



**OI-OPPA. European Social Fund
Prague & EU: We invest in your future.**



DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

WINDOWING

PETR FELKEL

FEL CTU PRAGUE

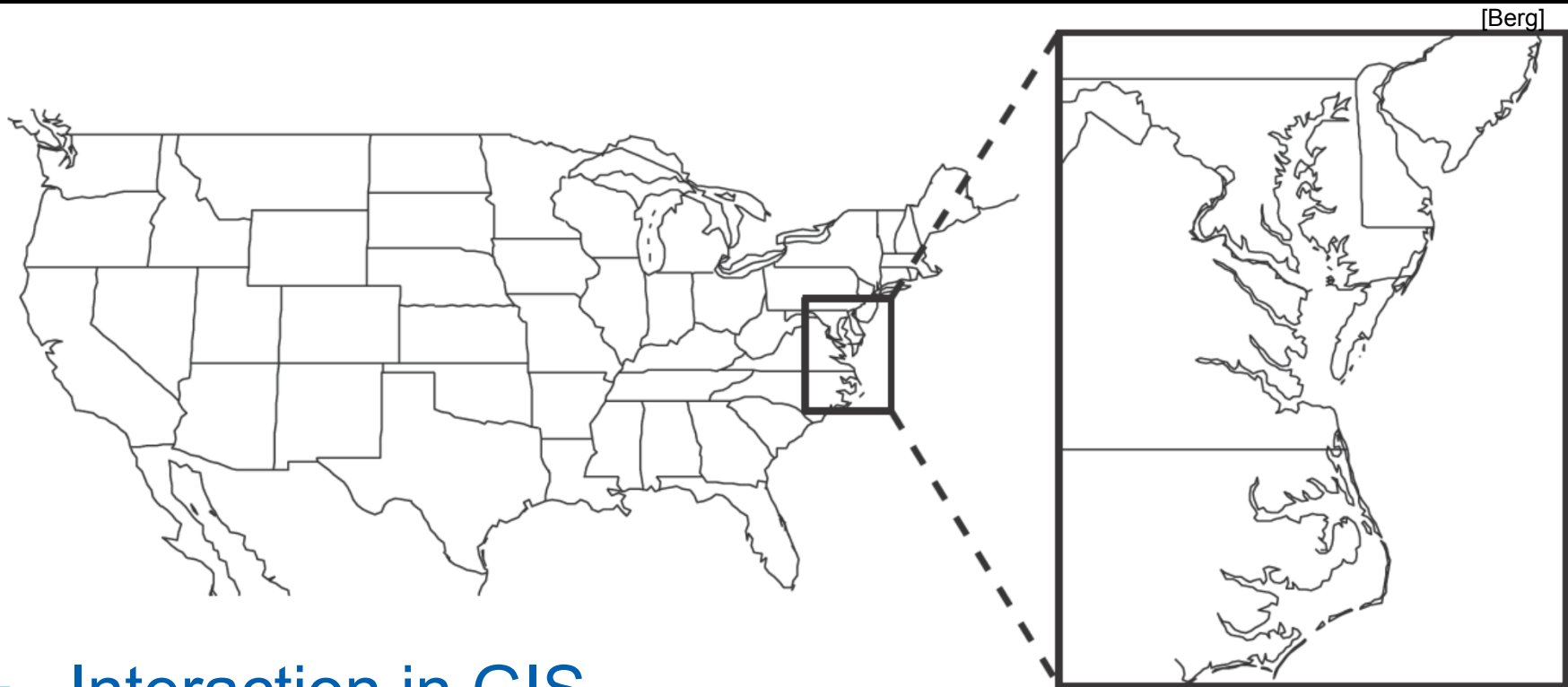
felkel@fel.cvut.cz

<https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start>

Based on [Berg], [Mount]

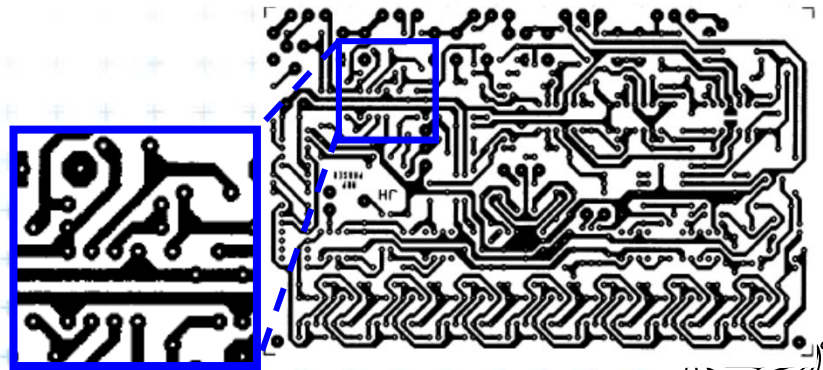
Version from 6.12.2012

Windowing queries - examples



- Interaction in GIS
 - Select subset by outlining
 - Zoom in and re-center

■ Circuit board inspection, ..



Talk overview

- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opak**) **Line** stabbing (*interval tree* with sorted lists)
 - **A**) **Line segment** stabbing (*IT* with *range trees*)
 - **B**) **Line segment** stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*



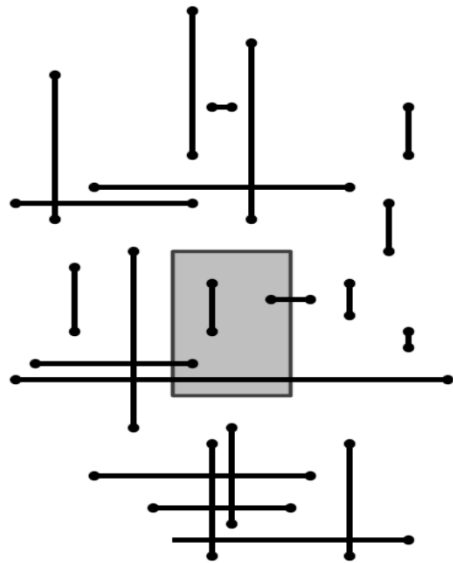
Windowing versus range queries

- Range queries (range trees in Lecture 03)
 - Points
 - Often in higher dimensions
- Windowing queries
 - Line segments, curves, ...
 - Usually in low dimension (2D, 3D)

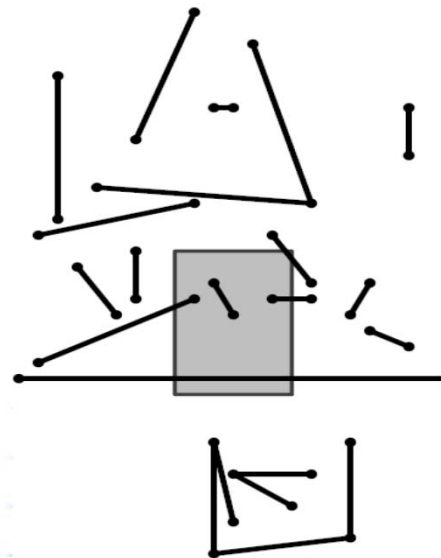


Windowing queries

- Preprocess the data into a data structure
 - so that the ones intersected by the query rectangle can be reported efficiently
- Two cases

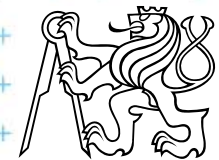


Axis parallel line segments



Arbitrary line segments
(non-crossing)

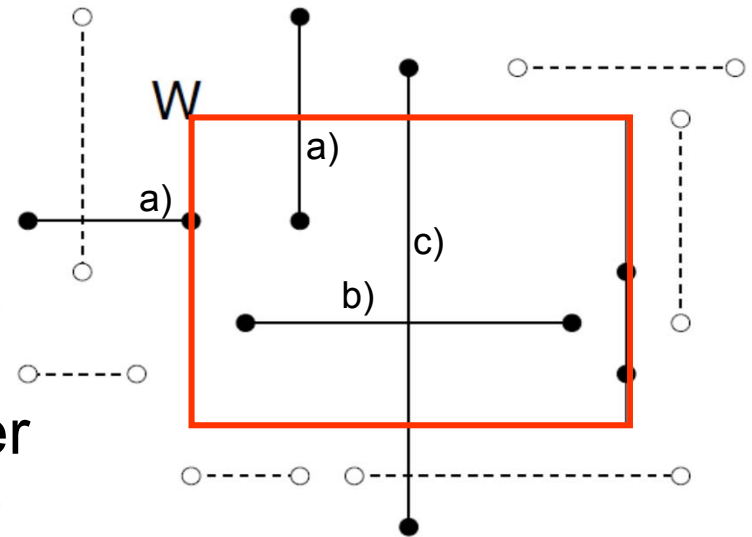
[Vakken]



Windowing of axis parallel line segments

Window query

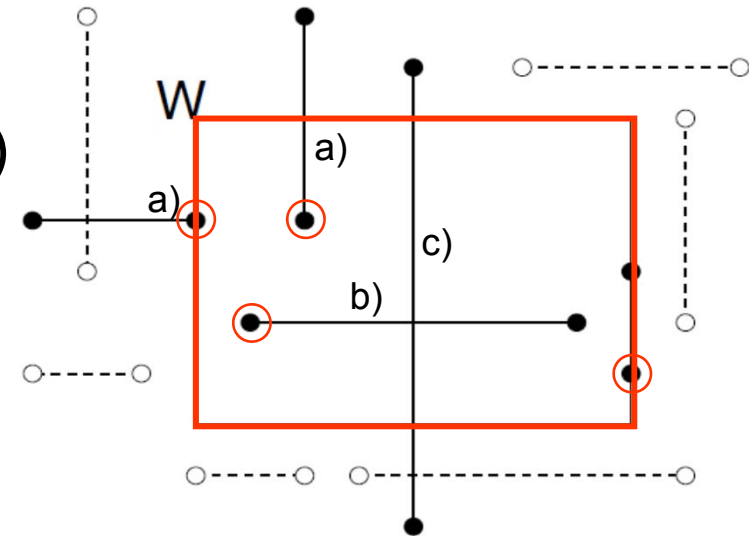
- Given
 - a set of **orthogonal line segments** S (preprocessed),
 - and orthogonal query rectangle $W = [x : x'] \times [y : y']$
- Count or report all the line segments of S that intersect W
- Such segments have
 - a) 1 endpoint in
 - b) 2 end points in – Included
 - c) no end point in – Cross over



Line segments with 1 or 2 points inside

a) 1 point inside

- Use a **range tree** (Lesson 3)
- $O(n \log n)$ storage
- $O(\log^2 n + k)$ query time or
- $O(\log n + k)$ with fractional cascading



b) 2 points inside – as a) 1 point inside

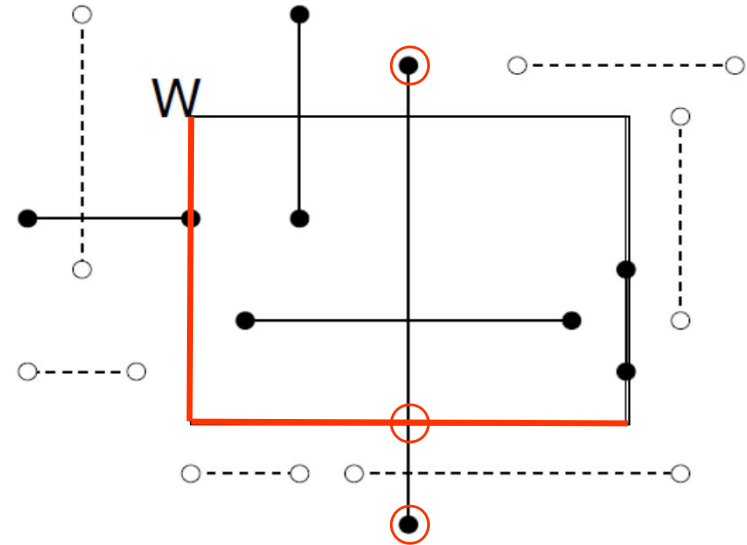
- Avoid reporting twice
 1. Mark segment when reported (clear after the query)
 2. When end point found, check the other end-point.
Report only the leftmost or bottom endpoint



