# ▶Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{d}_s \text{ such that } \quad -\sum_{r=1}^{k} \mathbf{L}_r^\top \nu_r(\theta^s) = \left(\sum_{r=1}^{k} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag} \mathbf{L}_r^\top \mathbf{L}_r\right) \mathbf{d}_s$$

This is a linear set of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

- $\mathbf{A}$ is very large — approx. $3 \cdot 10^4 \times 3 \cdot 10^4$ for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$ is sparse and symmetric, $\mathbf{A}^{-1}$ is dense — direct matrix inversion is prohibitive

> Choleski: Every symmetric positive definite matrix $\mathbf{A}$ can be decomposed to $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is lower triangular. If $\mathbf{A}$ is sparse then $\mathbf{L}$ is sparse, too.

1. decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ — transforms the problem to solving $\mathbf{L}\underbrace{\mathbf{L}^\top\mathbf{x}}_{\mathbf{c}} = \mathbf{b}$
2. solve for $\mathbf{x}$ in two passes:

$$\mathbf{L}\mathbf{c} = \mathbf{b} \qquad \mathbf{c}_i := \mathbf{L}_{ii}^{-1}\left(\mathbf{b}_i - \sum_{j<i} \mathbf{L}_{ij}\mathbf{c}_j\right) \qquad \text{forward substitution, } i = 1, \ldots, q$$

$$\mathbf{L}^\top\mathbf{x} = \mathbf{c} \qquad \mathbf{x}_i := \mathbf{L}_{ii}^{-1}\left(\mathbf{c}_i - \sum_{j>i} \mathbf{L}_{ji}\mathbf{x}_j\right) \qquad \text{back-substitution}$$

- Choleski decomposition is fast (does not touch zero blocks)
  non-zero elements are $9p + 121c + 66pc \approx 3.4 \cdot 10^6$; ca. $250\times$ fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse $\mathbf{A}$ and diagonal pivoting for semi-definite $\mathbf{A}$ [Triggs et al. 1999]
- $\lambda$ controls the definiteness

# Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%    L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%    for sparse square symmetric positive definite matrix A,
%    especially useful for arrowhead sparse matrices.

