Adversarial Search

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Games, man vs. algorithm

- Deep Blue
- Alpha Go
- Deep Stack
- Why Games, actually?

Games are interesting for AI *because* they are hard (to solve).

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More: Adversarial Learning

Video: Adversing visual segmentation

Elements of the game

- s_0: The initial state
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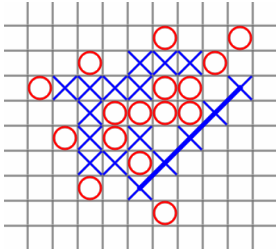
- TERMINAL-TEST(s). Game over?
- TERMINAL-UTILITY(s, p). What is prize?

Examples for some games ...



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Considering the notation, we are making slight transition from [1] to [2].

- Players: $P = \{1, 2, \ldots, N\}$ (often just $N = 2$)
- Transition functions: $S \times A \to S$.
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What are we loking for? A strategy/policy $S \to A$

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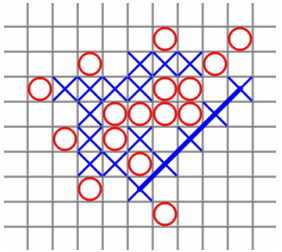

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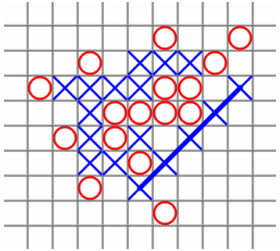

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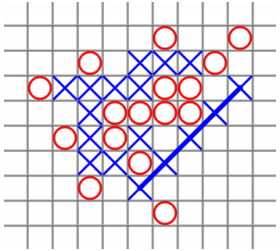

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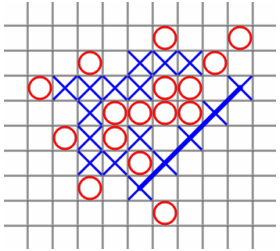

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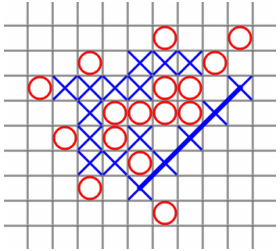

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Terminal utilitity: Zero-Sum and General games

- Zero-sum: players have opposite utilities (values)
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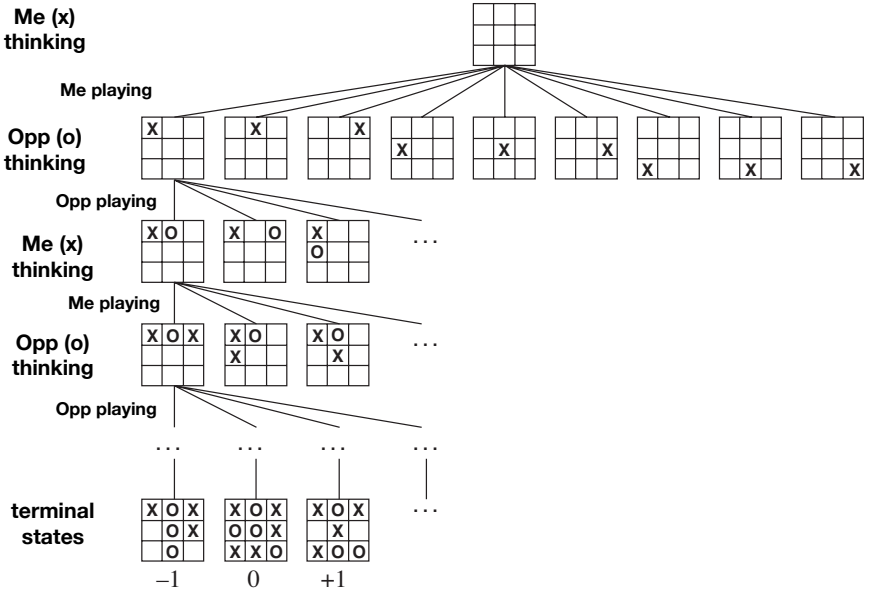
Game Tree(s)



TERMINAL-UTILITY$(s, \mathbf{x})$

Init state, ACTIONS function, and RESULT function defines game tree

State Value $V(s)$

$V(s)$ – value V of a state s : The best utility achievable from this state.

$$V(s) = \max_{s' \in \text{children}(s)} V(s')$$

Think about the State Value. It is a theoretical construct, definition. Depending on the problem, there may be various computational algorithms.
In a game, what State Values are known? Usually, only terminal states.
Think, for a moment, you are the only player. You can control every step.
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Two-ply game: **max** for me, **min** for the opponent.

$$a_1 = \underset{a \in \text{ACTIONS}(A)}{\arg\max} \ \text{RESULT}(A, a)$$

I'm player that starts (state A) and want to decide what to play, actions/plays/moves a_1, a_2, a_3 are the options. B, C, D are the possible outcomes of my moves. Now the opponent is about to play. The numbers in terminal states denote *my* profit/utility.

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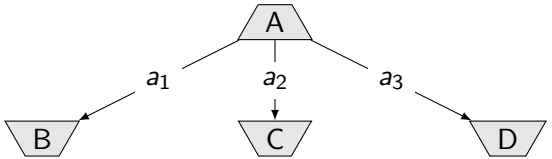

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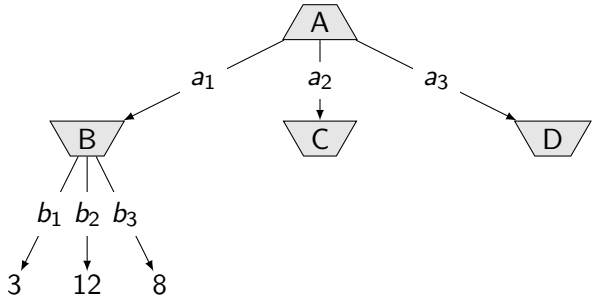

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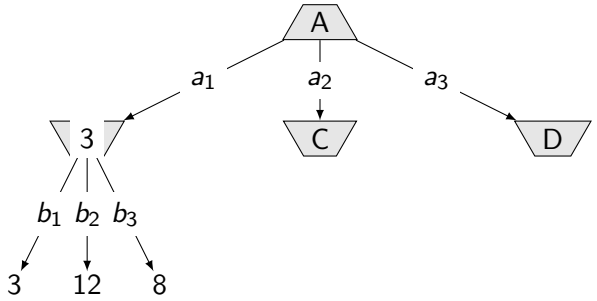

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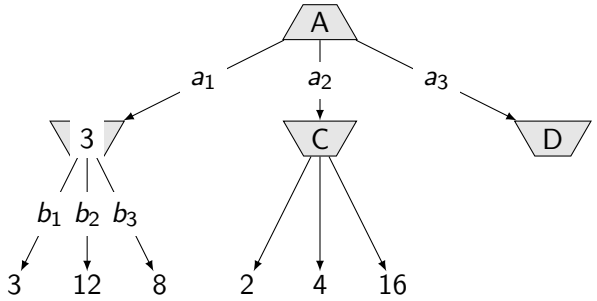