Line segments that cross over the window

c) No points inside

- not detected using a range tree
- Cross the boundary twice or contain one boundary edge
- It is enough to detect segments intersected by the **left** and **bottom boundary edges** (not having end point inside)
- For left boundary: Report the segments intersecting **vertical query line segment (B)**
- Let's discuss **vertical query line** first (A)
- Bottom boundary is rotated 90°



Talk overview

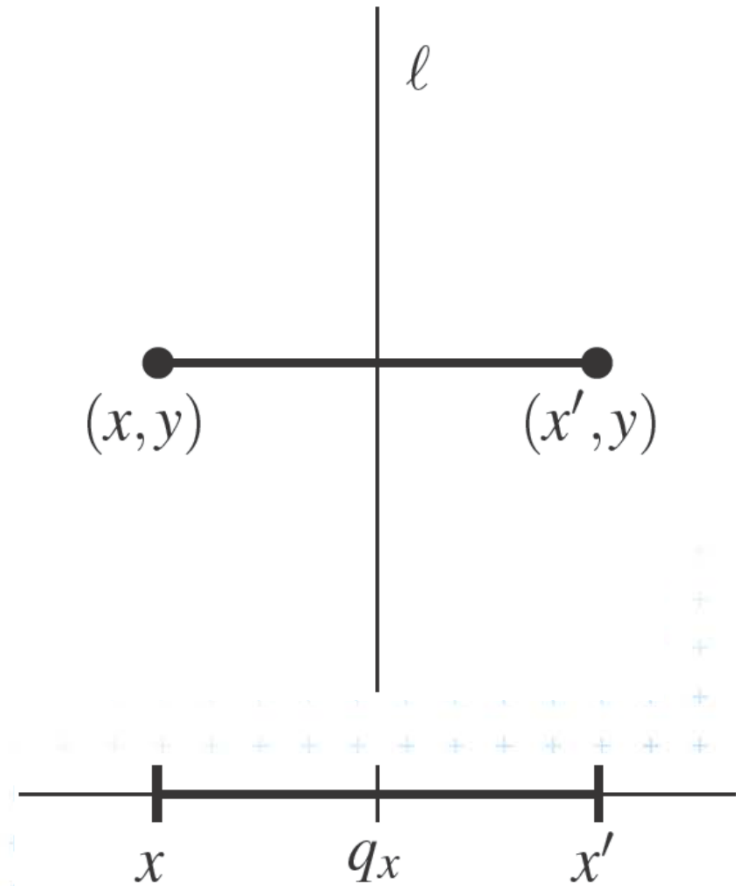
- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - A) Line segment stabbing (*IT* with *range trees*)
 - B) Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*



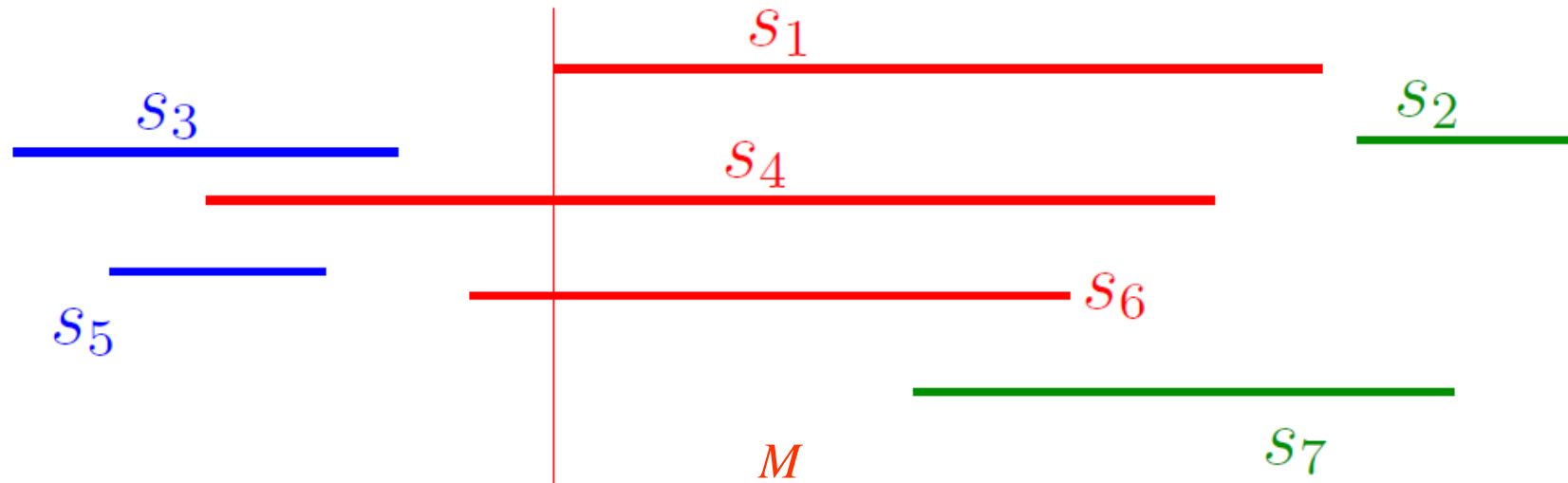
A: Segment intersected by vertical line \rightarrow 1D

- Query line $\ell := (x=q_x)$
Report the segments
stabbed by a vertical line
= 1 dimensional problem
(ignore y coordinate)

