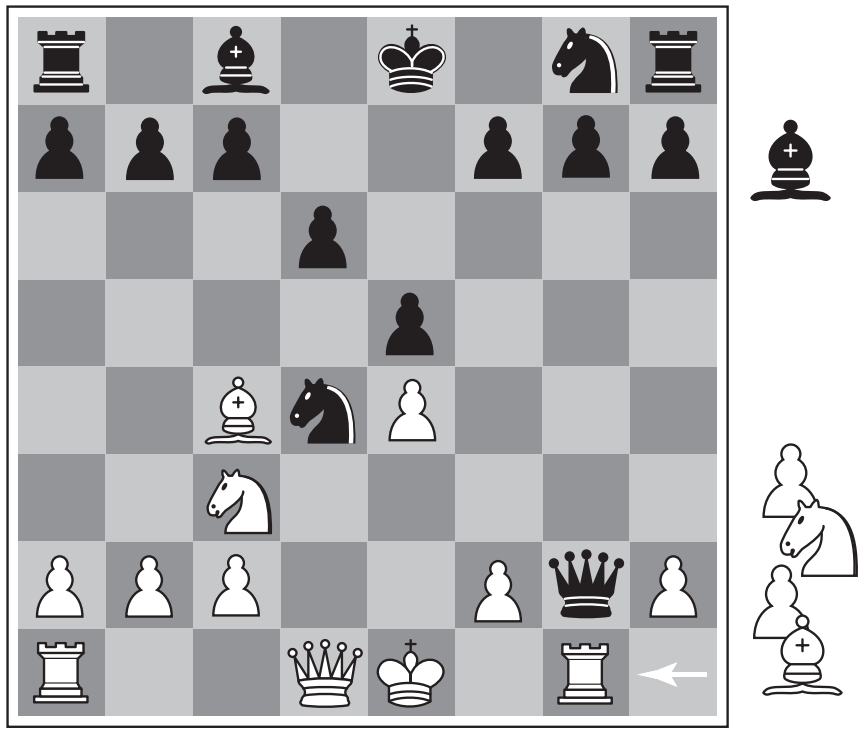

(b) White to move

$$\textsc{Eval}(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots w_n f_n(s)$$

References

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