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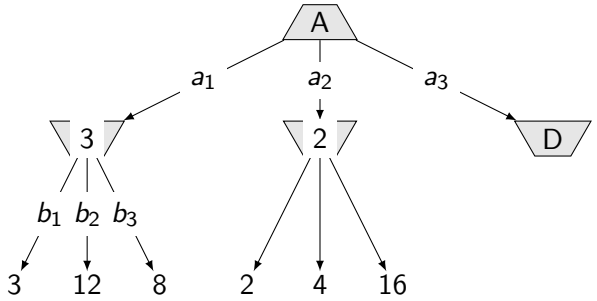

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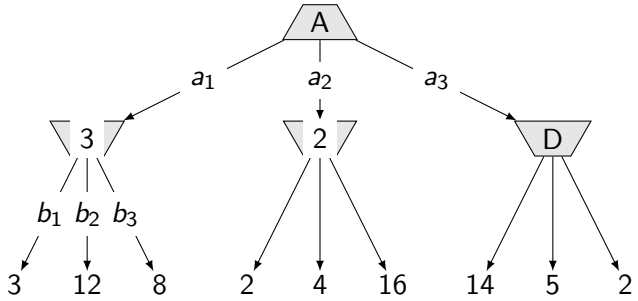

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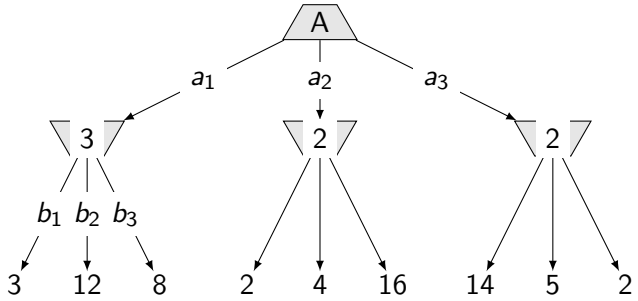

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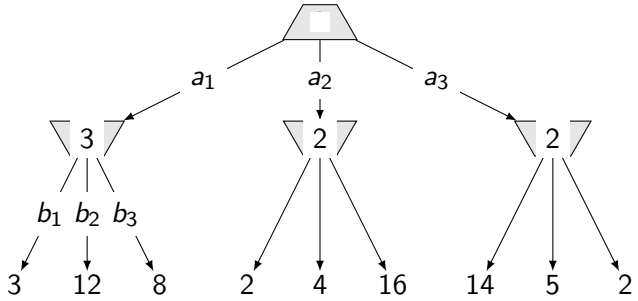

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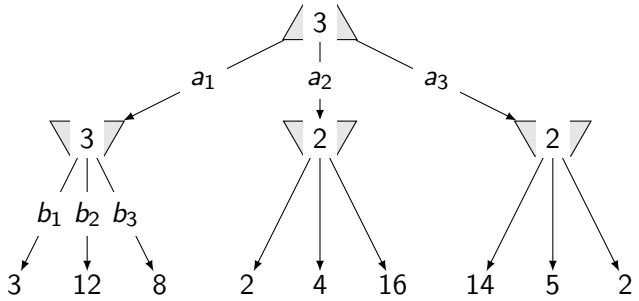

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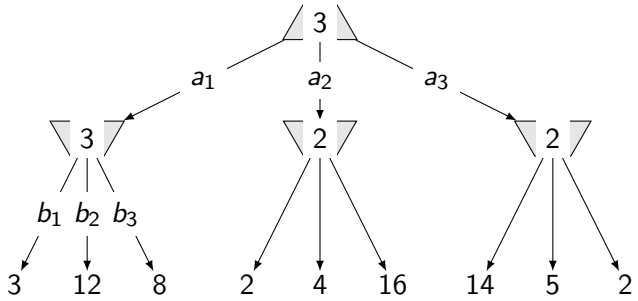

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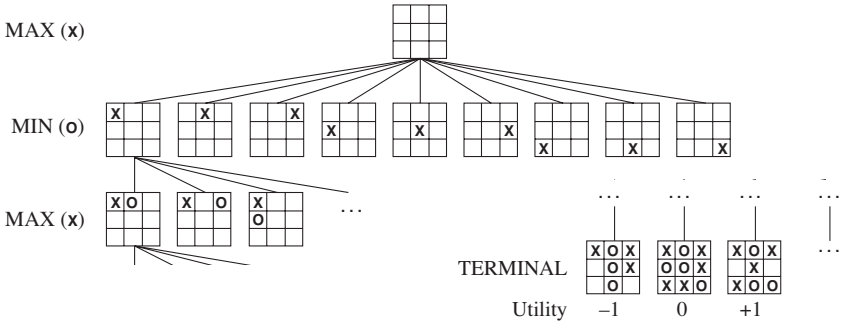

Max step: I want to maximize my outcome.

Min step: I want to minimize the outcome of the opponent.

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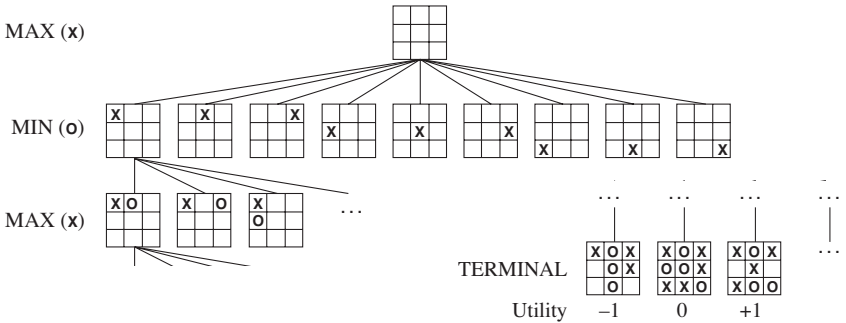

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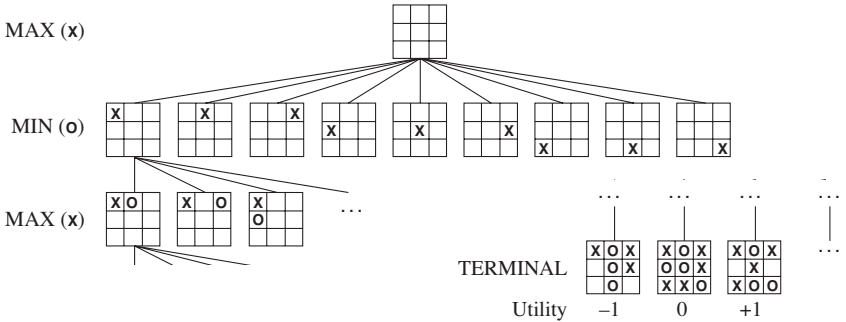