\Rightarrow Report the interval
containing query point q_x



Interval tree principle



$$M_l = (s_4, s_6, s_1)$$

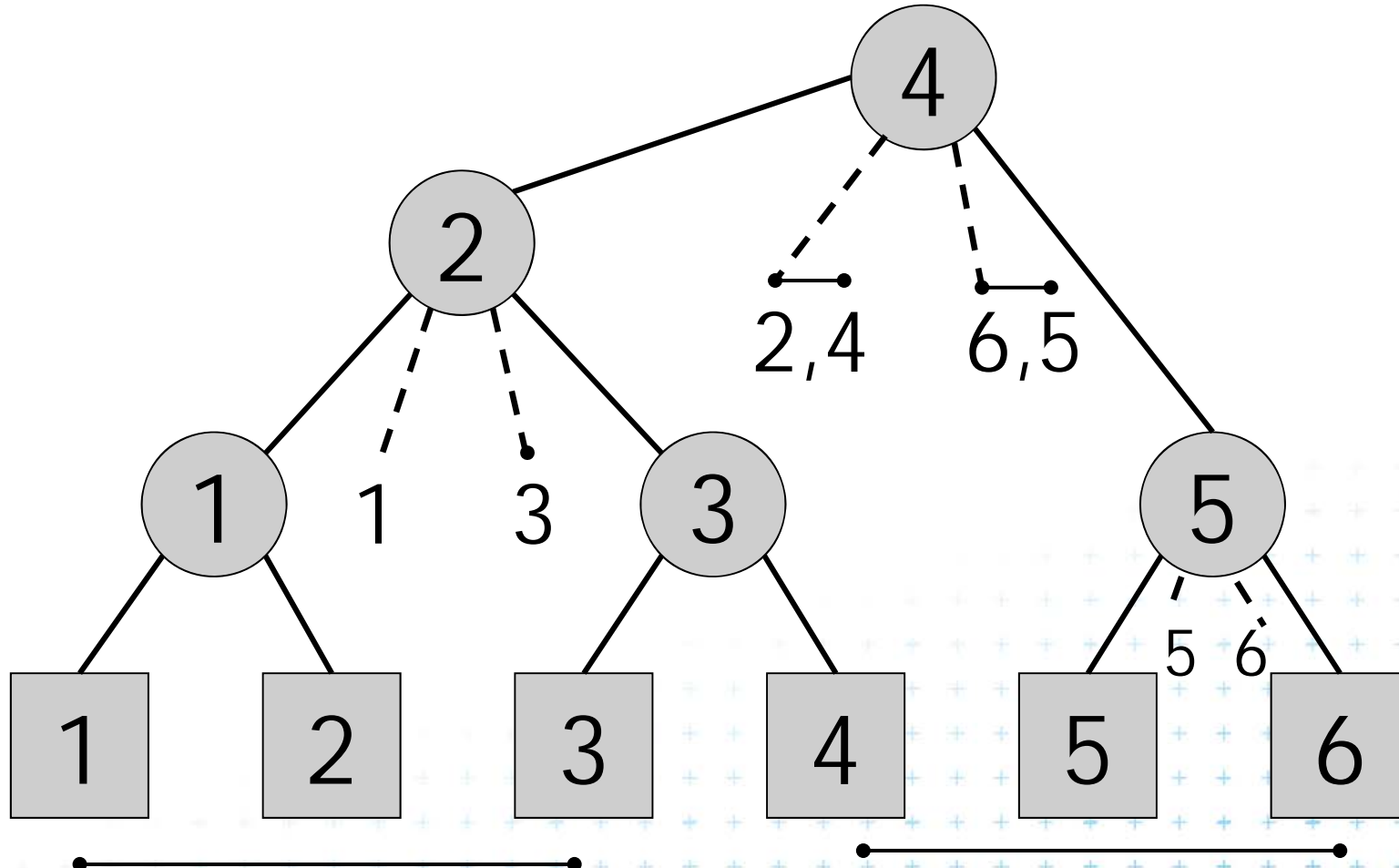
$$M_r = (s_1, s_4, s_6)$$

L
Interval tree on s_3 and s_5

R
Interval tree on s_2 and s_7



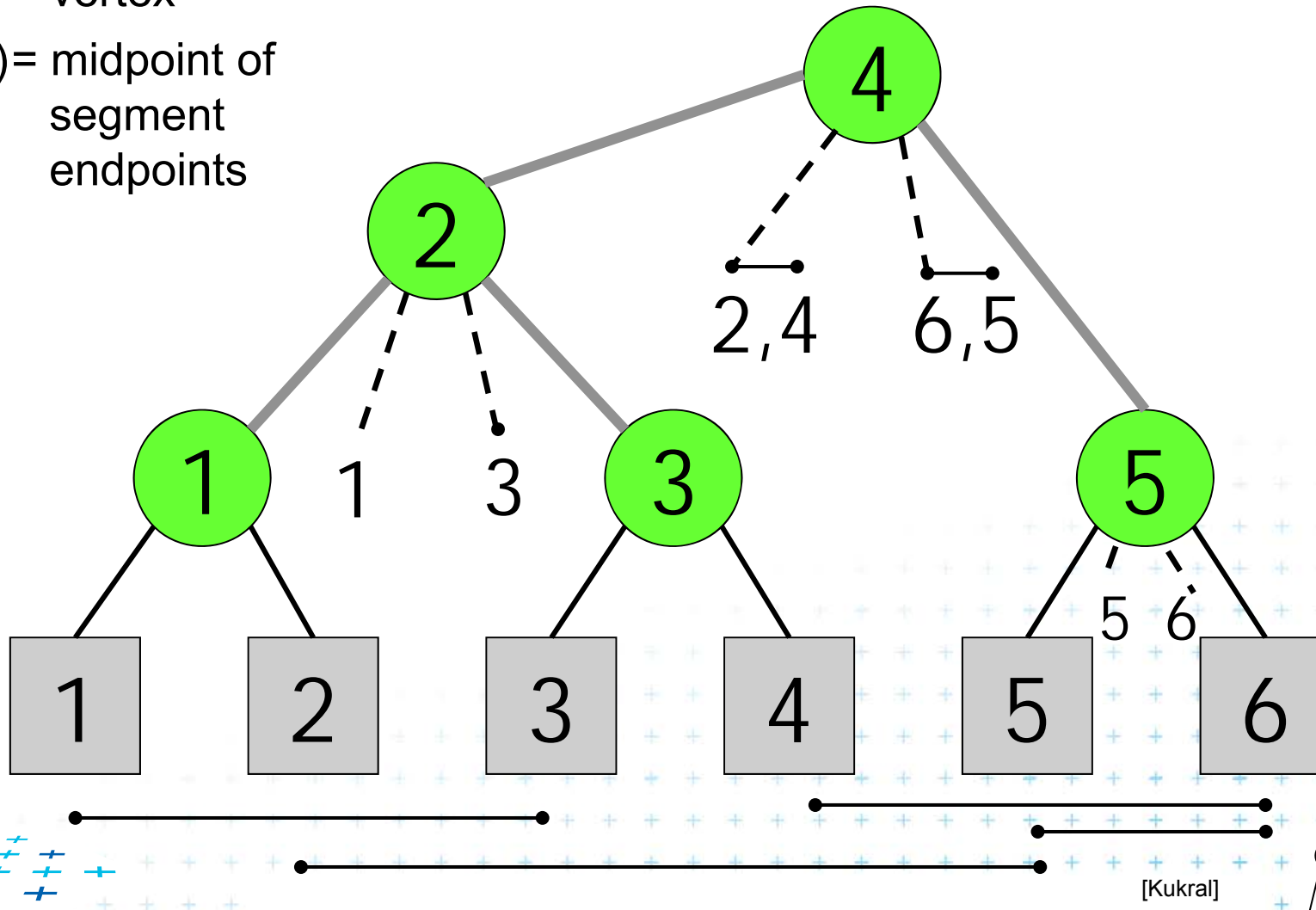
Static interval tree [Edelsbrunner80]



Primary structure – static tree for endpoints

v = vertex

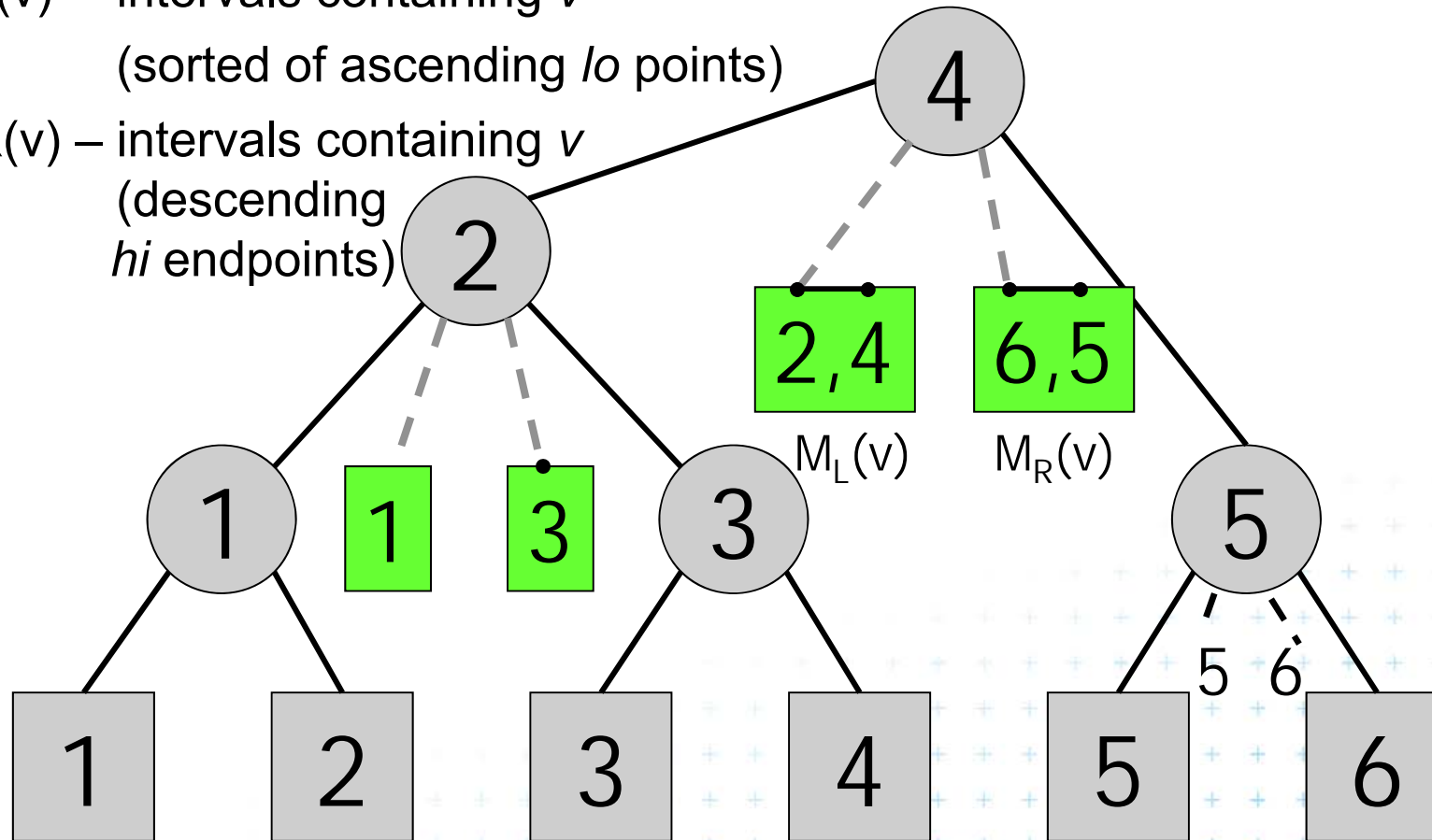
$d(v)$ = midpoint of segment endpoints



Secondary lists – sorted segments in M

ML(v) – intervals containing v
(sorted of ascending lo points)

MR(v) – intervals containing v
(descending hi endpoints)



Interval tree construction (all intervals at once)

ConstructIntervalTree(S) // Intervals all active – **no active lists**

Input: Set S of intervals on the real line – on *x-axis*

Output: The root of an interval tree for S

1. if ($|S| == 0$) return null // no more
2. else
3. $xMed$ = median endpoint of intervals in S // median endpoint
4. $L = \{ [xlo, xhi] \text{ in } S \mid xhi < xMed \}$ // left of median
5. $R = \{ [xlo, xhi] \text{ in } S \mid xlo > xMed \}$ // right of median
6. $M = \{ [xlo, xhi] \text{ in } S \mid xlo \leq xMed \leq xhi \}$ // contains median
7. ML = sort M in increasing order of xlo // sort M
8. MR = sort M in decreasing order of xhi
9. $t = \text{new IntTreeNode}(xMed, ML, MR)$ // this node
10. $t.left = \text{ConstructIntervalTree}(L)$ // left subtree
11. $t.right = \text{ConstructIntervalTree}(R)$ // right subtree
12. return t



[Mount]



Line stabbing query for an interval tree

Stab(t, xq)

Input: IntTreeNode t, Scalar xq

Output: prints the intersected intervals

```
1.  if (t == null) return // fell out of tree
2.  if (xq < t.xMed) // left of median?
3.      for (i = 0; i < t.ML.length; i++) // traverse ML
4.          if (t.ML[i].lo <= xq) print(t.ML[i]) // ..report if in range
5.          else break // ..else done
6.      stab(t.left, xq) // recurse on left
7.  else // (xq ≥ t.xMed) // right of or equal to median
8.      for (i = 0; i < t.MR.length; i++) { // traverse MR
9.          if (t.MR[i].hi ≥ xq) print(t.MR[i]) // ..report if in range
10.         else break // ..else done
11.         stab(t.right, xq) // recurse on right
```

Note: Small inefficiency for $xq == t.xMed$ – recurse on right



[Mount]



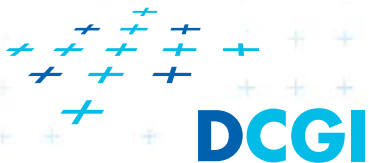
Complexity of **line** stabbing via interval tree

- Construction - $O(n \log n)$ time
 - Each step divides at maximum into two halves or less (minus elements of M) \Rightarrow tree height $O(\log n)$
 - If presorted the endpoints in three lists L, R, M then median in $O(1)$ and copy to new L, R, M in $O(n)$
- Vertical line stabbing query - $O(k + \log n)$ time
 - One node processed in $O(1 + k')$, k' =reported intervals
 - v visited nodes in $O(v + k)$, k =total reported intervals
 - $v =$ tree height $= O(\log n)$
- Storage - $O(n)$
 - Tree has $O(n)$ nodes, each segment stored twice (two endpoints)



Talk overview

- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - **A)** Line segment stabbing (*IT* with *range trees*)
 - **B)** Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*

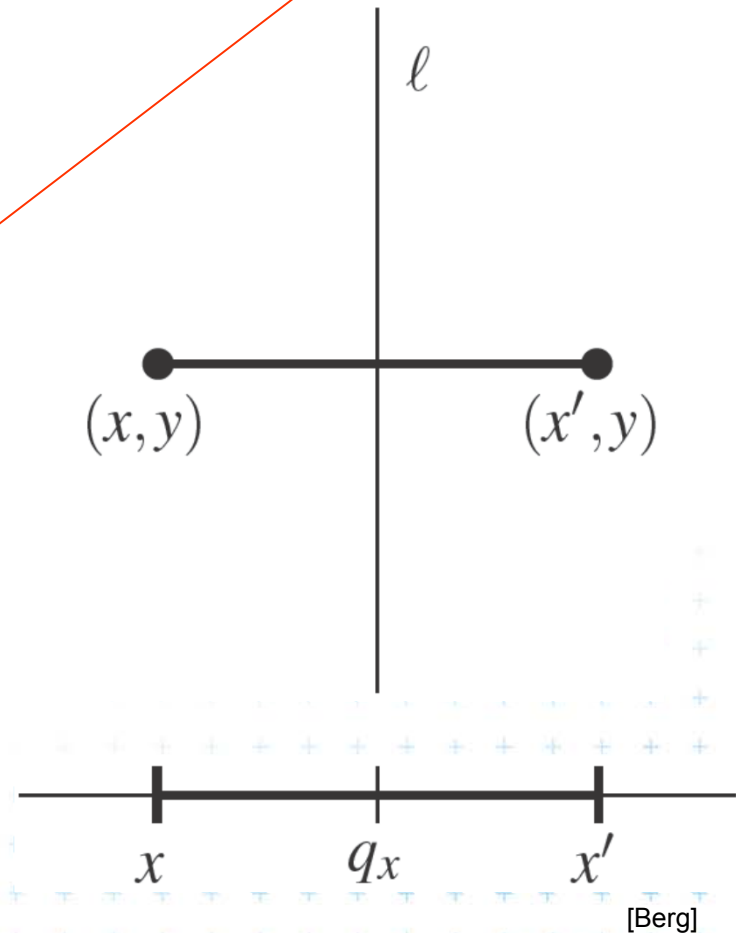


A: Segment intersected by vertical line - 1D

- Query line $\ell := (x = q_x)$
Report the segments
stabbed by a vertical line
= 1 dimensional problem
(ignore y coordinate)

