

Adversarial Search

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Games, man vs. algorithm

- ▶ Deep Blue
- ▶ Alpha Go
- ▶ Deep Stack
- ▶ Why Games, actually?

Games are interesting for AI *because* they are hard (to solve).

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More: Adversarial Learning

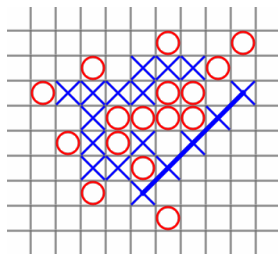


Video: Adversing visual segmentation

Vision for Robotics and Autonomous Systems, <http://cyber.felk.cvut.cz/vras>

Elements of the game

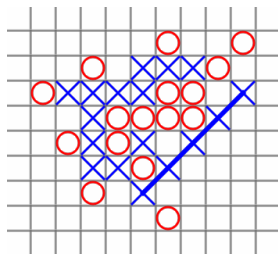
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 - ▶ $\text{PLAYER}(s)$. Which player has to move in s .
 - ▶ $\text{ACTIONS}(s)$. What are the legal moves?
 - ▶ $\text{RESULT}(s, a)$. Transition, result of a move.
 - ▶ $\text{TERMINAL-TEST}(s)$. Game over?
 - ▶ $\text{TERMINAL-UTILITY}(s, p)$. What is prize?
- Examples for some games ...



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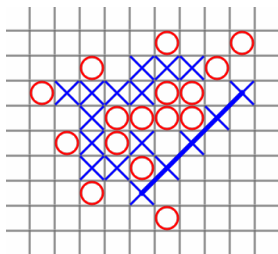
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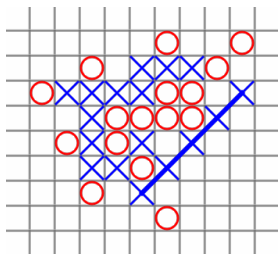
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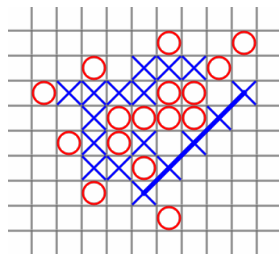
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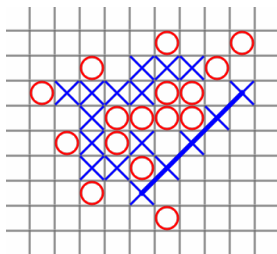
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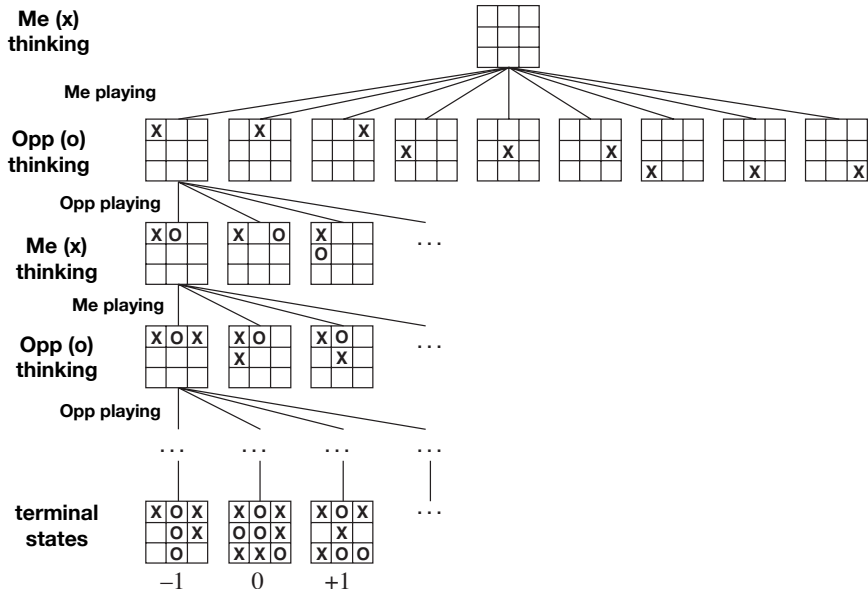
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Terminal utility: Zero-Sum and General games

Game Tree(s)



State Value $V(s)$

$V(s)$ – value V of a state s : The best utility achievable from this state.