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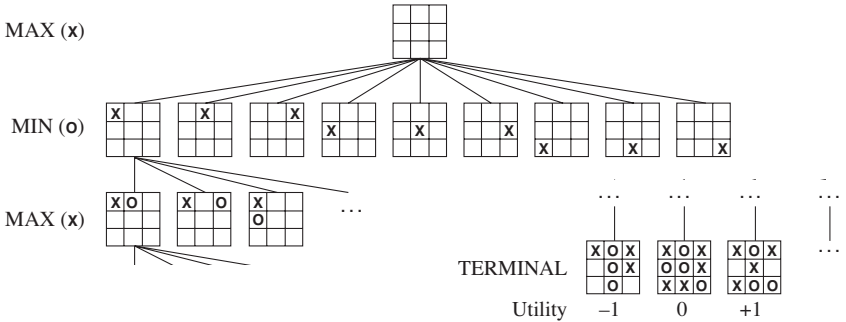

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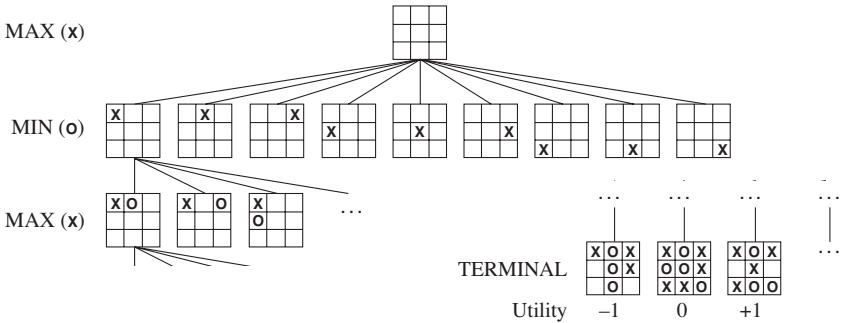

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Minimax algorithm

function MINIMAX(state) **returns** an action
  return $\underset{a \in \text{Actions}(s)}{\textbf{argmax}}$ MIN-VALUE(RESULT(state, a))
end function

function MIN-VALUE(state) **returns** a utility value v
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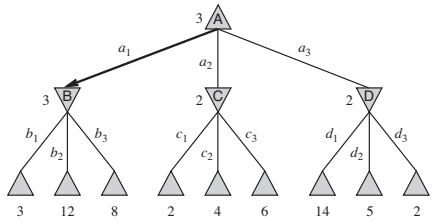
  if TERMINAL-TEST(state) **then return** UTILITY(state)
  end if
  $v \leftarrow -\infty$
  for all ACTIONS(state) **do**
    $v \leftarrow$ **max**(v, MIN-VALUE(RESULT(state,a)))
  end for
end function

A two ply game, down to terminal and back again . . .

```
function MINIMAX(s) returns a                                   MAX
    argmax_{a∈Actions(s)} MINVAL(RES(s, a))
end function
-------------------------------------------------------------   MIN
function MINVAL(s) returns v
    if TERMINAL(s) then UTIL(s)
    end if
    v ← ∞
    for all ACTIONS(s) do
        v ← min(v, MAXVAL(RES(s, a)))
    end for
end function
-------------------------------------------------------------
function MAXVAL(s) returns v
    if TERMINAL(s) then UTIL(s)
    end if
    v ← −∞
    for all ACTIONS(s) do
        v ← max(v, MINVAL(RES(s, a)))
    end for
end function
```



Before going to the animation on the next slide, try to follow the algorithm by a pencil and paper.

A two ply game, recursive run

What is the complexity? How many nodes to visit?

Can we do better? How?

Efficiency/complexity:

- Exhaustive DFS
- Time $O(b^m)$
- Space $O(bm)$

Chess $b \approx 35, m \approx 100$ ...

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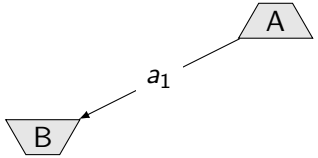
A two ply game, recursive run



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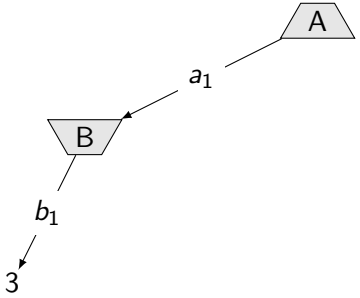

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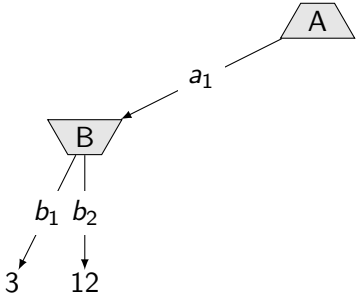

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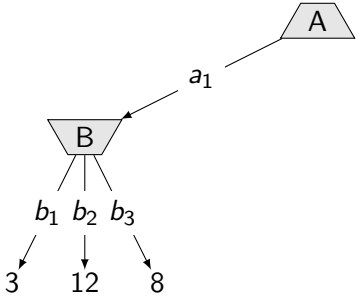

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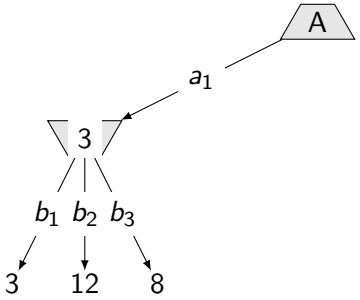

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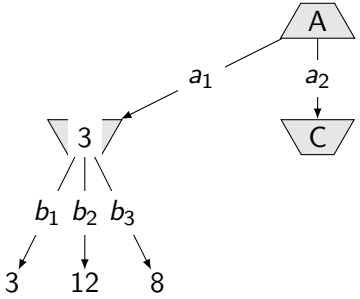

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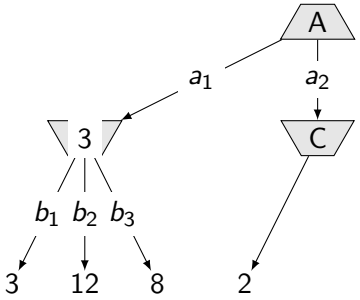

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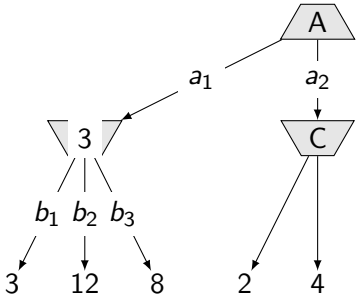