=> Report the interval
containing query point q_x

DS: Interval tree



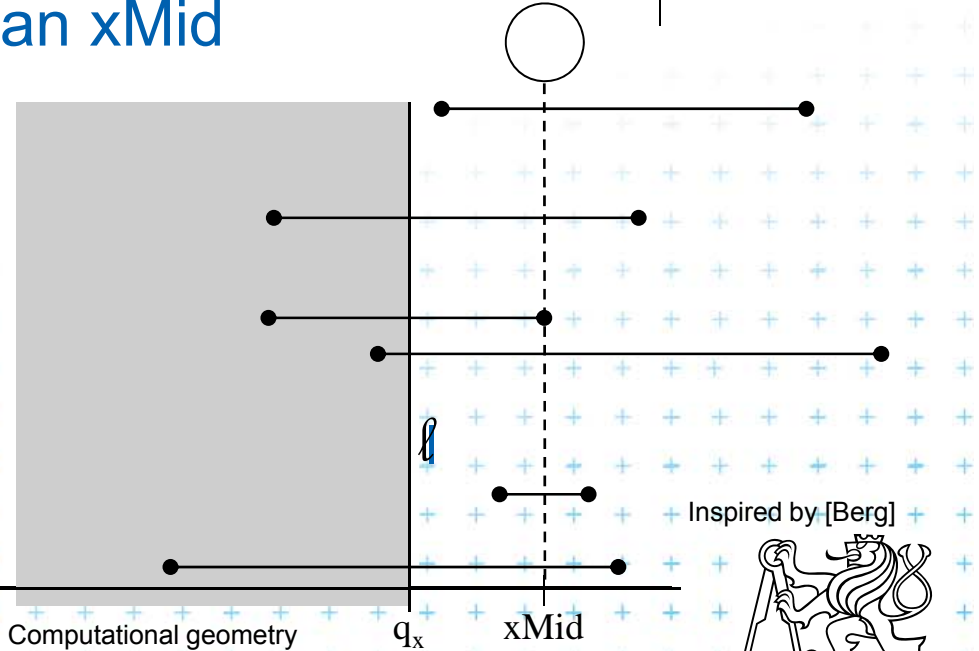
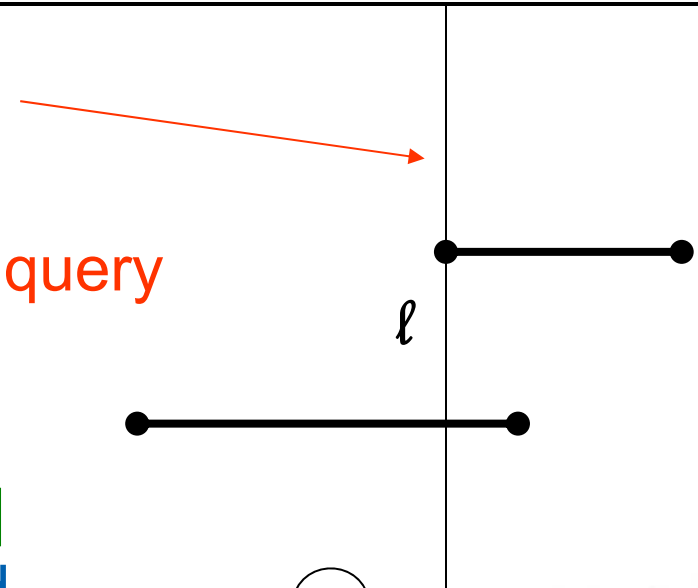
A: Segment intersected by vertical line - 2D

- Query line $\ell := q_x \times [-\infty : \infty]$
- Horizontal segment of M stabs the query line ℓ iff its left endpoint lies in halph-space

$$(-\infty : q_x] \times [-\infty : \infty]$$

- In IT node with stored median $xMid$ report all segments from M

- whose left point lies in $(-\infty : q_x]$ if ℓ lies left from $xMid$
- whose right point lies in $(q_x : +\infty]$ if ℓ lies right from $xMid$



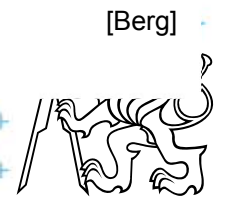
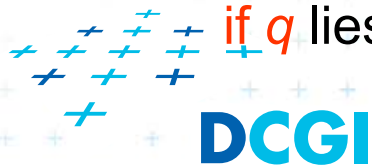
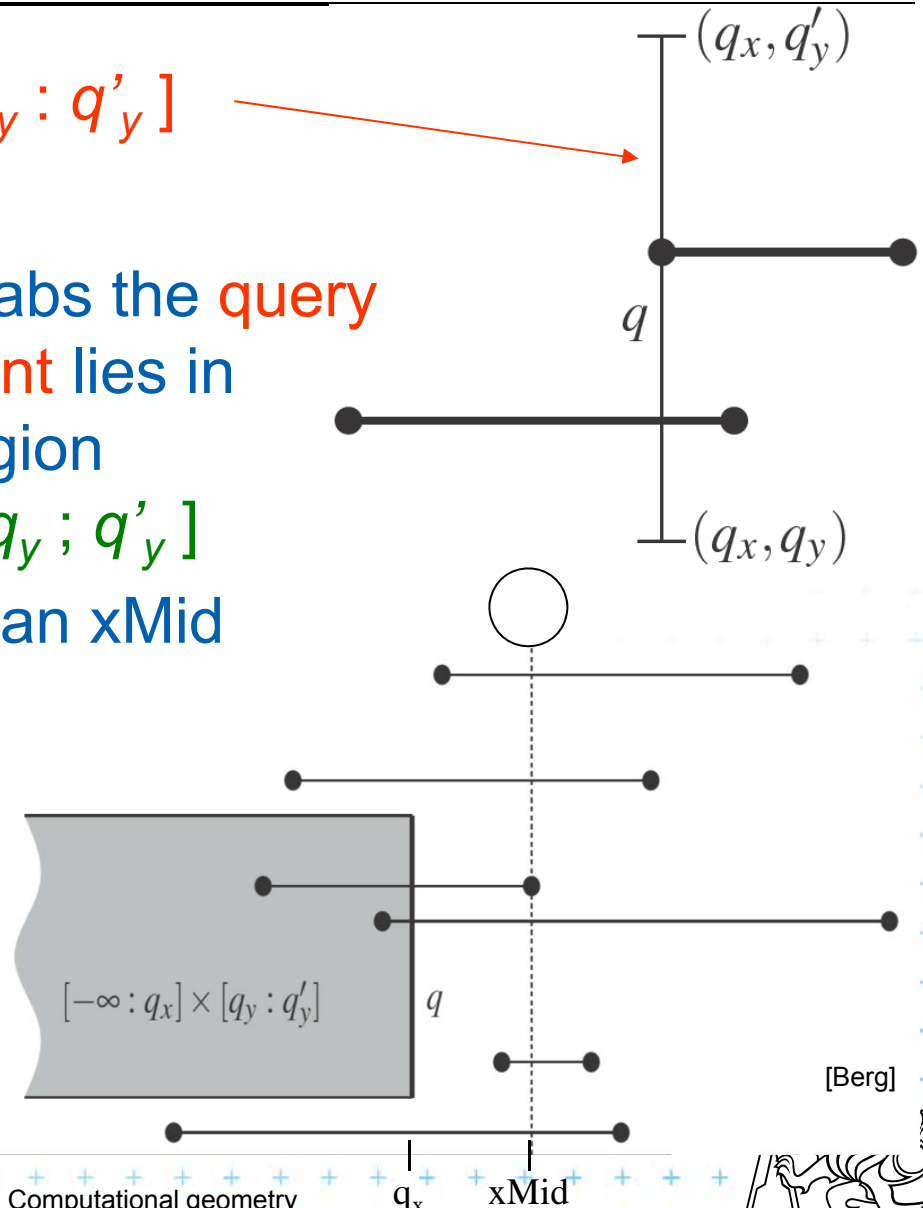
B: Segment intersected by vertical line segment

- Query segment $q := q_x \times [q_y : q'_y]$
- Horizontal segment of M stabs the query segment q iff its left endpoint lies in semi-infinite rectangular region

$$(-\infty : q_x] \times [q_y ; q'_y]$$

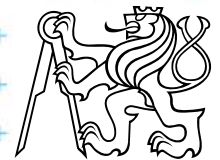
- In IT node with stored median $xMid$ report all segments

- whose left point lies in $(-\infty : q_x] \times [q_y ; q'_y]$ if q lies left from $xMid$
- whose right point lies in $(q_x : +\infty] \times [q_y ; q'_y]$ if q lies right from $xMid$

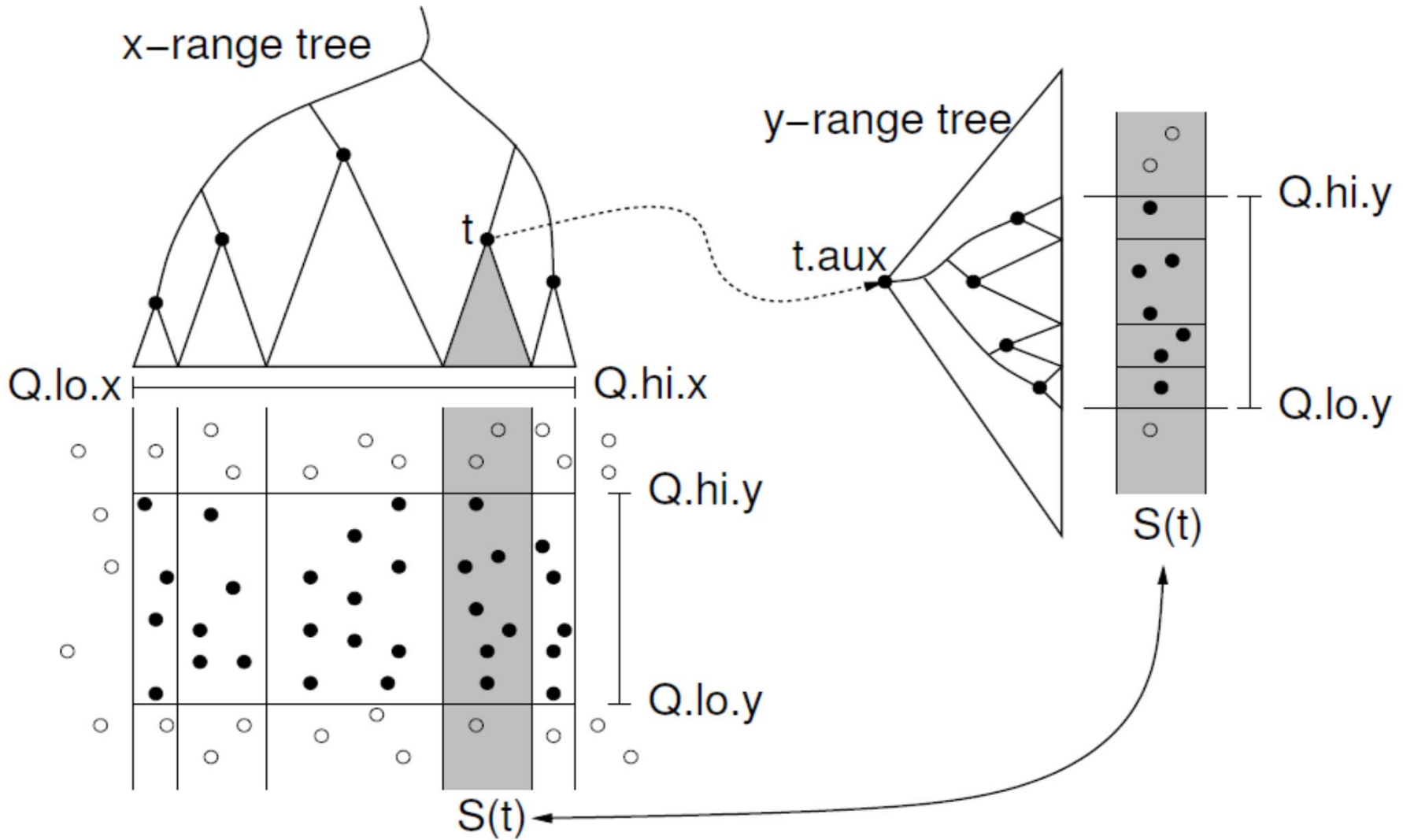


Data structure for endpoints

- Storage of ML and MR
 - Sorted lists not enough for line segments
 - Use **two range trees**
- Instead $O(n)$ sequential search in ML and MR perform $O(\log n)$ search in range tree with fractional cascading