$$V(s) = \max_{s' \in \text{children}(s)} V(s')$$

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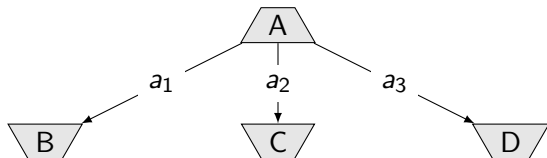
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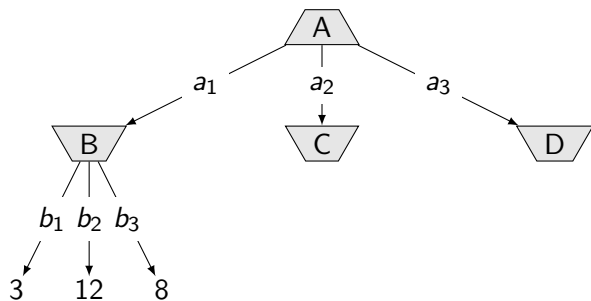
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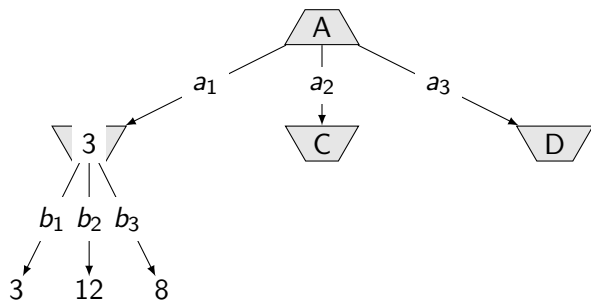
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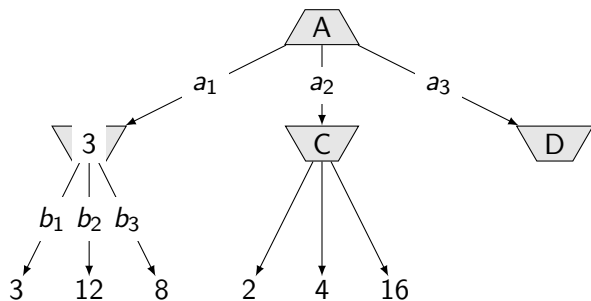
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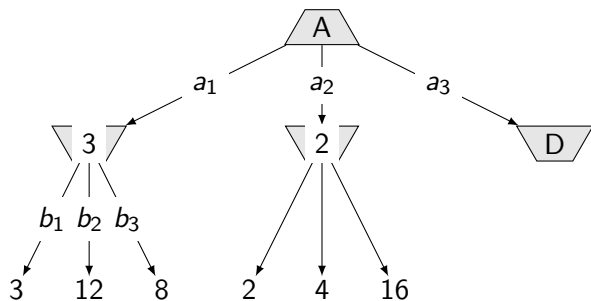
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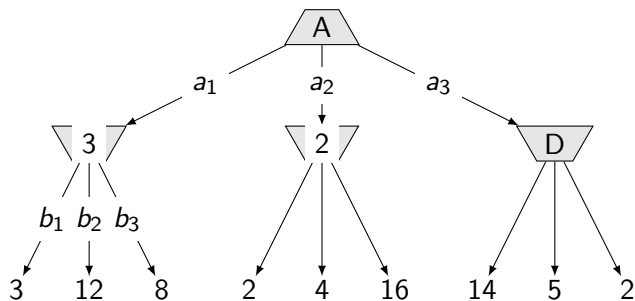
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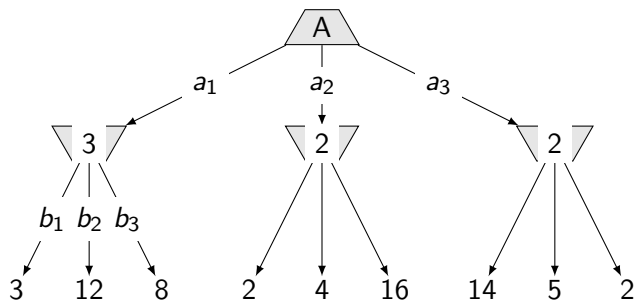
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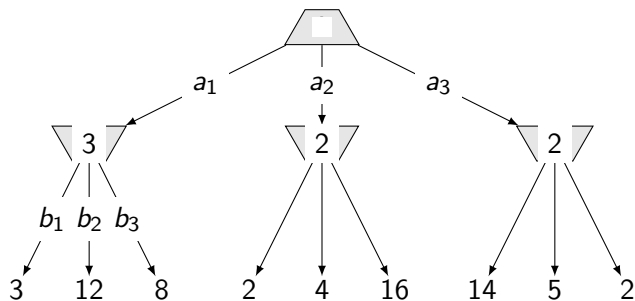
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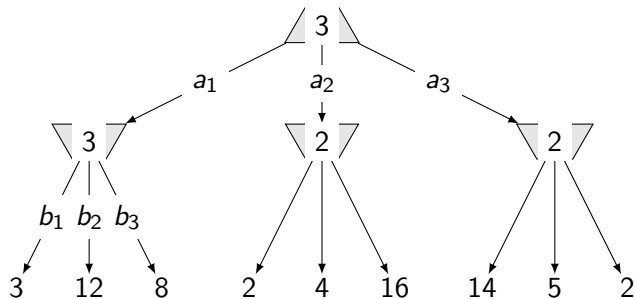
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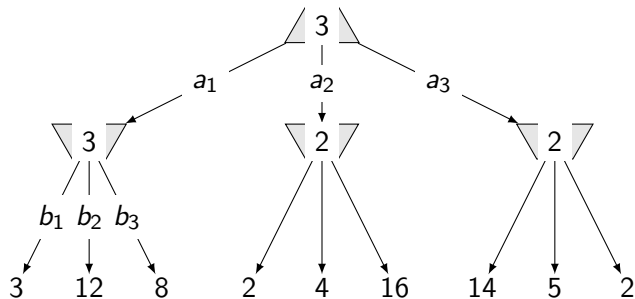
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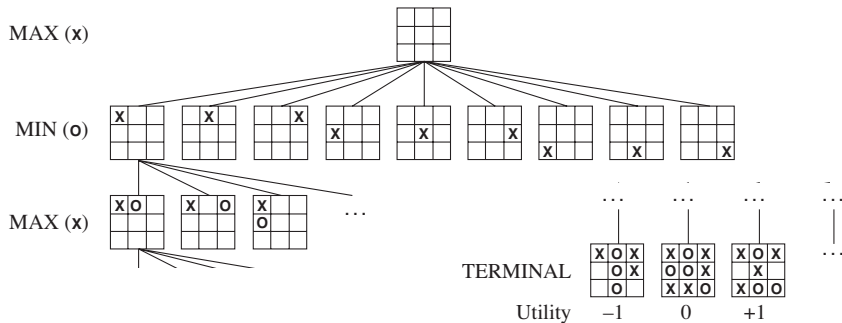
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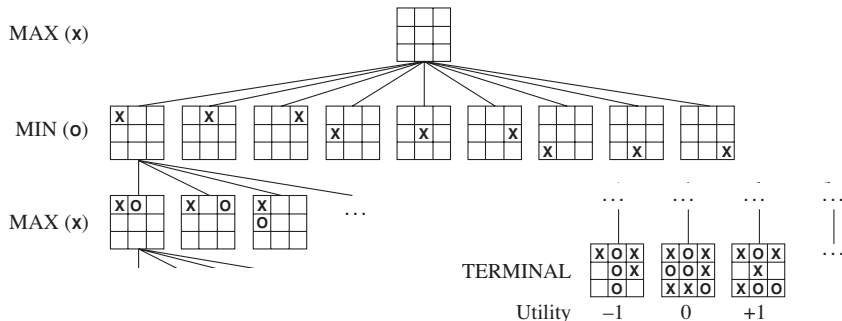
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Zero-Sum game: **max** for me, **min** for the opponent.