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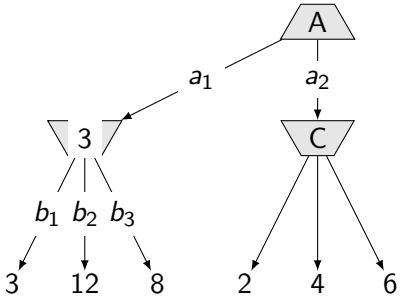

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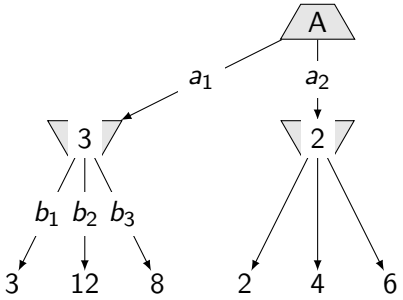

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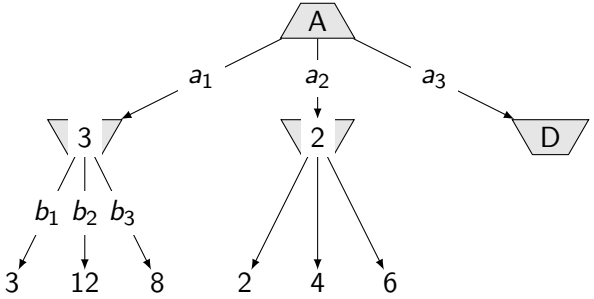

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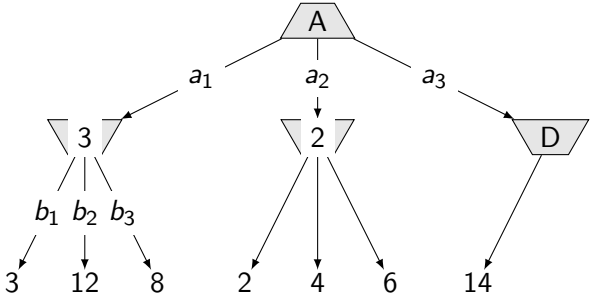

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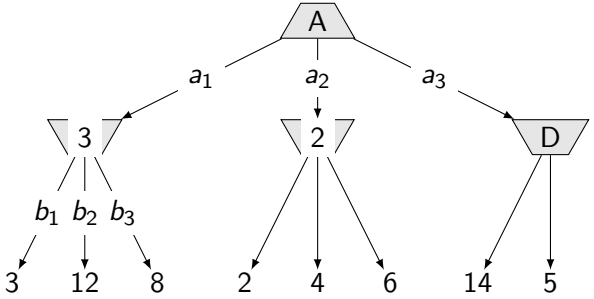

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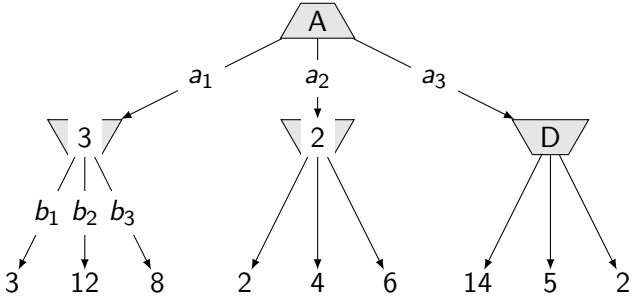

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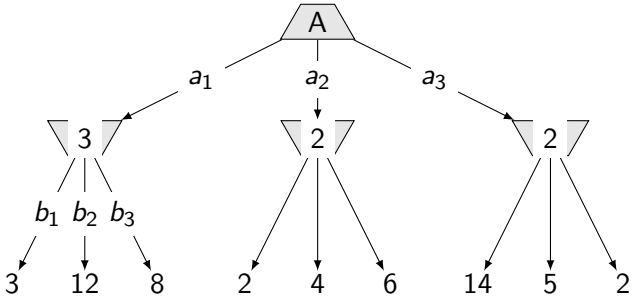

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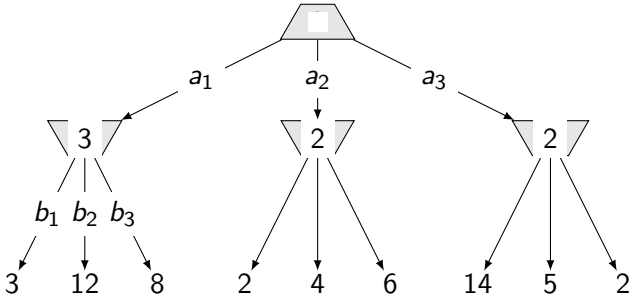

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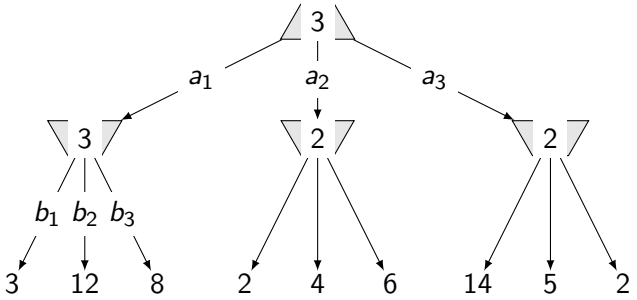

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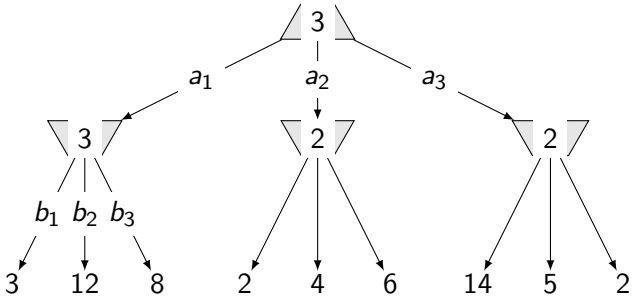

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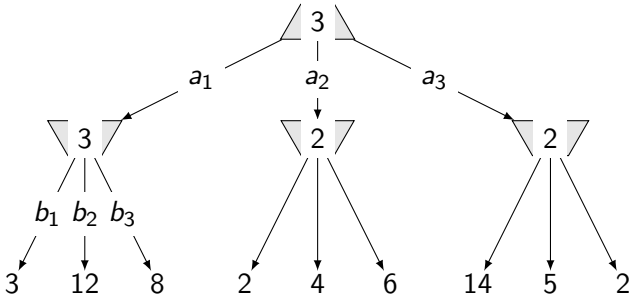

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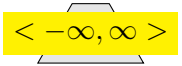

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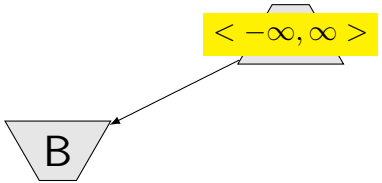