 [p,q] = size(A);
 if p ~= q, error 'Matrix must be square'; end

 L = sparse(q,q);
 F = ones(q,1);
 for i=1:q
  F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
  for j = F(i):i-1
   k = max(F(i),F(j));
   a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
   L(i,j) = a/L(j,j);
  end
  a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
  if a < 0, error 'Matrix must be positive definite'; end
  L(i,i) = sqrt(a);
 end
end
```

# ▶Gauge Freedom

1. The external frame is not fixed: See Projective Reconstruction Theorem, Slide 124

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.

    - $\mathbf{P}$ is 12 numbers for 11 parameters
    - we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
    - but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

## As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

## Solutions

- fixing the external frame (e.g. a selected camera frame) explicitly or by constraints
- imposing constraints on projective entities
    - cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
    - points, e.g. $\|\underline{\mathbf{X}}_i\|^2 = 1$ — this way we can represent points at infinity
- using minimal representations
    - points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
    - rotation matrix can be represented by Cayley transform see next

# ▶Minimal Representations for Rotation

- $\mathbf{o}$ – rotation axis, $\|\mathbf{o}\| = 1$, $\varphi$ – rotation angle
- **wanted**: simple mapping to/from rotation matrices

1. Rodrigues' representation

$$\mathbf{R} = \mathbf{I} + \sin\varphi[\mathbf{o}]_\times + (1-\cos\varphi)[\mathbf{o}]_\times^2$$

$$\sin\varphi\,[\mathbf{o}]_\times = \frac{1}{2}(\mathbf{R}-\mathbf{R}^\top), \quad \cos\varphi = \frac{1}{2}(\operatorname{tr}\mathbf{R} - 1)$$

   - hiding $\varphi$ in the vector $\mathbf{o}$ as in $[\sin\varphi\,\mathbf{o}]_\times$ is not so easy
   - Cayley tried:

2. Cayley's representation; let $\mathbf{a} = \mathbf{o}\tan\frac{\varphi}{2}$, then

$$\mathbf{R} = (\mathbf{I} + [\mathbf{a}]_\times)(\mathbf{I} - [\mathbf{a}]_\times)^{-1}$$

$$[\mathbf{a}]_\times = (\mathbf{R}+\mathbf{I})^{-1}(\mathbf{R}-\mathbf{I})$$

$$\mathbf{a}_1 \circ \mathbf{a}_2 = \frac{\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_1 \times \mathbf{a}_2}{1 - \mathbf{a}_1^\top \mathbf{a}_2}$$

composition of rotations $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2$

   - no trigonometric functions
   - cannot represent rotation by $180^\circ$
   - explicit composition formula

3. exponential map $\mathbf{R} = \exp[\varphi\,\mathbf{o}]_\times$, inverse by Rodrigues' formula

# Minimal Representations for Other Entities

1. with the help of rotation we can minimally represent
   - fundamental matrix

     $$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top, \quad \mathbf{D} = \mathrm{diag}(d, 1, 0), \quad \mathbf{U}, \mathbf{V} \text{ are rotations}, \quad 3 + 1 + 3 = 7 \text{ DOF}$$