$$\begin{aligned}
 \text{MINIMAX}(s) = & \\
 & \text{UTILITY}(s) \quad \text{if } \text{TERMINAL-TEST}(s) \\
 & \max_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s, a)) \quad \text{if } \text{PLAYER}(s) = \text{MAX} \\
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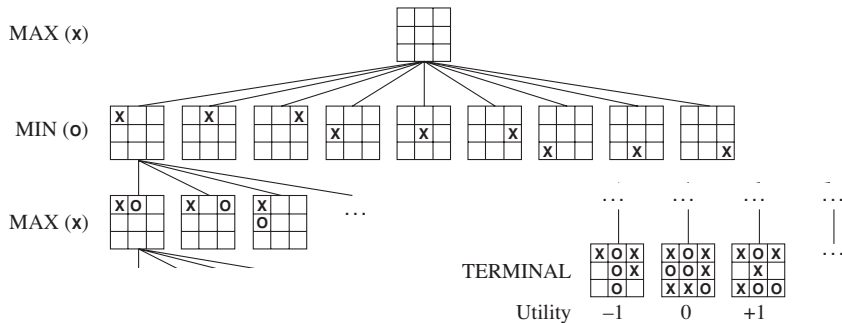
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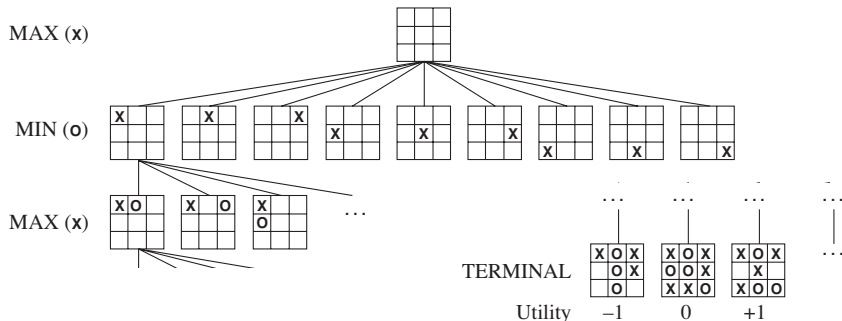
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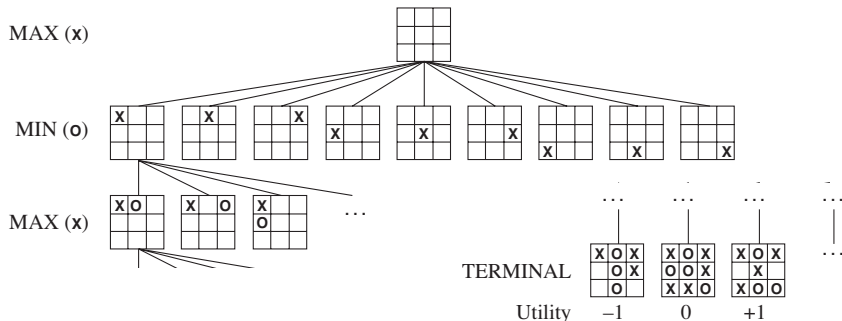
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Minimax algorithm

```
function MINIMAX(state) returns an action
    return argmaxa ∈ Actions(s) MIN-VALUE(RESET(state, a))
end function
```

```
function MIN-VALUE(state) returns a utility value  $v$ 
    if TERMINAL-TEST(state) then return UTILITY(state)
    end if
     $v \leftarrow \infty$ 
    for all ACTIONS(state) do
         $v \leftarrow \min(v, \text{MAX-VALUE}(\text{RESULT}(\text{state}, a)))$ 
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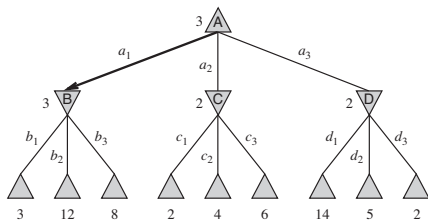
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A two ply game, down to terminal and back again ...

function MINIMAX(s) **returns** a MAX
 $\operatorname{argmax}_{a \in \text{Actions}(s)} \text{MINVAL}(\text{RES}(s, a))$
end function

function MINVAL(s) **returns** v MIN
 if TERMINAL(s) **then** UTIL(s)
 end if
 $v \leftarrow \infty$
 for all ACTIONS(s) **do**
 $v \leftarrow \min(v, \text{MAXVAL}(\text{RES}(s, a)))$