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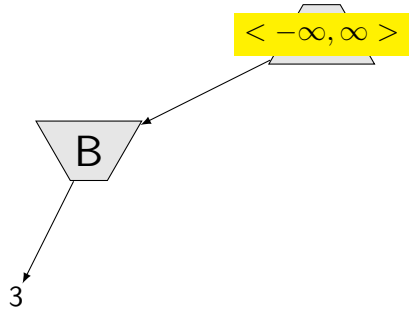

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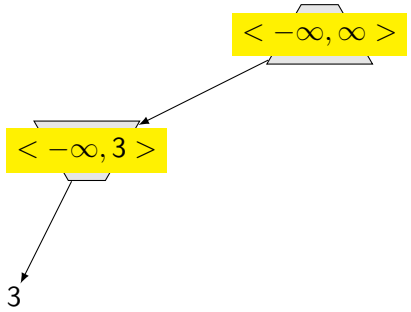

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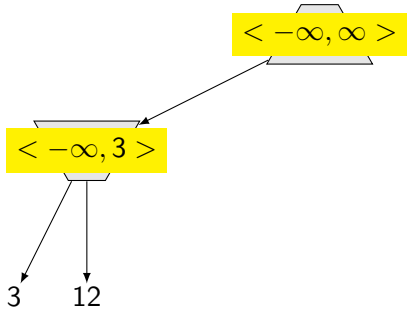

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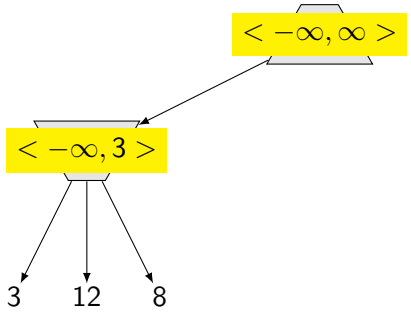

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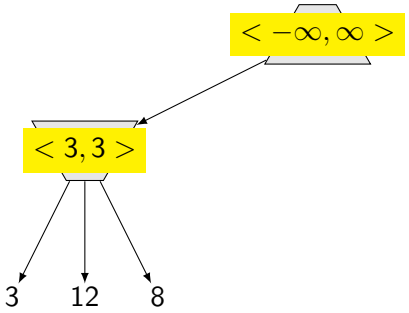

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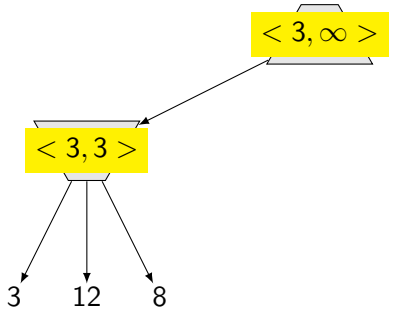

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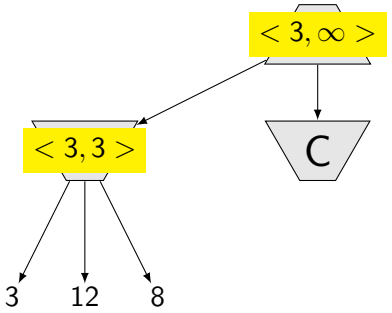

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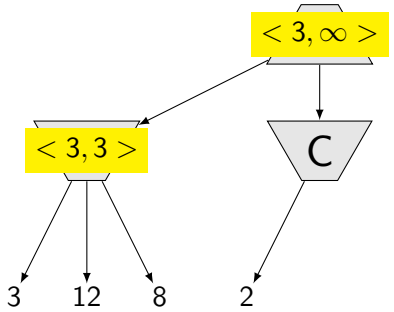

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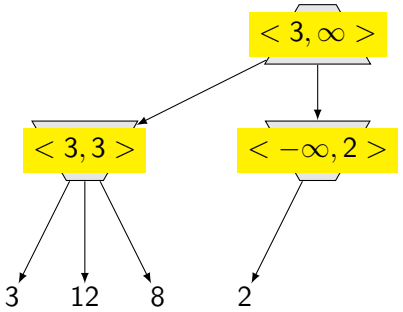

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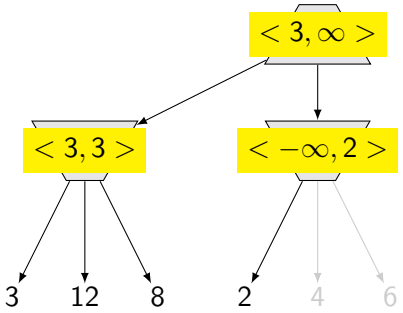

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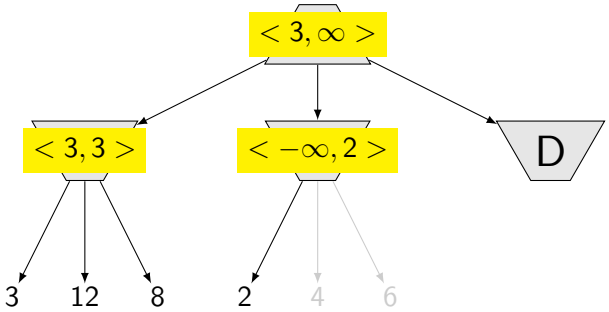

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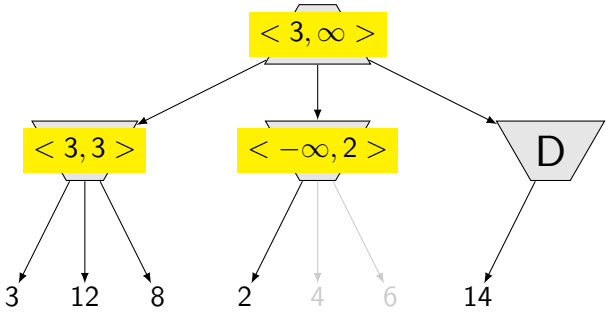

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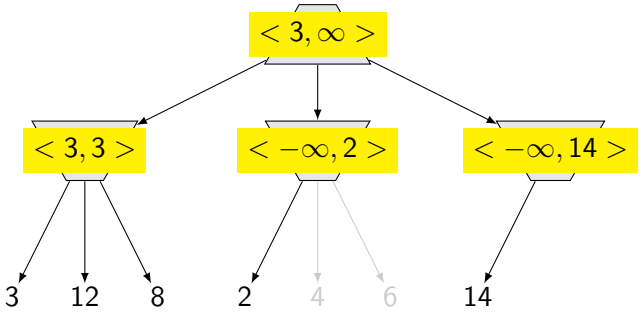