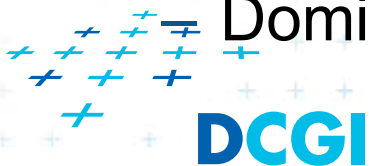
2D range tree (without fractional casc. - see more in Lecture 3)



Complexity of **line segment stabbing**

- Construction - $O(n \log n)$ time
 - Each step divides at maximum into two halves L,R or less (minus elements of M) => tree height $O(\log n)$
 - If the **range trees** are efficiently build in $O(n)$
- Vertical line segment stab. q. - $O(k + \log^2 n)$ time
 - One node processed in $O(\log n + k')$, k' =reported inter.
 - v visited nodes in $O(v \log n + k)$, k =total reported inter.
 - $v =$ tree height = $O(\log n)$
 - $O(k + \log^2 n)$ time - range tree with fractional cascading
 - $O(k + \log^3 n)$ time - range tree without fractional casc.
- Storage - $O(n \log n)$

Dominated by the range trees



Talk overview

- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - **A)** Line segment stabbing (*IT* with *range trees*)
 - **B)** Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*



- Priority search trees – in case c) on slide 8
 - Exploit the fact that **query rectangle** in range queries is **unbounded**
 - Can be used as **secondary data structures** for both left and right endpoints (ML and MR) of segments (intervals) in nodes of interval tree
 - Improve the **storage** to $O(n)$ for horizontal segment intersection with window edge (Range tree has $O(n \log n)$)
- For cases a) and b) - $O(n \log n)$ remains
 - we need range trees for windowing segment endpoints



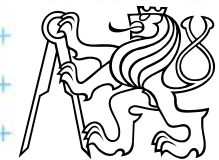
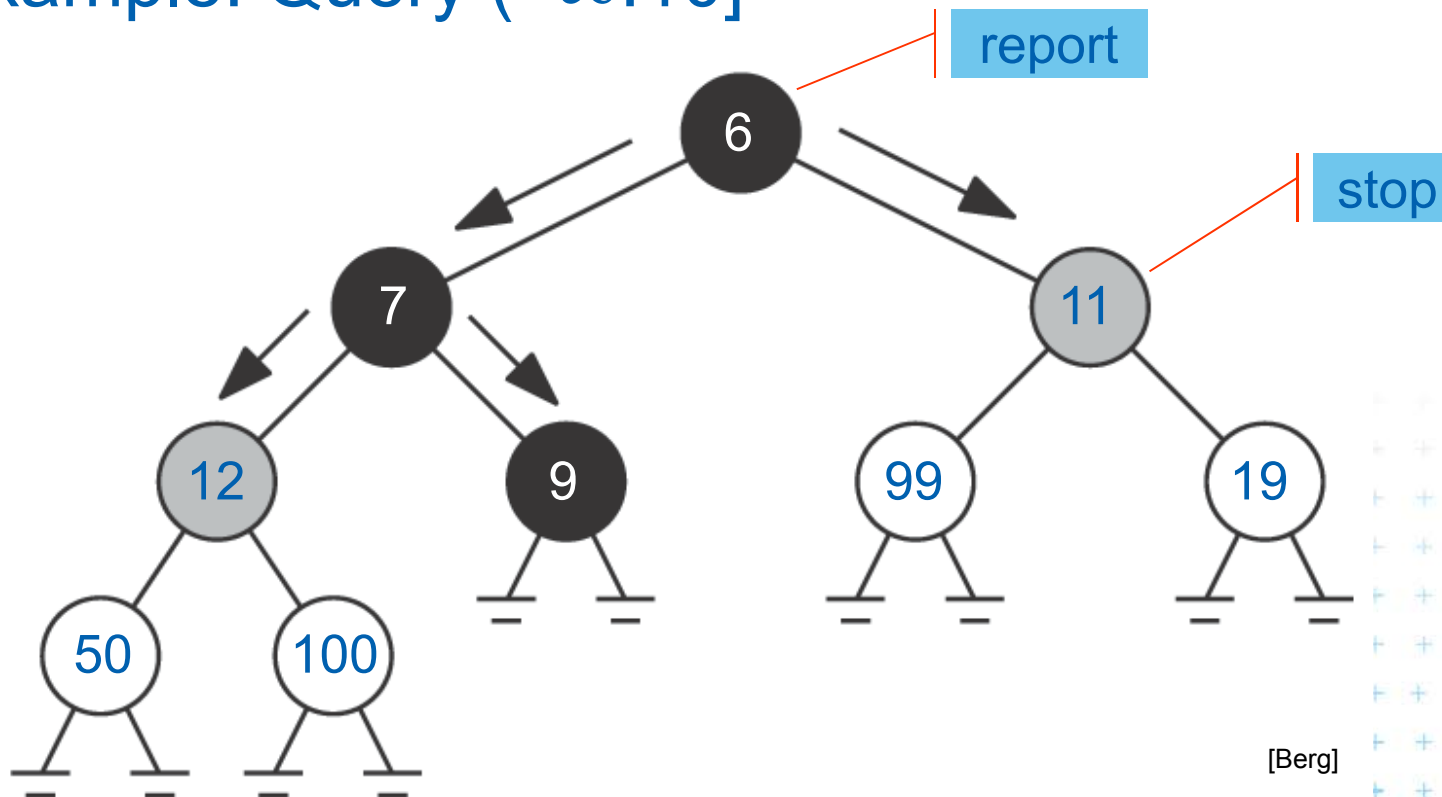
Rectangular range queries variants

- Let $P = \{ p_1, p_2, \dots, p_n \}$ is set of points in plane
- Goal: rectangular range queries of the form $(-\infty : q_x] \times [q_y ; q'_y]$
- **In 1D**: search for nodes v with $v_x \in (-\infty : q_x]$
 - range tree $O(\log n + k)$ time
 - ordered list $O(1 + k)$ time
(start in the leftmost, stop on v with $v_x > q_x$)
 - use heap $O(1 + k)$ time
(traverse all children, stop when $v_x > q_x$)
- **In 2D** – use heap for points with $x \in (-\infty : q_x]$
+ integrate information about y-coordinate