 end for
end function

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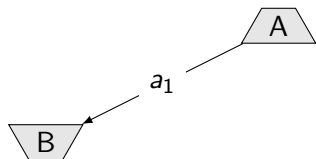
A two ply game, recursive run



What is the complexity? How many nodes to visit?

Can we do better? How?

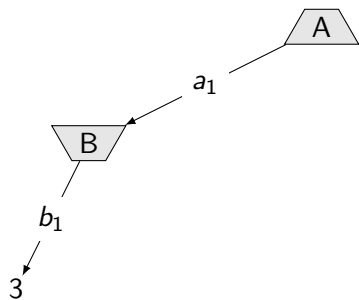
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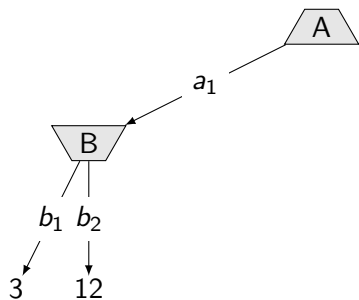
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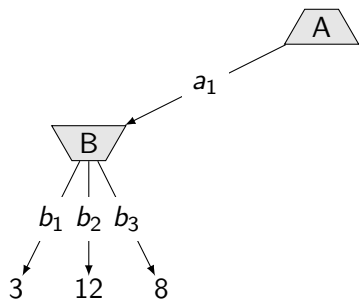
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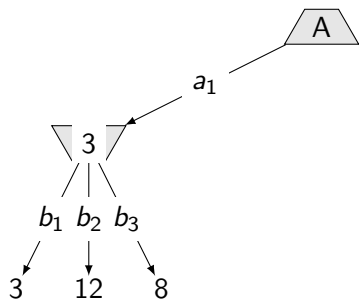
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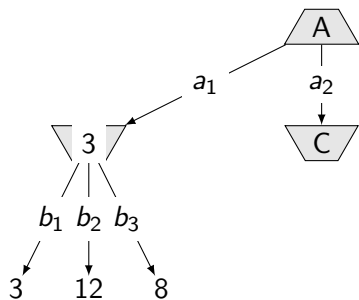
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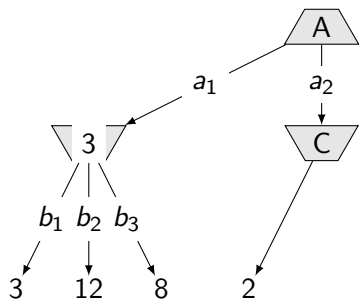
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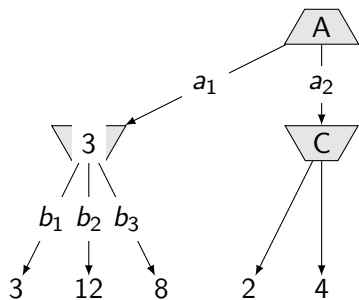
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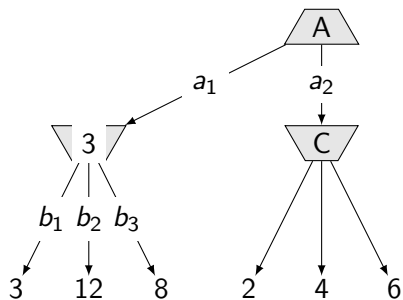
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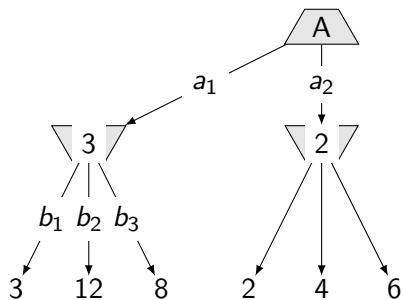
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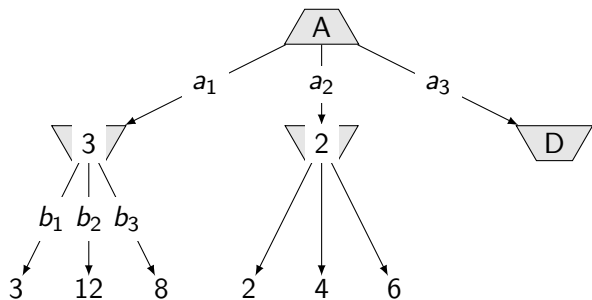
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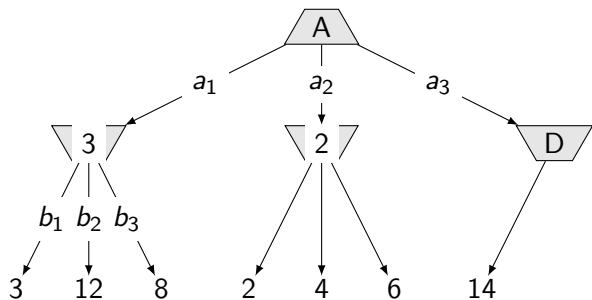
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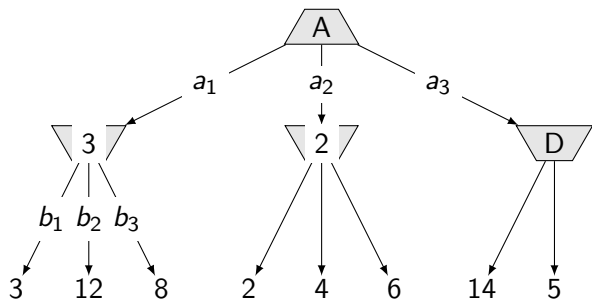
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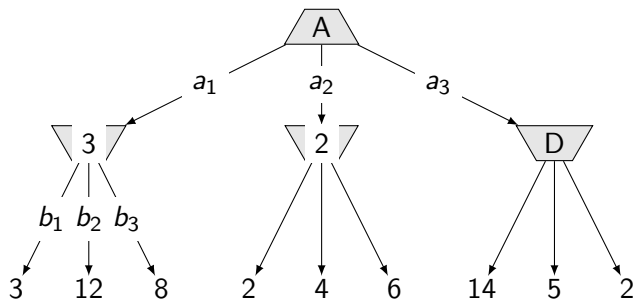
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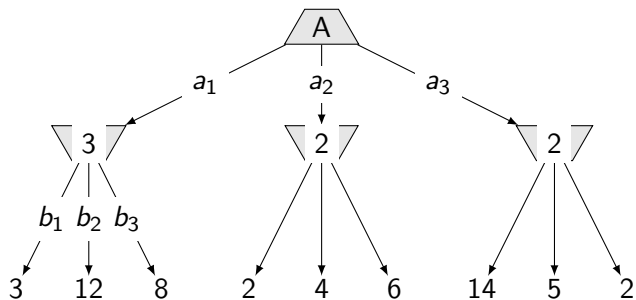