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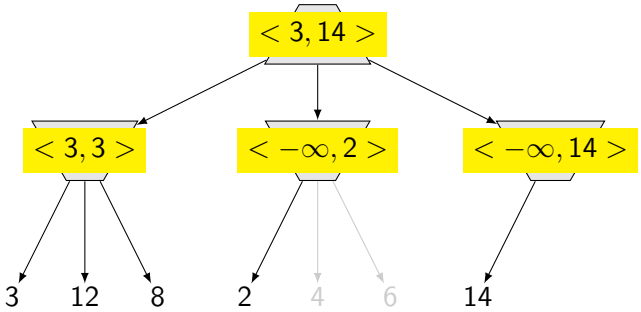

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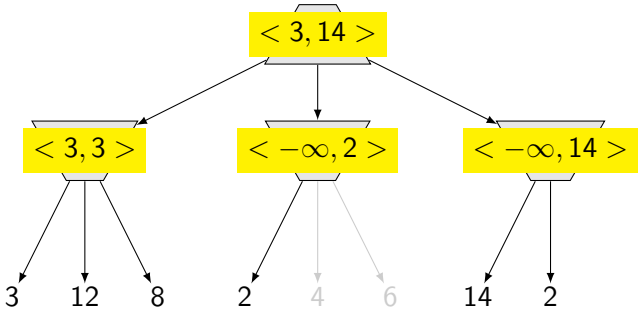

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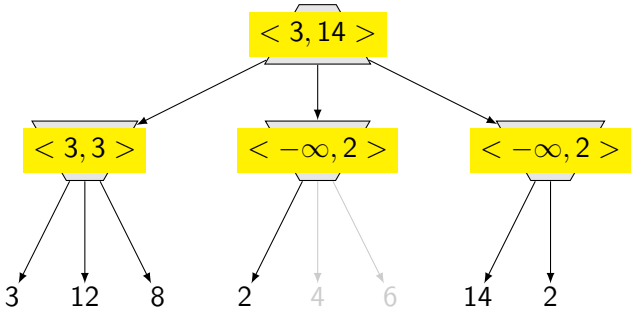

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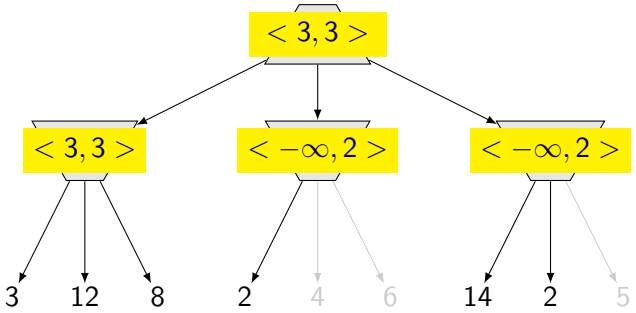

α-β prunning

α highest (best) value choice found so far for any choice along MAX
β lowest (best) value choice found so far for any choice along MIN
v value of the state

In MIN-VAL: $v \leftarrow 2$
$v \leq \alpha$ then: return v!

Functions scope: MAX-VALUE MIN-VALUE. The terminal nodes are served/answered within the MAX-VALUE function.
In MAX nodes α is changing and β is stopping, in MIN nodes β is changing and α is stopping.
It is clear that ordering of child nodes matters. Draw tree of α-β search in case of perferct ordering. Effective branching factor becomes $\sqrt{b}$ instead of b which effectively doubles the depth can be searched.

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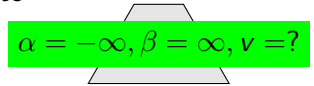

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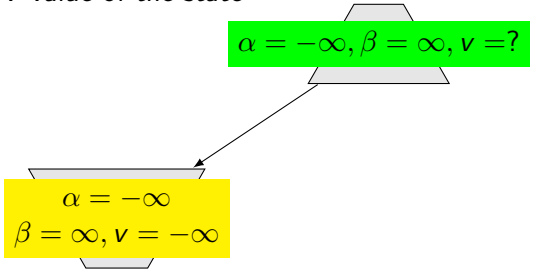
β lowest (best) value choice found so far for any choice along MIN
v value of the state



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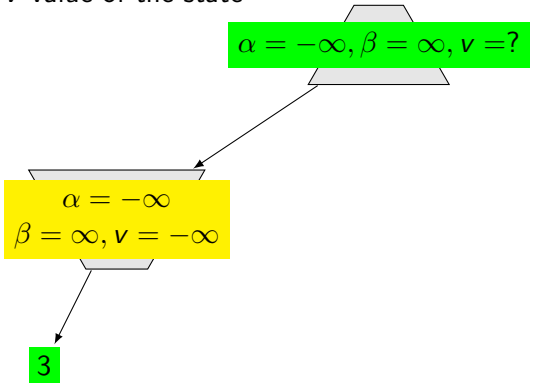

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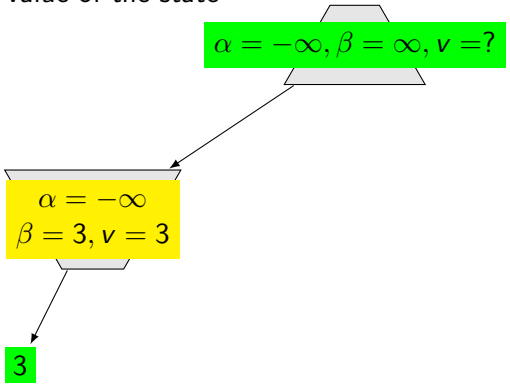

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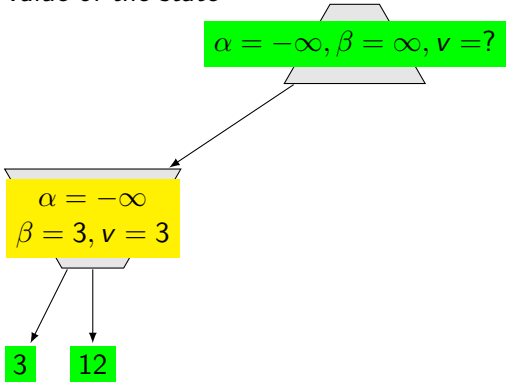

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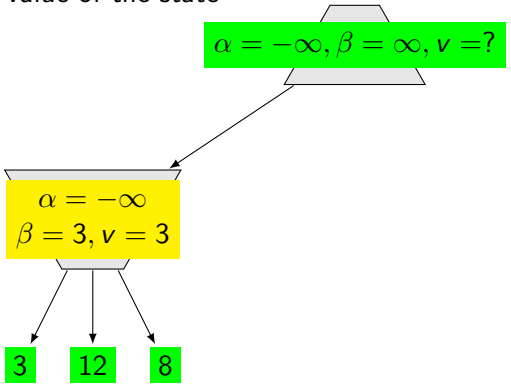