   - essential matrix

     $$\mathbf{E} = [-\mathbf{t}]_\times \mathbf{R}, \quad \mathbf{R} \text{ is rotation}, \quad \|\mathbf{b}\| = 1, \qquad 3 + 2 = 5 \text{ DOF}$$

   - camera

     $$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}, \qquad 5 + 3 + 3 = 11 \text{ DOF}$$

2. homography can be represented via exponential map

$$\exp \mathbf{A} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \qquad \text{note: } \mathbf{A}^0 = \mathbf{I}$$

some properties

$$\exp \mathbf{0} = \mathbf{I}, \quad \exp(-\mathbf{A}) = \left(\exp \mathbf{A}\right)^{-1}, \quad \exp(\mathbf{A} + \mathbf{B}) \neq \exp(\mathbf{A}) \exp(\mathbf{B})$$

$\exp(\mathbf{A}^\top) = (\exp \mathbf{A})^\top$ hence if $\mathbf{A}$ antisymmetric then $\exp \mathbf{A}$ orthogonal

$$\left(\exp(\mathbf{A})\right)^\top = \exp(\mathbf{A}^\top) = \exp(-\mathbf{A}) = \left(\exp(\mathbf{A})\right)^{-1}$$

$\det \exp \mathbf{A} = \exp(\mathrm{tr}\, \mathbf{A})$ a key to homography representation:

$$\mathbf{H} = \exp \mathbf{Z} \text{ such that } \mathrm{tr}\, \mathbf{Z} = 0, \text{ eg. } \mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & -(z_{11} + z_{22}) \end{bmatrix}, \quad 8 \text{ DOF}$$

# ▶Implementing Simple Constraints

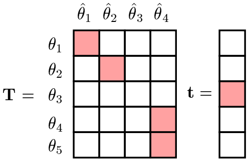
## What for?

1. fixing external frame $\rightarrow \theta_i = \theta_i^0$ 'trivial gauge'
2. representing additional knowledge $\rightarrow \theta_i = \theta_j$ e.g. cameras share calibration matrix $\mathbf{K}$

We introduce reduced parameters $\hat{\theta}$:

$$\theta = \mathbf{T}\,\hat{\theta} + \mathbf{t}, \quad \mathbf{T} \in \mathbb{R}^{p,\hat{p}}, \quad \hat{p} \leq p$$

Then $\mathbf{L}_r$ in LM changes to $\mathbf{L}_r\,\mathbf{T}$ and everything else stays the same




| these $\mathbf{T}$, $\mathbf{t}$ represent | |
|---|---|
| $\theta_1 = \hat{\theta}_1$ | no change |
| $\theta_2 = \hat{\theta}_2$ | no change |
| $\theta_3 = t_3$ | constancy |
| $\theta_4 = \theta_5 = \hat{\theta}_4$ | equality |

- $\mathbf{T}$ deletes columns of $\mathbf{L}_r$ that correspond to fixed parameters it reduces the problem size
- consistent initialisation: $\theta^0 = \mathbf{T}\,\hat{\theta}^0 + \mathbf{t}$
  or filter the initialization by pseudoinverse $\theta^0 \mapsto \mathbf{T}^{\dagger}\theta^0$
- we need not compute derivatives for $\theta_j$ that correspond to all-zero rows $\mathbf{T}_j$
  fixed params

- constraining projective entities $\rightarrow$ minimal representations
- more complex constraints tend to make normal equations dense
- implementing constraints is safer than explicit renaming of the parameters, gives a flexibility to experiment
- other methods are much more involved, see [Triggs et al. 1999]
- **BA resource:** `http://www.ics.forth.gr/~lourakis/sba/`

# Part VII

## Stereovision



**mostly covered by**

Šára, R. How To Teach Stereoscopic Vision. Proc. ELMAR 2010 referenced as [SP]