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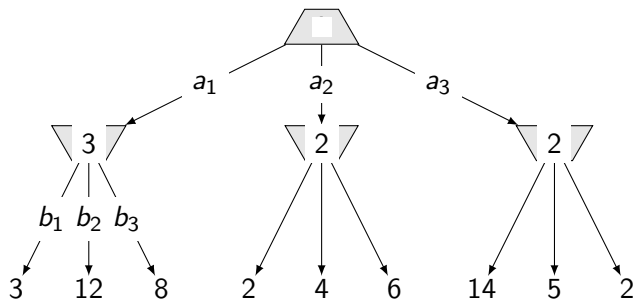
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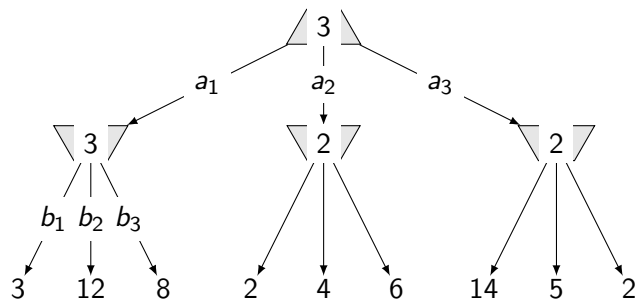
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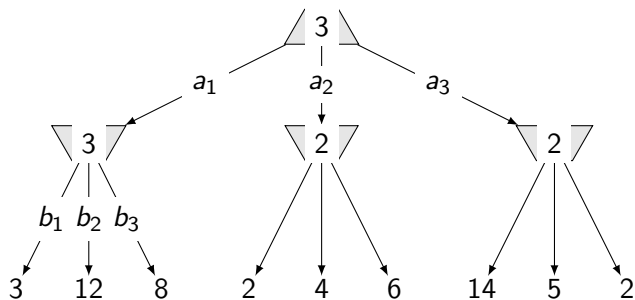
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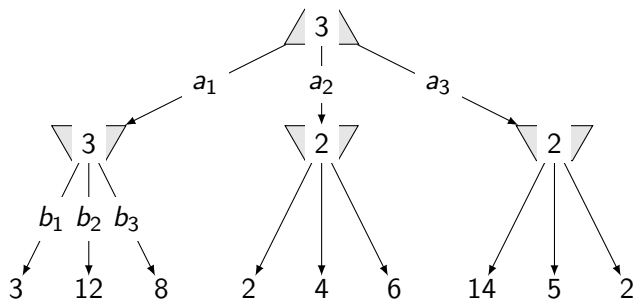
A two ply game, recursive run



What is the complexity? How many nodes to visit?

Can we do better? How?

A two ply game, recursive run



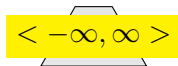
What is the complexity? How many nodes to visit?

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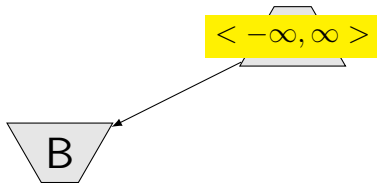
Nodes (sub-trees) worth visiting



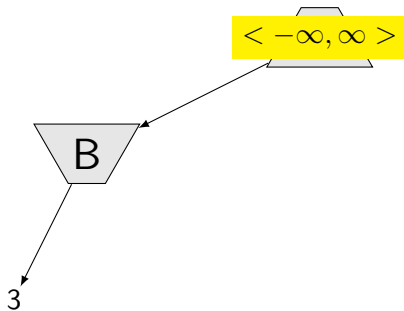
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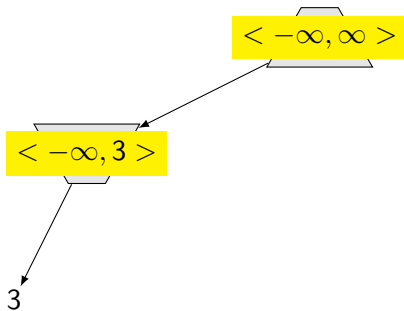
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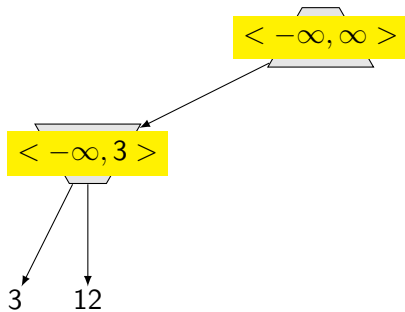
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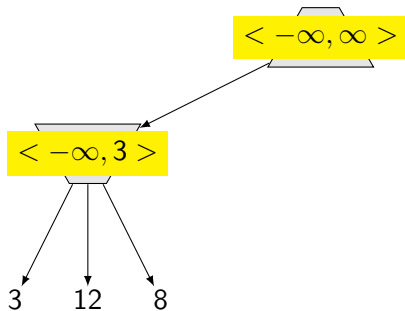
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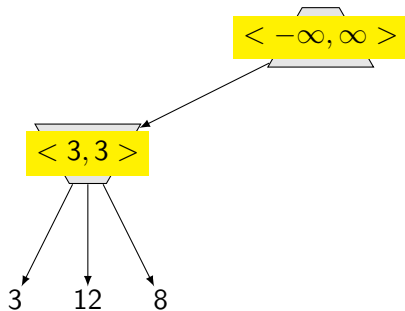
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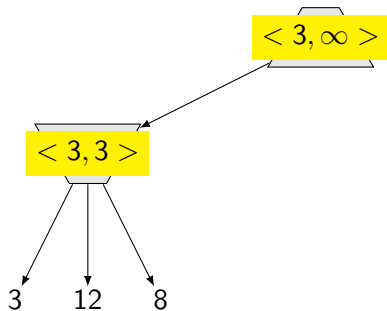
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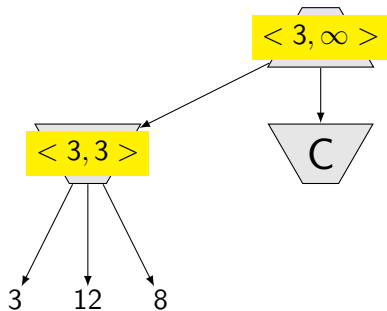
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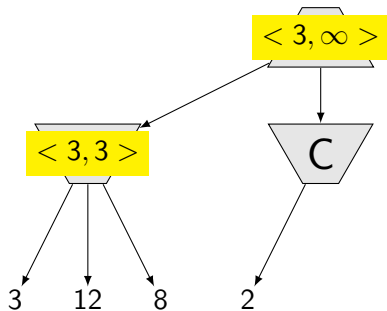