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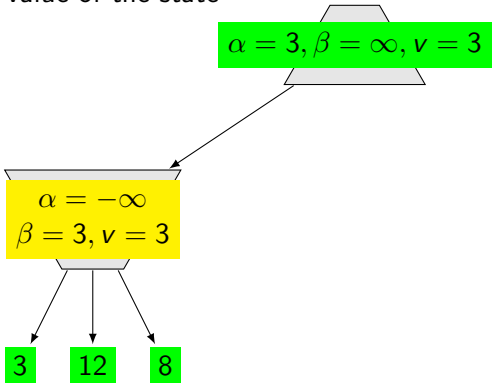

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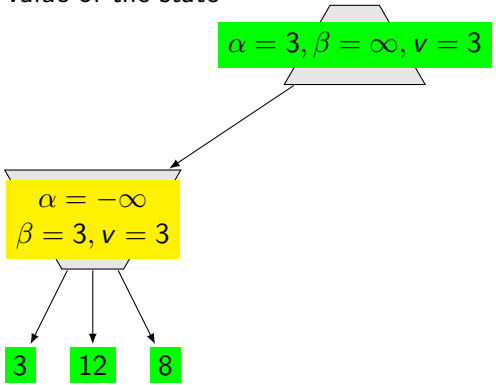

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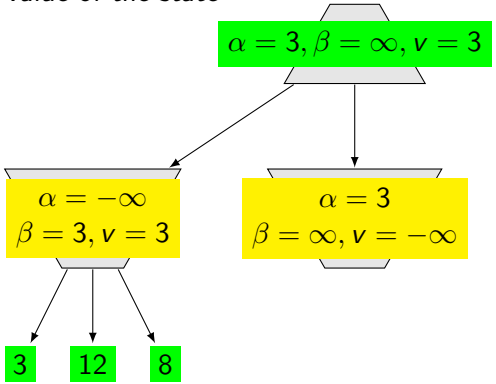

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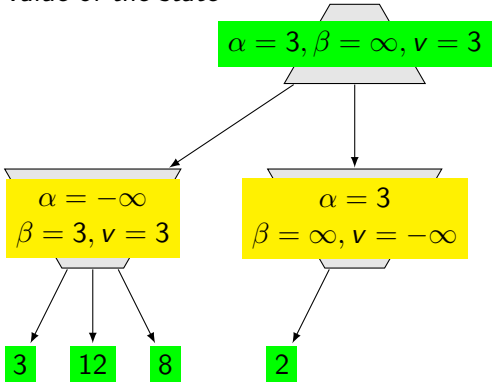

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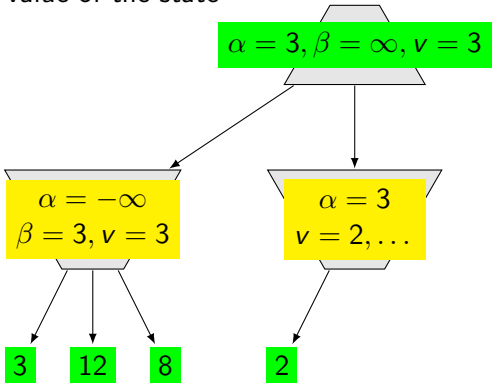

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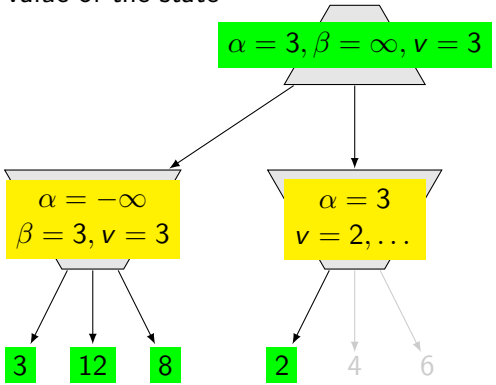

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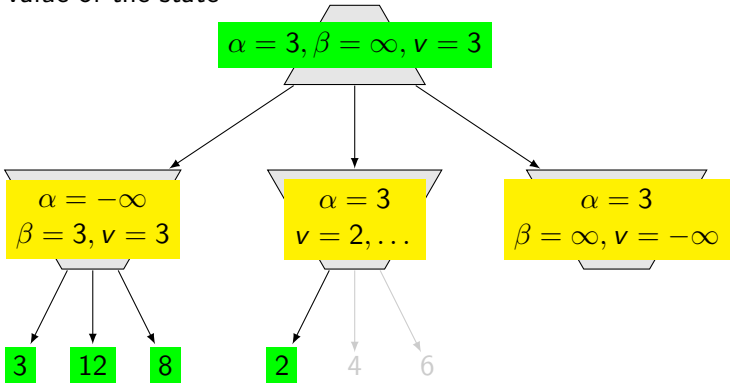

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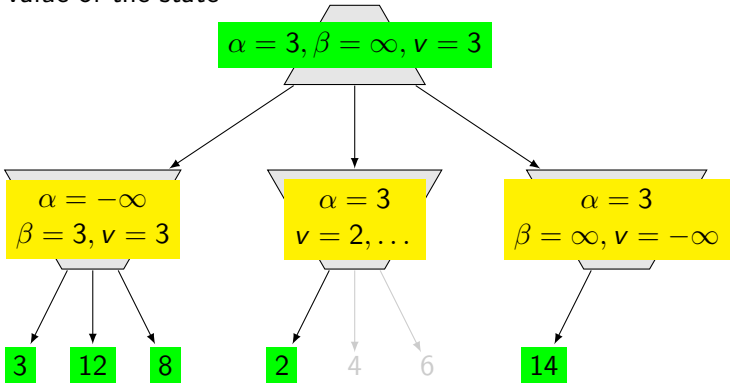

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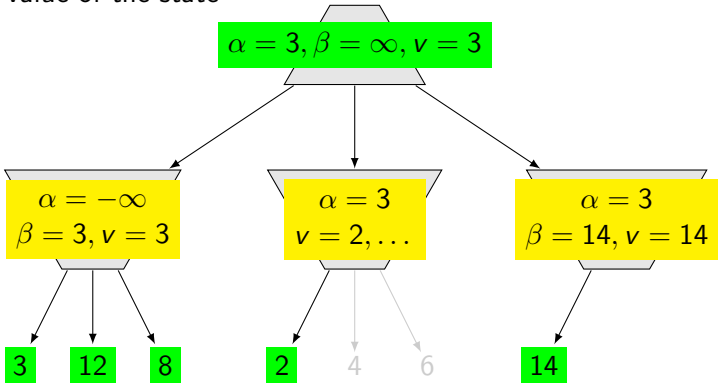