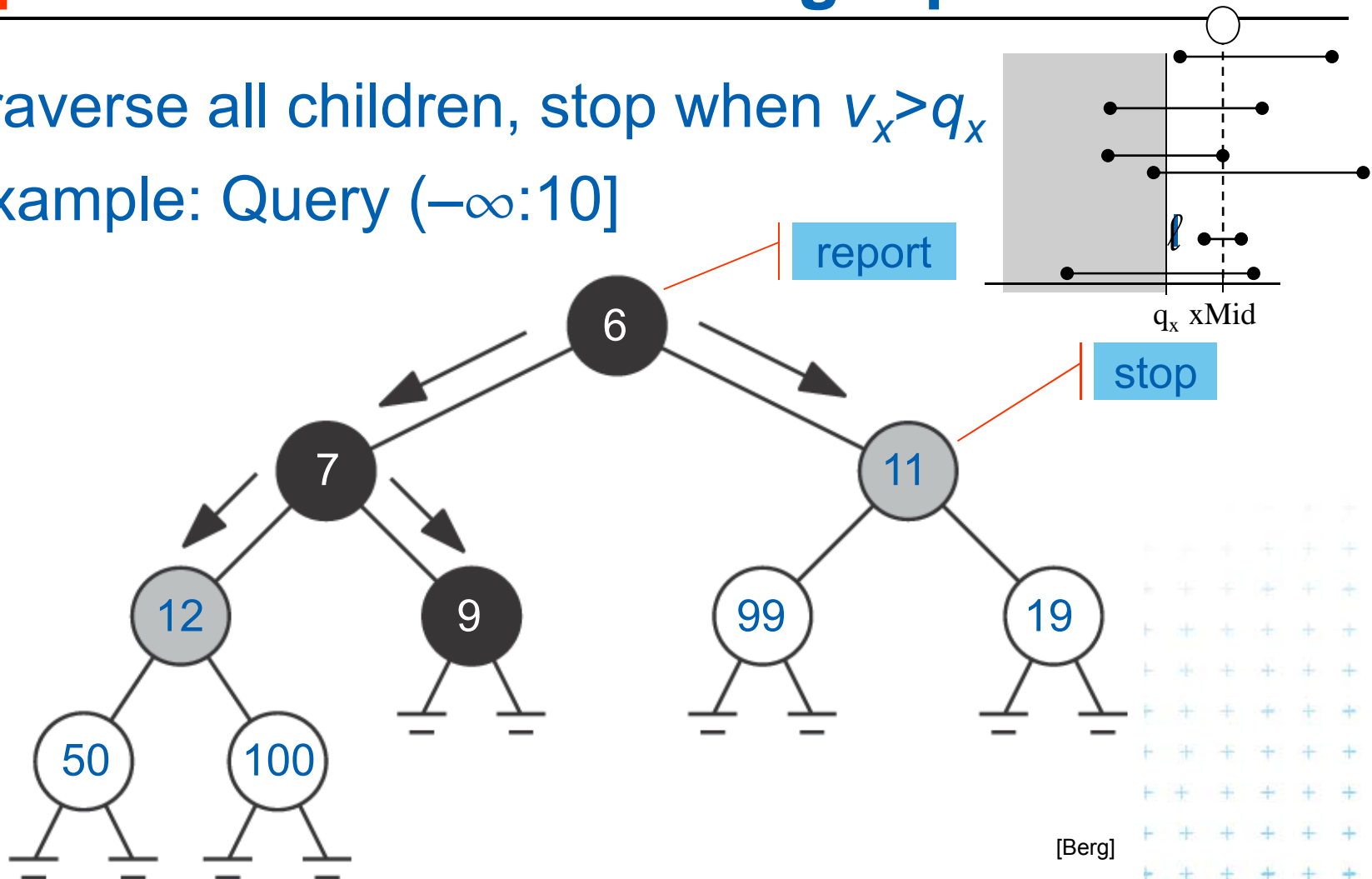
Heap for 1D unbounded range queries

- Traverse all children, stop when $v_x > q_x$
- Example: Query $(-\infty:10]$



Heap for 1D unbounded range queries

- Traverse all children, stop when $v_x > q_x$
- Example: Query $(-\infty:10]$

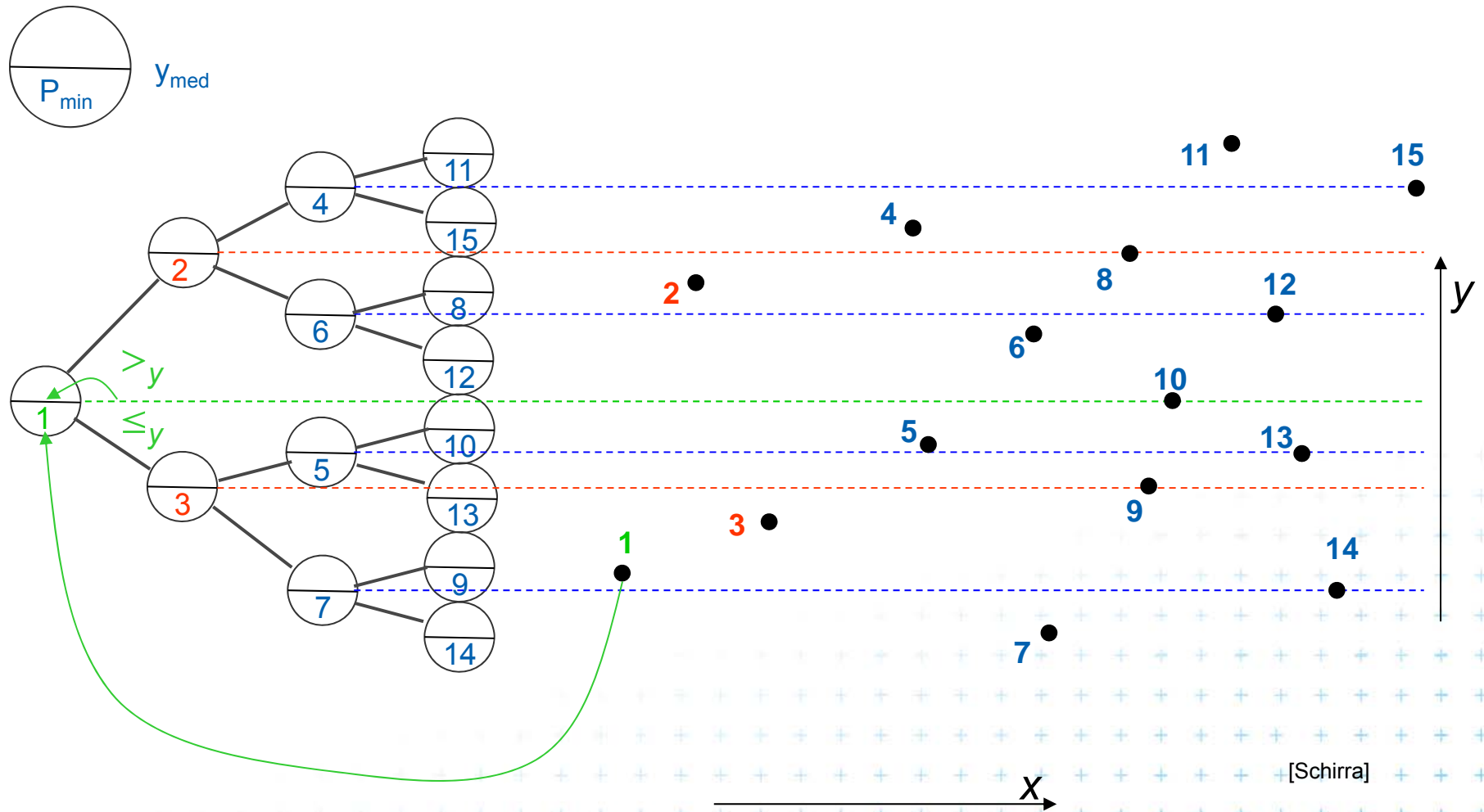


Priority search tree (PST)

- Heap in 2D can incorporate info about both x, y
 - **BST on y -coordinate** (horizontal slabs) \sim range tree
 - **Heap on x -coordinate** (minimum x from slab along x)
- If P is empty, PST is empty leaf
- else
 - p_{min} = point with **smallest x -coordinate** in P --- a heap root
 - y_{med} = **y -coord. median** of points $P \setminus \{p_{min}\}$ --- BST root
 - $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med} \}$
 - $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med} \}$
- Point p_{min} and scalar y_{med} are stored in the PST root
- The left subtree is PST of P_{below}
- The right subtree is PST of P_{above}



Priority search tree construction example



Priority search tree definition

PrioritySearchTree(P)

Input: set P of points in plane

Output: priority search tree T

1. if $P = \emptyset$ then PST is an empty leaf
2. else
3. p_{min} = point with smallest x-coordinate in P // heap on x root
4. y_{med} = y-coord. median of points $P \setminus \{p_{min}\}$ // BST on y root
5. Split points $P \setminus \{p_{min}\}$ into two subsets – according to y_{med}
6. $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med} \}$
7. $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med} \}$
8. $T = \text{newTreeNode}()$ Notation in alg:
9. $T.p = p_{min}$ // point [x, y] ... p(v)
10. $T.y = y_{mid}$ // skalar ... y(v)
11. $T.left = \text{PrioritySearchTree}(P_{below})$... lc(v)
12. $T.rigft = \text{PrioritySearchTree}(P_{above})$... rc(v)

13. $O(n \log n)$, but $O(n)$ if presorted on y-coordinate and bottom up



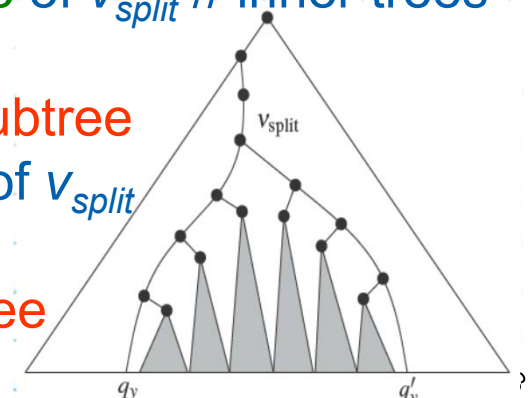
Query Priority Search Tree

QueryPrioritySearchTree($T, (-\infty : q_x] \times [q_y ; q'_y]$)

Input: A priority search tree and a **range, unbounded to the left**

Output: All **points** lying in the range

1. Search with q_y and q'_y in T // BST on y-coordinate – select y range
Let v_{split} be the node where the two search paths split (split node)
2. for each node v on the search path of q_y or q'_y // points along the paths
3. if $p(v) \in (-\infty : q_x] \times [q_y ; q'_y]$ then **report** $p(v)$ // starting in tree root
4. for each node v on the path of q_y in the **left subtree** of v_{split} // inner trees
5. if the search **path goes left** at v
6. **ReportInSubtree**($rc(v), q_x$) // **report right subtree**
7. for each node v on the path of q'_y in **right subtree** of v_{split}
8. if the search **path goes right** at v
9. **ReportInSubtree**($lc(v), q_x$) // **rep. left subtree**



Reporting of subtrees between the paths

ReportInSubtree(v , q_x)

Input: The root v of a subtree of a priority search tree and a value q_x .

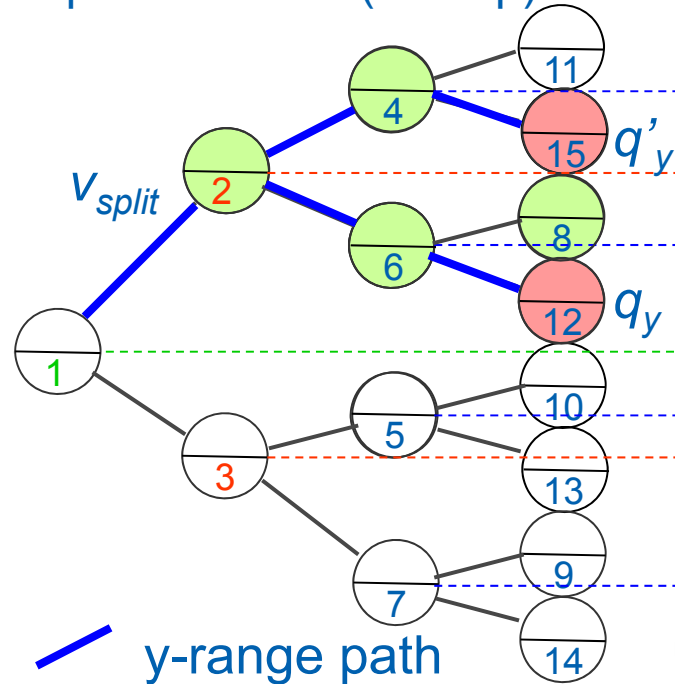
Output: All points in the subtree with x -coordinate at most q_x .

1. if v is not a leaf and $x(p(v)) \leq q_x$ // $x \in (-\infty : q_x]$ -- heap condition
2. Report $p(v)$.
3. ReportInSubtree($lc(v)$, q_x)
4. ReportInSubtree($rc(v)$, q_x)

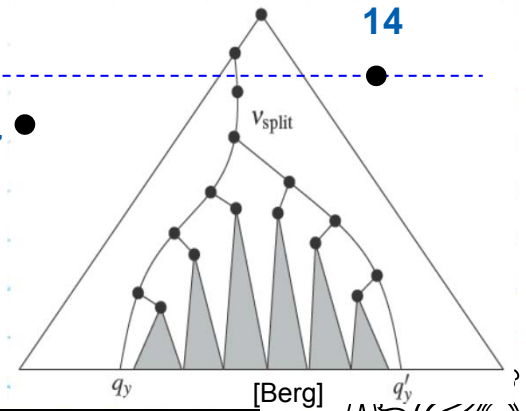
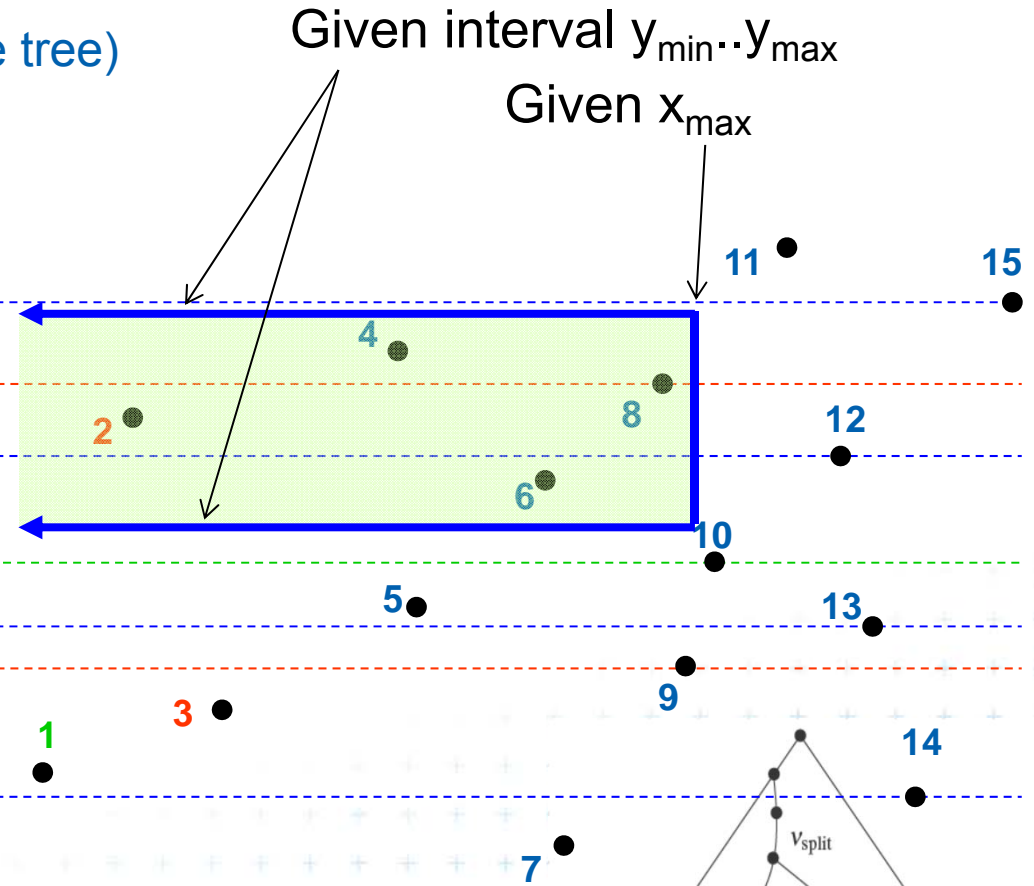