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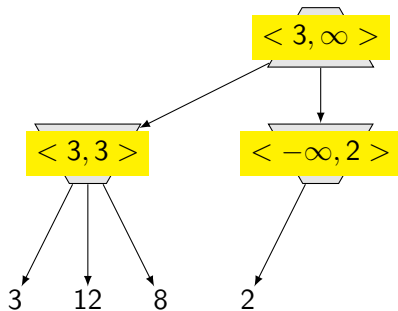
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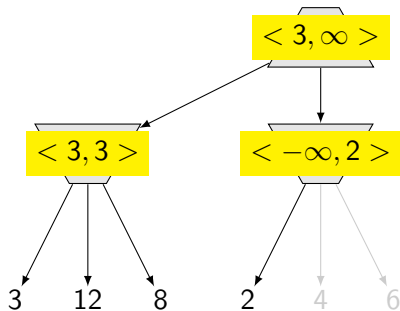
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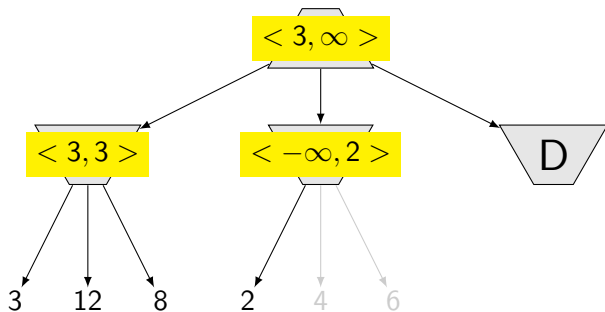
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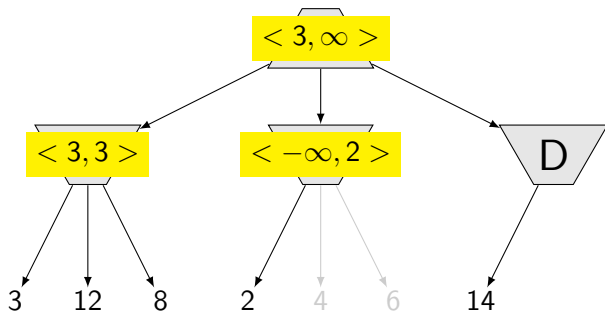
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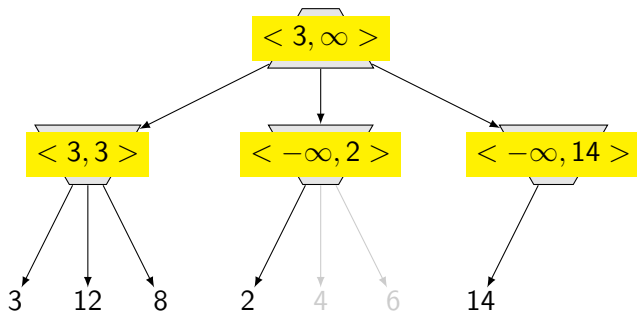
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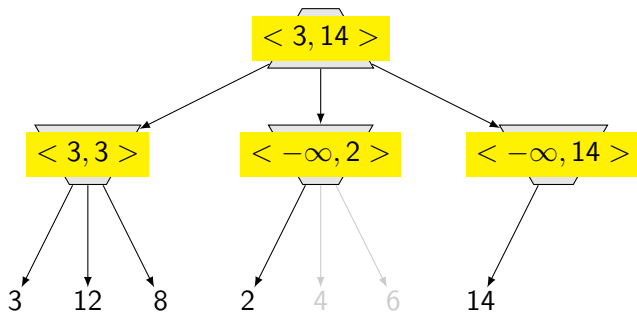
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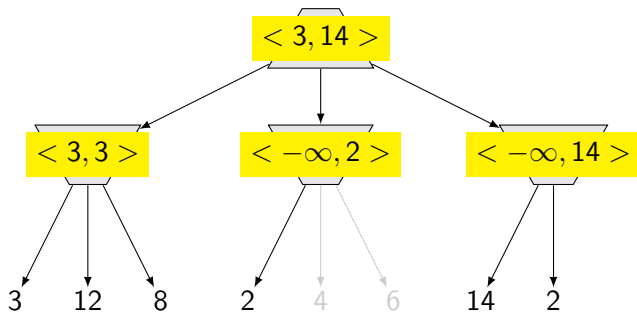
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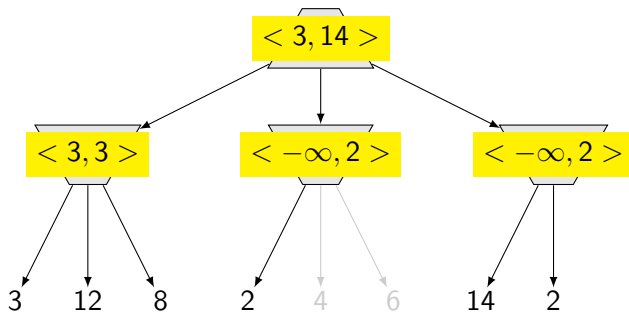
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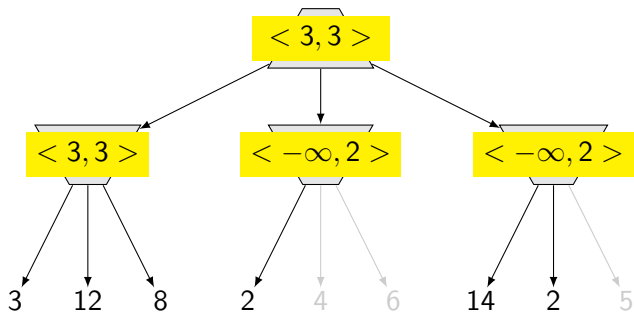
Nodes (sub-trees) worth visiting



Nodes (sub-trees) worth visiting



Nodes (sub-trees) worth visiting



α - β pruning

α highest (best) value choice found so far for any choice along MAX

β lowest (best) value choice found so far for any choice along MIN



In MIN-VAL: $v \leftarrow 2$
 $v \leq \alpha$ then: return v !

α - β pruning

α highest (best) value choice found so far for any choice along MAX

β lowest (best) value choice found so far for any choice along MIN

$$\alpha = -\infty, \beta = \infty$$

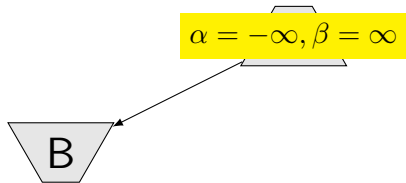
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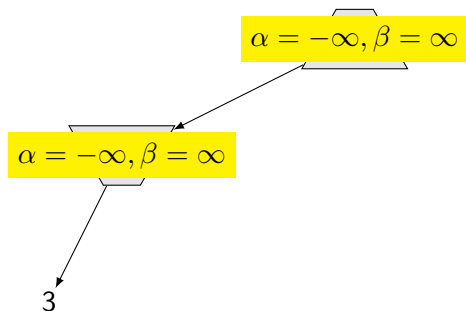