**additional references**


C. Geyer and K. Daniilidis. Conformal rectification of omnidirectional stereo pairs. In *Proc Computer Vision and Pattern Recognition Workshop*, p. 73, 2003.

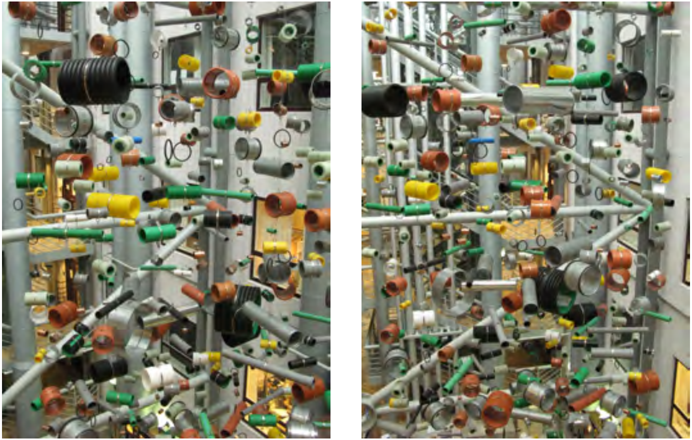
J. Gluckman and S. K. Nayar. Rectifying transformations that minimize resampling effects. In *Proc IEEE CS Conf on Computer Vision and Pattern Recognition*, vol. 1:111–117. 2001.

M. Pollefeys, R. Koch, and L. V. Gool. A simple and efficient rectification method for general motion. In *Proc Int Conf on Computer Vision*, vol. 1:496–501, 1999.

# What Are The Relative Distances?

- monocular vision already gives a rough 3D sketch because we understand the scene

# What Are The Relative Distances?



Centrum för teknikstudier at Malmö Högskola, Sweden

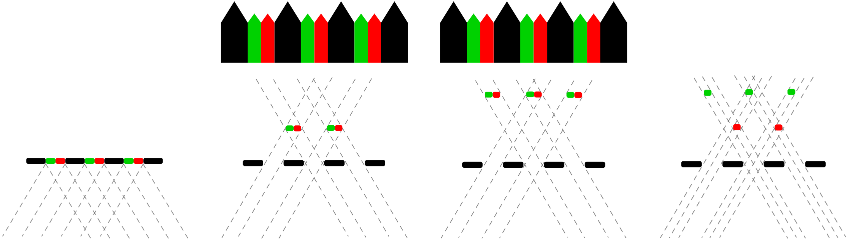
- we have no help from image interpretation here
- this is how difficult is low-level stereo we will attempt to solve

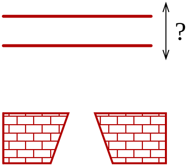
# What Are The Relative Distances? (Why?)

- a combination of lack of texture and occlusion $\longrightarrow$ ambiguous interpretation

# Repetition: How Many Scenes Correspond to a Stereopair?

Consider the fence and the fortress worlds . . .



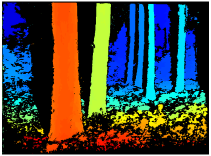



- lack of texture is a limiting case of repetition

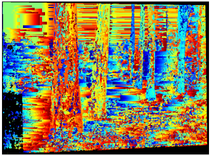
# How Difficult Is Stereo?

- when we do not recognize the scene and cannot use high-level constraints the problem seems difficult (right, less so in the center)
- most stereo matching algorithms do not require scene understanding prior to matching
- the success of a model-free stereo matching algorithm is unlikely:

left image



disparity map



disparity map from WTA

**WTA Matching:**

- for every left-image pixel find the most similar right-image pixel along the corresponding epipolar line [Marroquin 83]

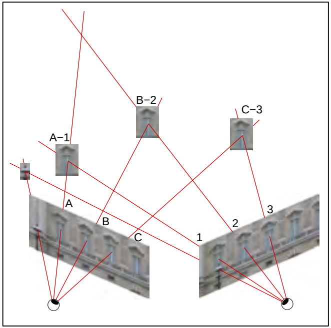
# Why Model-Free Stereo Fails?

- lack of an occlusion model
- lack of a continuity model

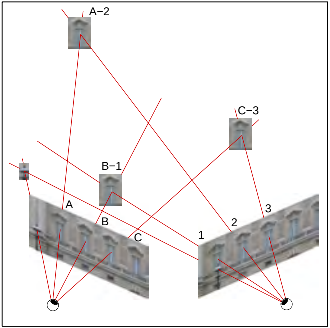
$\Rightarrow$ structural ambiguity

left image

right image

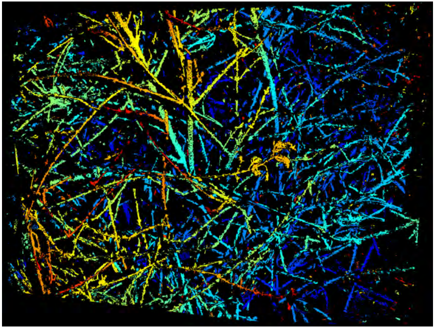



interpretation 1




interpretation 2

# But What Kind of Continuity Model Applies Here?



- continuity alone is not a sufficient model
- occlusion model is more primal
- but occlusion model alone is insufficient, since it does not solve structural ambiguity

# A Summary of Our Observations and an Outlook

- simple matching algorithms do not work
- decisions on matches are not independent due to occlusions

  occlusion constraint works along epipolars only
- occlusion model alone is insufficient — does not resolve the structural ambiguity
- a continuity model can resolve structural ambiguity

  but continuity is piecewise due to object boundaries