Priority search tree query

1. select y range (y-BVS ~ 1D range tree)
2. report points on paths (x-heap)
3. report subtrees (x-heap)



- y-range path
- x ok – report this point
- x too high – stop



Based on [Schirra]



Priority search tree complexity

For set of n points in the plane

- Build $O(n \log n)$
- Storage $O(n)$
- Query $O(k + \log n)$
 - points in query range $(-\infty : q_x] \times [q_y ; q'_y]$
 - k is number of reported points
- Use PST as associated data structure for interval trees for storage of M



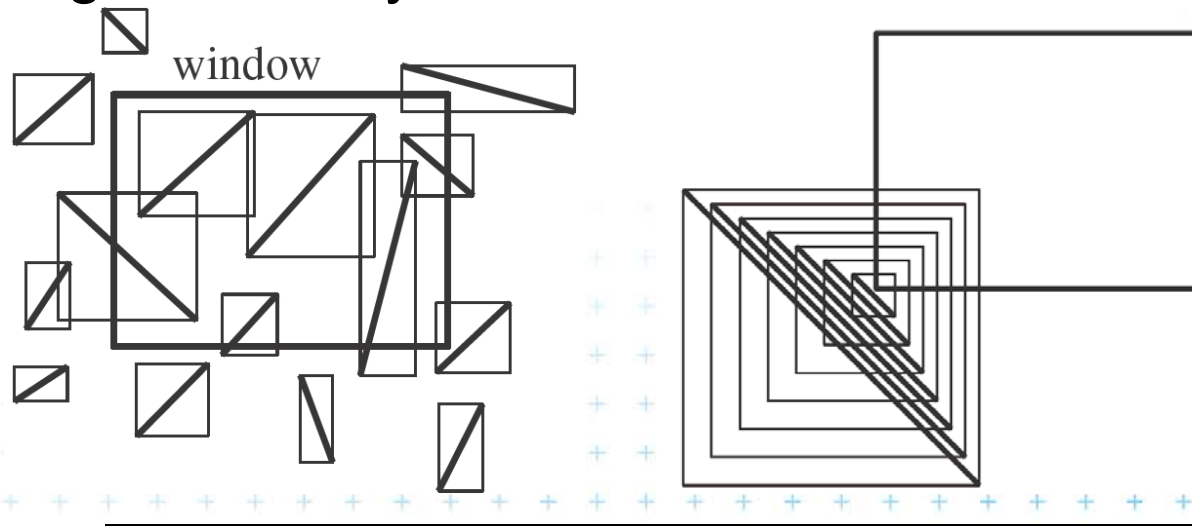
Talk overview

- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - A) Line segment stabbing (*IT* with *range trees*)
 - B) Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*
 - windowing of segments in general position



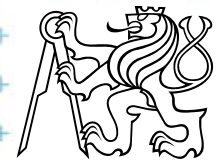
Windowing of arbitrary oriented line segments

- Two cases of intersection
 - a,b) Endpoint inside the query window => range tree
 - c) Segment intersects side of query window => ???
- Intersection with BBOX (segment bounding box)?
 - Intersection with $4n$ sides
 - But segments may not intersect the window → query y

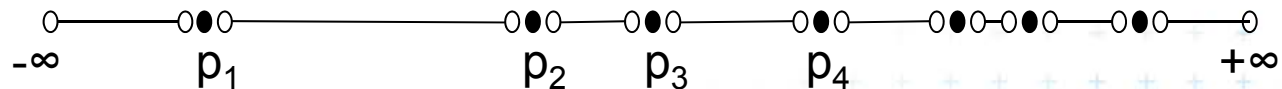


Talk overview

- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - A) Line segment stabbing (*IT* with *range trees*)
 - B) Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*
 - windowing of segments in general position



- Exploits locus approach
 - Partition parameter space into regions of same answer
 - Localization of such region = knowing the answer
- For given set S of n intervals (segments) on real line
 - Finds m elementary intervals (induced by interval end-points)
 - Partitions 1D parameter space into these elementary intervals
 - Stores intervals s_i with the elementary intervals
 - Reports the intervals s_i containing query point q_x .



$(-\infty : p_1), [p_1 : p_1], (p_1 : p_2), [p_2 : p_2], \dots,$
 $(p_{m-1} : p_m), [p_m : p_m], (p_m : +\infty)$

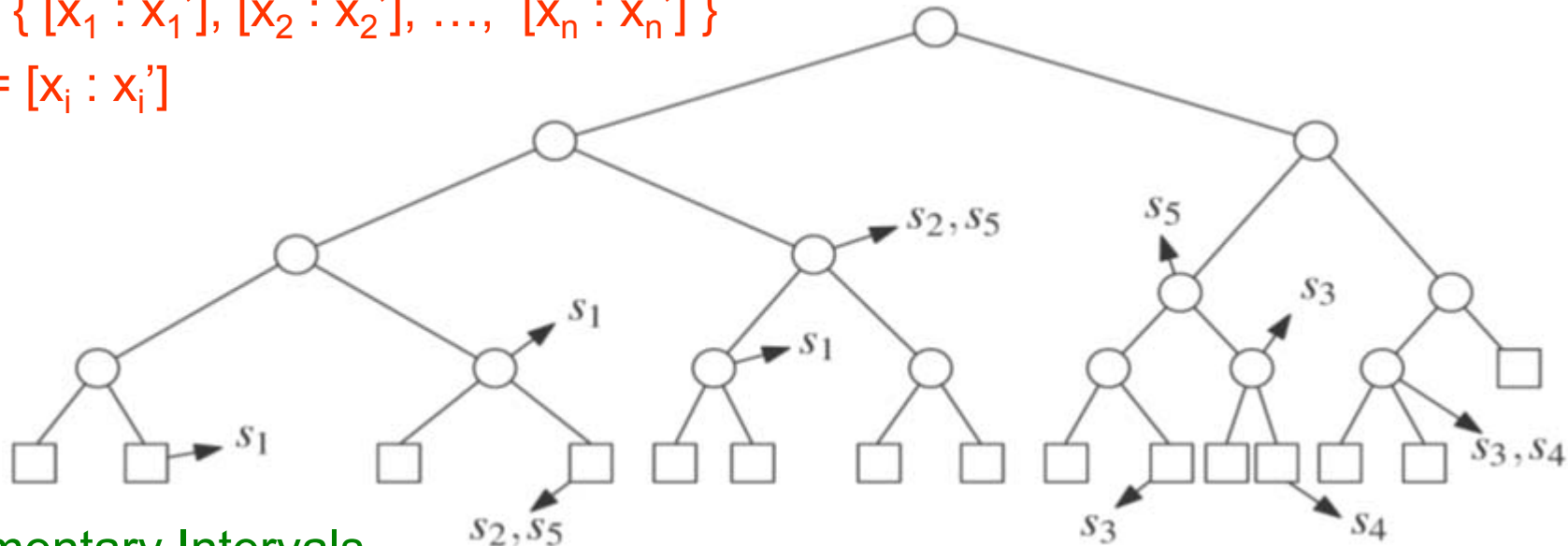


Segment tree example

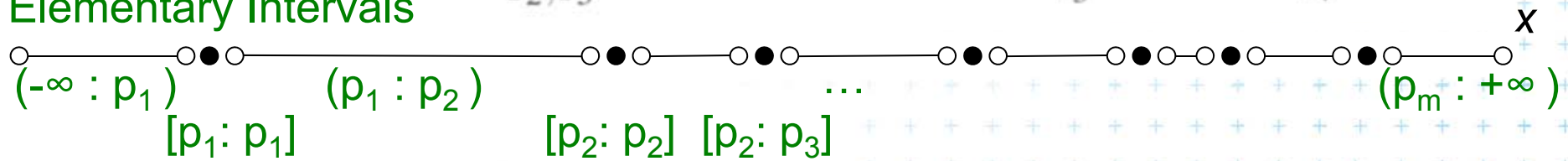
Intervals

$$S = \{ [x_1 : x_1'], [x_2 : x_2'], \dots, [x_n : x_n'] \}$$

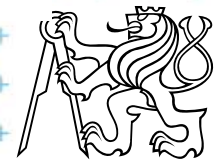
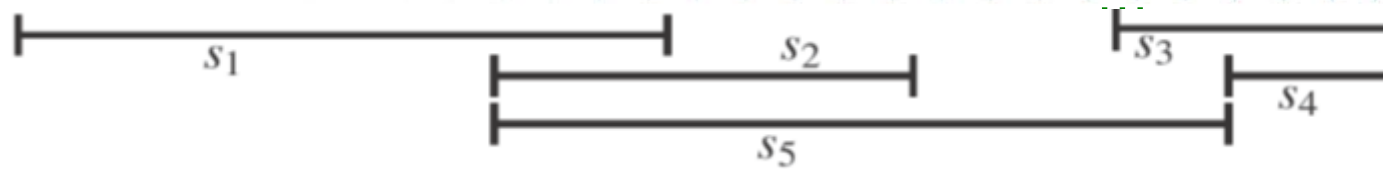
$$s_i = [x_i : x_i']$$