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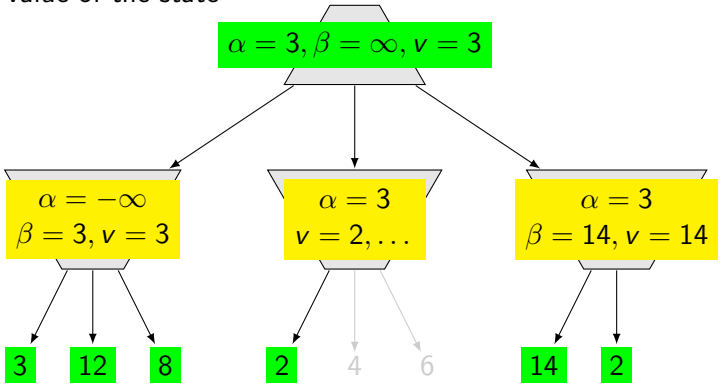

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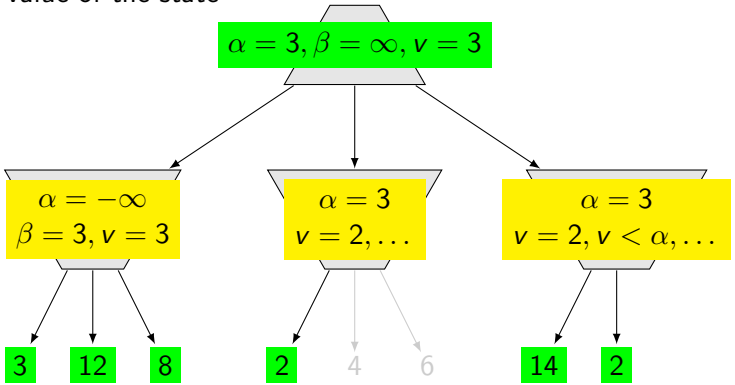

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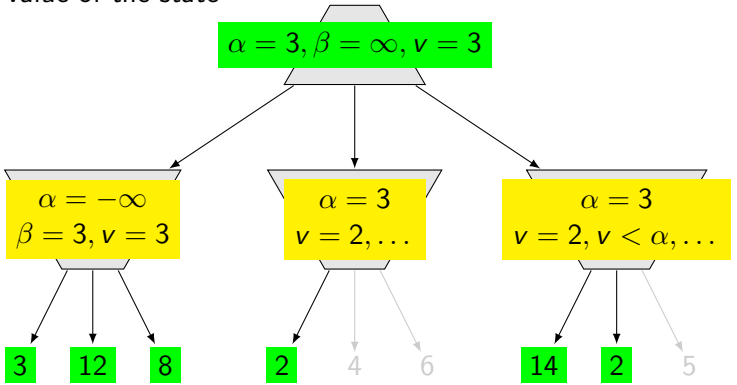

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function ALPHA-BETA-SEARCH(state) **returns** an action
  $v \leftarrow$ MAX-VALUE(state, $\alpha = -\infty$, $\beta = \infty$)
  return action corresponding to v
end function

---

function MAX-VALUE(state,α, β) **returns** a utility value v
  if TERMINAL-TEST(state) **return** UTILITY(state)
  $v \leftarrow -\infty$
  for all ACTIONS(state) **do**
    $v \leftarrow$ **max**(v, MIN-VALUE(RESULT(state,a),α, β))
    if $v \geq \beta$ **return** v
    $\alpha \leftarrow \max(\alpha, v)$
  end for
end function

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    if $v \leq \alpha$ **return** v
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  end for
end function

Take the tree from the previous slide and try to go step-by-step, watch α, β and v

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function ALPHA-BETA-SEARCH(state) returns an action
    v ← MAX-VALUE(state, α = −∞, β = ∞)
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function MAX-VALUE(state,α, β) returns a utility value v
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        if v ≥ β return v
        α ← max(α, v)
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    end for
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```


Take the tree from the previous slide and try to go step-by-step, watch α, β and v

Imperfect but real-time decisions - iterative deepening

$$\text{H-MINIMAX}(s, d) =$$
$$\begin{cases} \text{EVAL}(s) & \text{if } \text{CUTOFF-TEST}(s, d) \\ \max_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d+1) & \text{if } \text{PLAYER}(s) = \text{MAX} \\ \min_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a, d+1)) & \text{if } \text{PLAYER}(s) = \text{MIN} \end{cases}$$

Even with perfect ordering, α-β prunning does not save us.

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\end{aligned}
$$

Even with perfect ordering, α-β prunning does not save us.

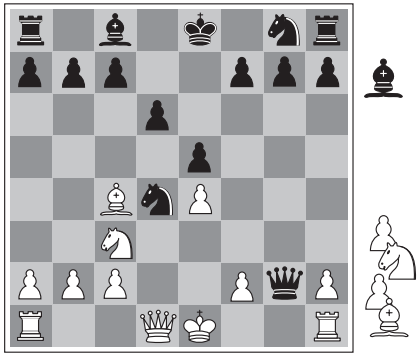
Cutting off search

Replace
if TERMINAL-TEST(s) **then return** TERMINAL-UTILITY(s)
with:
if CUTOFF-TEST(s,d) **then return** EVAL(s)

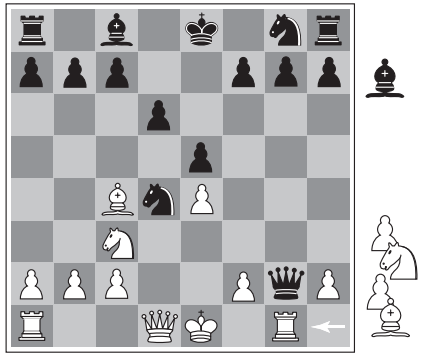
Cutting depends on d only, why we need s as the input parameter?

EVAL(s) – Evaluation functions

(estimate of) State value for non-terminal states



(a) White to move



(b) White to move

$$\text{EVAL}(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots w_n f_n(s)$$

For many problems it is not so easy to find/construct proper function. We may try more functions and combine them conveniently.

$$f_1(s) = \text{number of white pawns} - \text{number of black pawns}$$

How to tune weights w_i?
or Deep Nets! Yeah!
How the get training data for supervised learning? More later.

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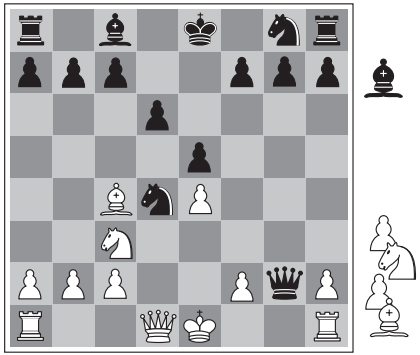

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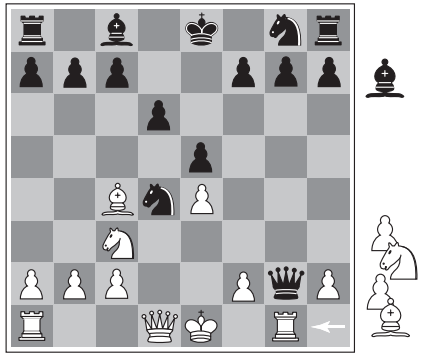

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References

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