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$$\alpha = -\infty, \beta = \infty$$

$$\alpha = -\infty, \beta = 3$$

3

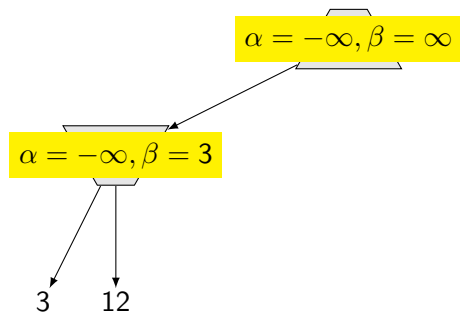
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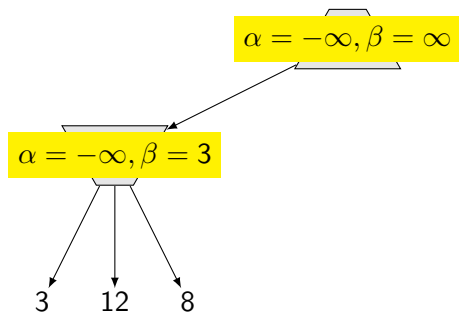
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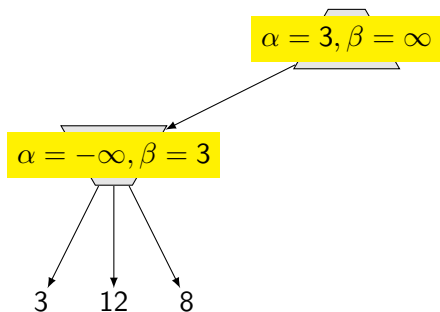
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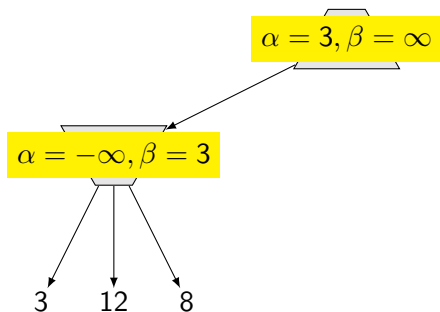
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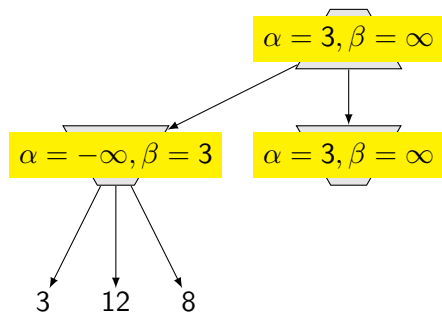
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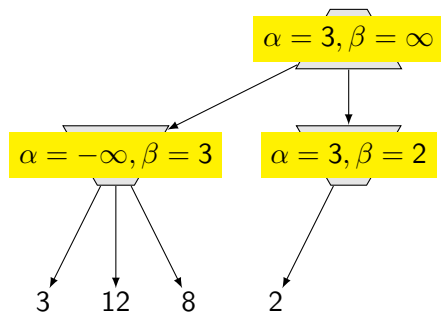
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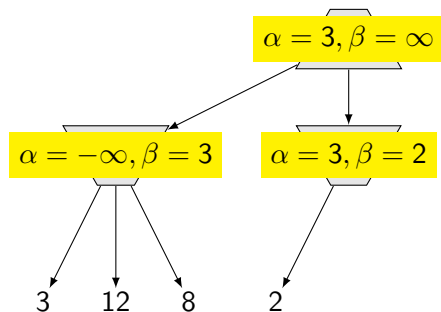
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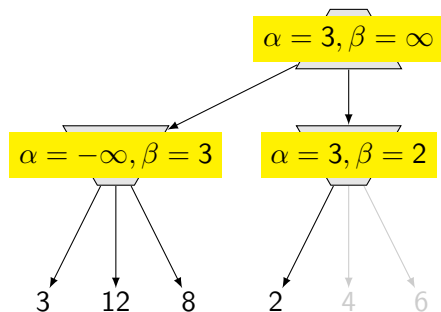
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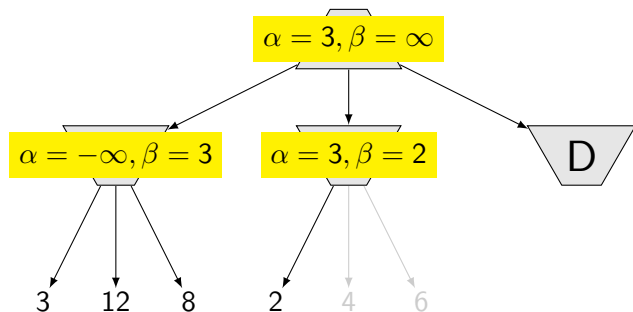
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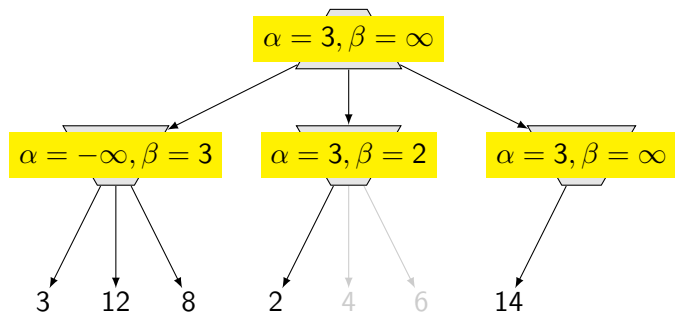
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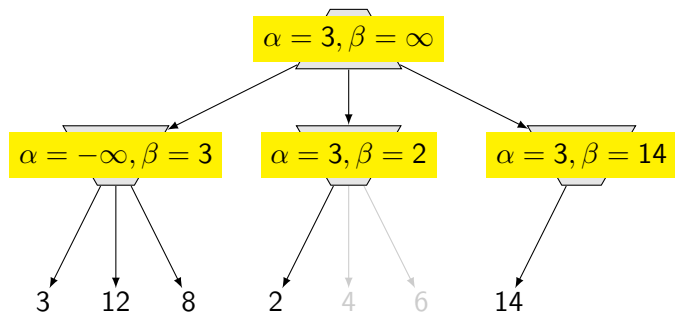
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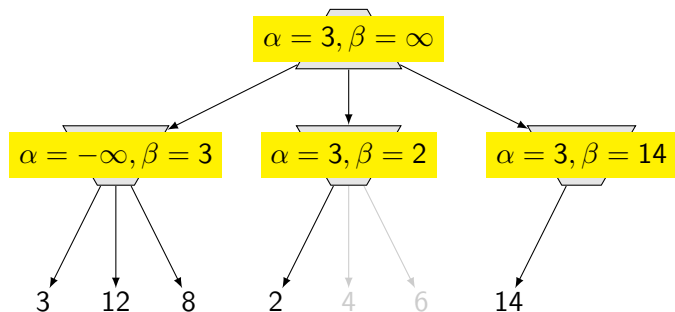
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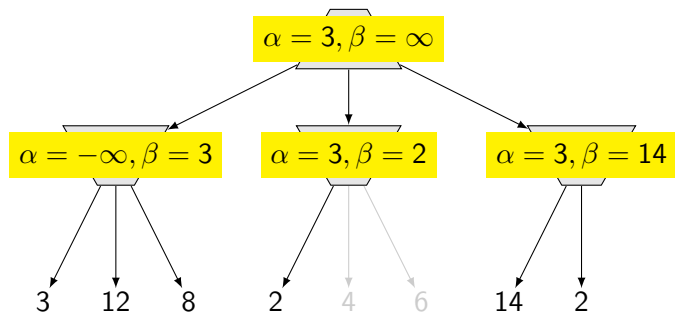
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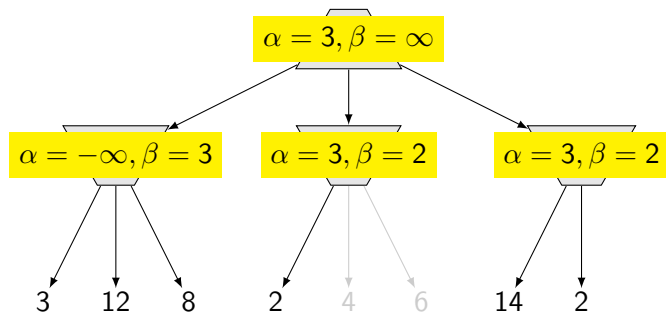
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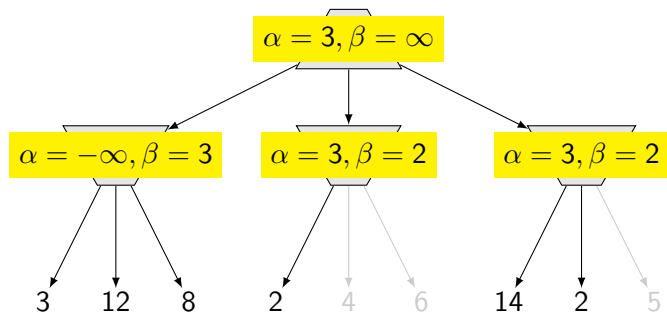