Elementary Intervals



Intervals



Segment tree definition

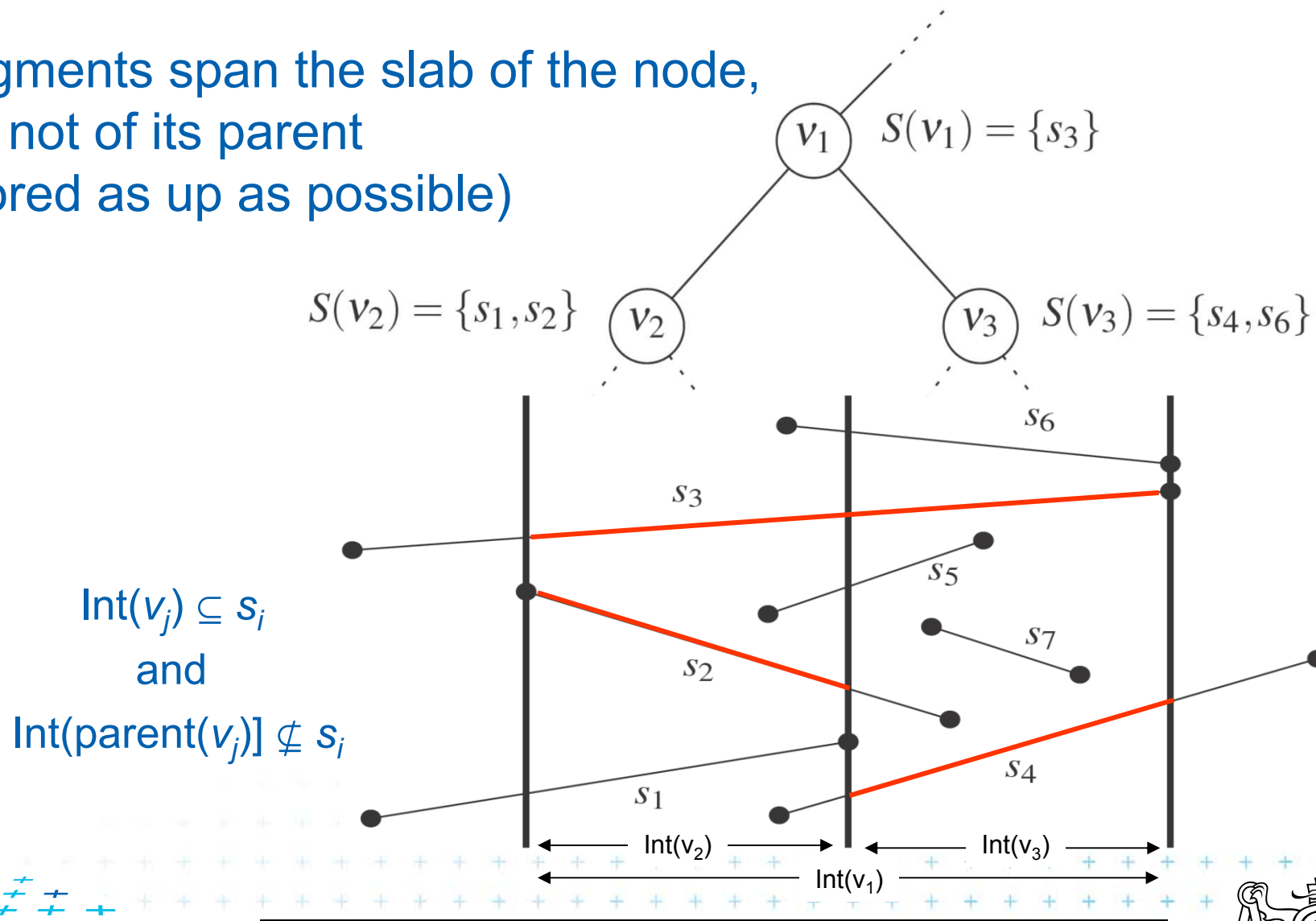
Segment tree

- Skeleton is a balanced binary tree T
- Leaves \sim elementary intervals $\text{Int}(v)$
- Internal nodes v
 - \sim union of elementary intervals of its children
 - Store: 1. interval $\text{Int}(v)$ = union of elementary intervals of its children segments s_i
 - 2. canonical set $S(v)$ of intervals $[x : x'] \in S$
 - Holds $\text{Int}(v) \subseteq [x : x']$ and $\text{Int}(\text{parent}(v)) \not\subseteq [x : x']$
(node interval is not larger than a segment)
 - Intervals $[x : x']$ are stored as high as possible, such that $\text{Int}(v)$ is completely contained in the segment



Segments span the slab

Segments span the slab of the node,
but not of its parent
(stored as up as possible)



Query segment tree

QuerySegmentTree(v, q_x)

Input: The root of a (subtree of a) segment tree and a query point q_x

Output: All **intervals** in the tree containing q_x .

1. Report all the **intervals** s_i in $S(v)$. // current node
2. **if** v is not a leaf
3. **if** $q_x \in \text{Int}(lc(v))$ // go left
4. QuerySegmentTree($lc(v), q_x$)
5. **else** // or go right
6. QuerySegmentTree($rc(v), q_x$)

Query time $O(\log n + k)$, where k is the number of reported **intervals**

Height $O(\log n)$, $O(1 + k_v)$ for node

Storage $O(n \log n)$



Segment tree construction

ConstructSegmentTree(S)

Input: Set of **intervals** S - **segments**

Output: segment tree

1. Sort endpoints of **segments** in S -> get **elementary intervals** ... $O(n \log n)$
2. Construct a binary search tree T on elementary intervals ... $O(n)$
(bottom up) and determine the interval $\text{Int}(v)$ it represents
3. Compute the canonical subsets for the nodes (lists of their segments):
4. $v = \text{root}(T)$
5. for all **segments** $s_i = [x : x'] \in S$
6. $\text{InsertSegmentTree}(v, [x : x'])$



Segment tree construction – interval insertion

InsertSegmentTree(v , $[x : x']$)

Input: The root of (a subtree of) a segment tree and an **interval**.

Output: The **interval** will be stored in the subtree.

1. **if** $\text{Int}(v) \subseteq [x : x']$ // $\text{Int}(v)$ contains $s_i = [x : x']$
2. store $[x : x']$ at v
3. **else if** $\text{Int}(lc(v)) \cap [x : x'] \neq \emptyset$
4. **InsertSegmentTree**($lc(v)$, $[x : x']$)
5. **if** $\text{Int}(rc(v)) \cap [x : x'] \neq \emptyset$
6. **InsertSegmentTree**($rc(v)$, $[x : x']$)

One **interval** is stored at most twice in one level =>

Single **interval** insert $O(\log n)$

Construction total $O(n \log n)$



Segment tree complexity

A segment tree for set S of n intervals in the plane,

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log n)$
 - Report all intervals that contain a query point
 - k is number of reported intervals



Segment tree versus Interval tree

- Segment tree

- $O(n \log n)$ storage x $O(n)$ of Interval tree
- But returns exactly the intersected segments s_i , interval tree must search the lists ML and/or MR

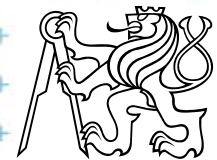
- Good for

1. extensions (allows different structuring of intervals)
2. stabbing counting queries
 - store number of intersected intervals in nodes
 - $O(n)$ storage and $O(\log n)$ query time = optimal
3. higher dimensions – multilevel segment trees
(Interval and priority search trees do not exist in \wedge dims)



Talk overview

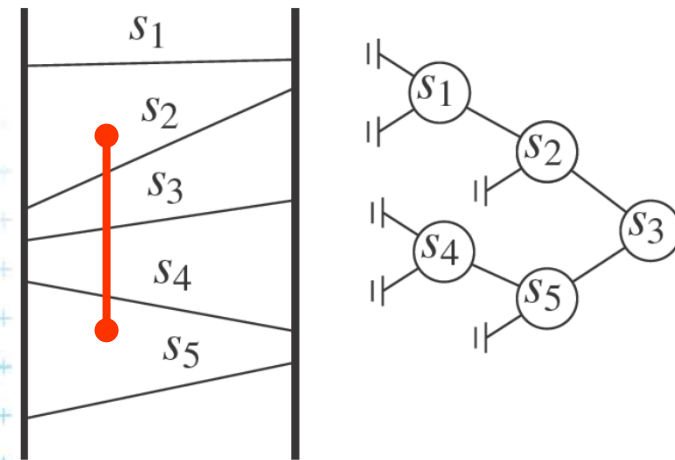
- Windowing intro
- Windowing of **axis parallel** line segments in 2D (interval tree - IT)
 - **Opakování**) Line stabbing (*interval tree* with sorted lists)
 - **A)** Line segment stabbing (*IT* with *range trees*)
 - **B)** Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in **general position**
 - *segment tree*
 - windowing of segments in general position



Windowing of arbitrary oriented line segments

- Let S be a set of arbitrarily oriented line segments in the plane.
- Report the segments intersecting a vertical query segment $q := q_x \times [q_y : q'_y]$
- Segment tree T on x intervals of segments in S
 - node v of T corresponds to vertical slab $\text{Int}(v) \times (-\infty : \infty)$
 - segments span the slab of the node, but not of its parent
 - segments do not intersect

=> segments in the slab (node) can be vertically ordered – BST



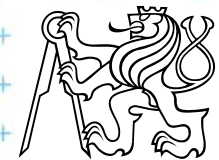
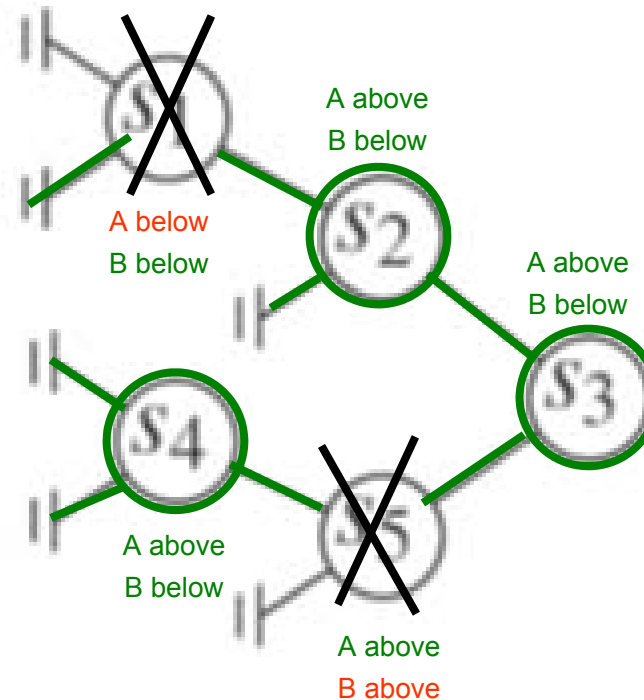
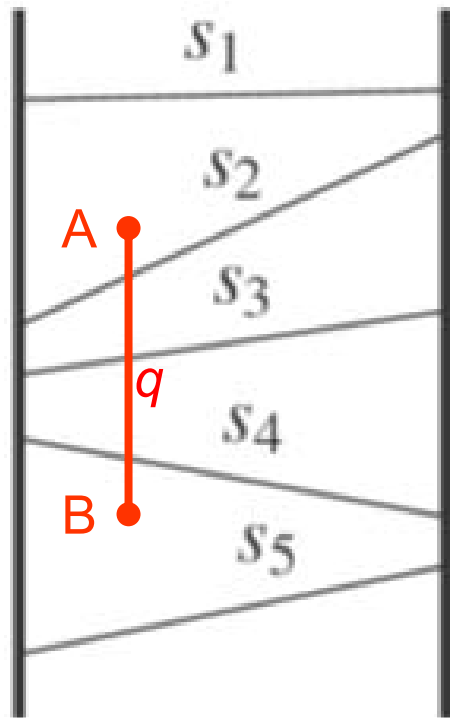
Segments between vertical segment endpoints

- Segments (in the slab) do not mutually intersect
 - => segments can be vertically ordered and stored in BST
 - Each node v of the x segment tree has an associated y BST
 - BST $T(v)$ of node v stores the canonical subset $S(v)$ according to the vertical order
 - Intersected segments can be found by searching $T(v)$ in $O(k_v + \log n)$, k_v is the number of intersected segments



Segments between vertical segment endpoints

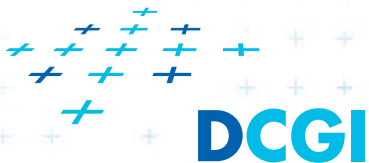
- Segment s is intersected by vert.query segment q iff
 - The lower endpoint (B) of q is below s and
 - The upper endpoint (A) of q is above s



Windowing complexity

Structure associated to node (BST) uses storage linear in the size of $S(v)$

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log^2 n)$
 - Report all segments that contain a query point
 - k is number of reported segments



References

- [Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: **Computational Geometry: Algorithms and Applications**, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapters 3 and 9, <http://www.cs.uu.nl/geobook/>
- [Mount] David Mount, - **CMSC 754: Computational Geometry**, Lecture Notes for Spring 2007, University of Maryland, Lectures 7,22, 13,14, and 30.
<http://www.cs.umd.edu/class/spring2007/cmsc754/lectures.shtml>
- [Rourke] Joseph O'Rourke: **Computational Geometry in C**, Cambridge University Press, 1993, ISBN 0-521- 44592-2
<http://maven.smith.edu/~orourke/books/compgeom.html>
- [Vigneron] **Segment trees and interval trees**, presentation, INRA, France,
<http://w3.jouy.inra.fr/unites/miaj/public/vigneron/cs4235/slides.html>
- [Schirra] **Stefan Schirra. Geometrische Datenstrukturen. Sommersemester 2009** <http://www.isg.cs.uni-magdeburg.de/ag/lehre/SS2009/GDS/slides/S10.pdf>





**OI-OPPA. European Social Fund
Prague & EU: We invest in your future.**