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function ALPHA-BETA-SEARCH(state) **returns** an action

$v \leftarrow \text{MAX-VALUE}(\text{state}, -\infty, \infty)$

return the action in $\text{ACTIONS}(\text{state})$ with value v

end function

function MAX-VALUE(state, α , β) **returns** a utility value v

if $\text{TERMINAL-TEST}(\text{state})$ **return** $\text{UTILITY}(\text{state})$

$v \leftarrow -\infty$

for all $\text{ACTIONS}(\text{state})$ **do**

$v \leftarrow \max(v, \text{MIN-VALUE}(\text{RESULT}(\text{state}, a), \alpha, \beta))$

if $v \geq \beta$ **return** v

$\alpha \leftarrow \max(\alpha, v)$

end for

end function

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 if $v \leq \alpha$ **return** v
 $\beta \leftarrow \text{min}(\beta, v)$
 end for
end function

Imperfect but real-time decisions

$$\begin{aligned} \text{H-MINIMAX}(s, d) = & \text{EVAL}(s) \quad \text{if } \text{CUTOFF-TEST}(s, d) \\ & \max_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d + 1) \quad \text{if } \text{PLAYER}(s) = \text{MAX} \\ & \min_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d + 1) \quad \text{if } \text{PLAYER}(s) = \text{MIN} \end{aligned}$$

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Cutting off search

Replace

if TERMINAL-TEST(*s*) **then return** UTILITY(*s*)

with:

if CUTOFF-TEST(*s*,*d*) **then return** EVAL(*s*)

EVAL(s) – Evaluation functions

References

- [1] Stuart Russell and Peter Norvig.
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