- in sufficiently complex scenes the only possibility is that stereopsis uses scene interpretation (or another-modality measurement)

**Outlook:**

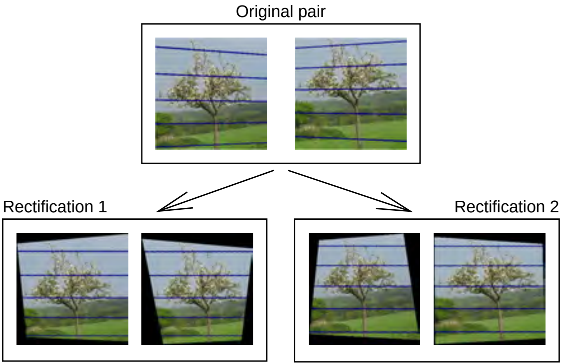
1. represent the occlusion constraint:
   - epipolar rectification
   - disparity
   - uniqueness as an occlusion constraint
2. represent piecewise continuity
   - ordering as a weak continuity model
3. use a consistent framework
   - looking for the most probable solution (MAP)

# ▶Epipolar Rectification

**Problem:** Given fundamental matrix $\mathbf{F}$ or camera matrices $\mathbf{P}_1$, $\mathbf{P}_2$, transform images so that epipolar lines become horizontal with the same row coordinate. The result is a standard stereo pair. for easier correspondence search

**Procedure:**

1. find a pair of rectification homographies $\mathbf{H}_1$ and $\mathbf{H}_2$.
2. warp images using $\mathbf{H}_1$ and $\mathbf{H}_2$ and modify fundamental matrix $\mathbf{F} \mapsto \mathbf{H}_2^{-\top} \mathbf{F} \mathbf{H}_1^{-1}$ or cameras $\mathbf{P}_1 \mapsto \mathbf{H}_1 \mathbf{P}_1$, $\mathbf{P}_2 \mapsto \mathbf{H}_2 \mathbf{P}_2$.




- there is a 9-parameter family of rectification homographies for binocular rectification, see next

# Rectification Example

Four cameras in general position

cam 1

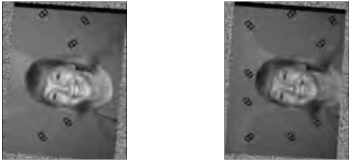
cam 2

cam 3

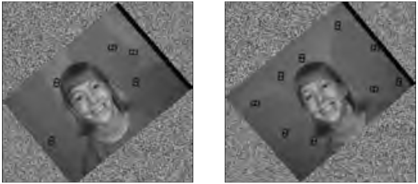
cam 4

Rectified pairs

pair 1 – 2



pair 2 – 4



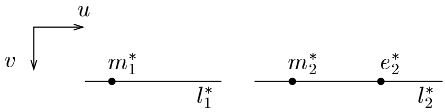
pair 1 – 4

# ▶Rectification Homographies

Cameras $(\mathbf{P}_1, \mathbf{P}_2)$ are rectified by a homography pair $(\mathbf{H}_1, \mathbf{H}_2)$:

$$\mathbf{P}_i^* = \mathbf{H}_i \mathbf{P}_i = \mathbf{H}_i \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix}, \quad i = 1, 2$$

rectified entities: $\mathbf{F}^*$, $\underline{\mathbf{l}}_2^*$, $\underline{\mathbf{l}}_1^*$, etc:




**corresponding epipolar lines must be:**

1. parallel to image rows $\Rightarrow$ epipoles become $e_1^* = e_2^* = (1, 0, 0)$
2. equivalent $l_2^* = l_1^*$ $\Rightarrow$ $\underline{\mathbf{l}}_2^* \simeq \underline{\mathbf{l}}_1^* \simeq \underline{\mathbf{e}}_1^* \times \underline{\mathbf{m}}_1 = [\underline{\mathbf{e}}_1^*]_\times \underline{\mathbf{m}}_1 = \mathbf{F}^* \underline{\mathbf{m}}_1$

both conditions together give the rectified fundamental matrix

$$\mathbf{F}^* \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

**A two-step rectification procedure**

1. Find some pair of primitive rectification homographies $\hat{\mathbf{H}}_1$, $\hat{\mathbf{H}}_2$
2. Upgrade them to a pair of optimal rectification homographies from the class preserving $\mathbf{F}^*$.

# ▶Geometric Interpretation of Linear Rectification

What pair of cameras is compatible with $\mathbf{F}^*$?

- we know that $\mathbf{F} = (\mathbf{Q}_1\mathbf{Q}_2^{-1})^\top [\underline{\mathbf{e}}_1]_\times$ Slide 77
- we choose $\mathbf{Q}_1^* = \mathbf{K}_1^*$, $\mathbf{Q}_2^* = \mathbf{K}_2^*\mathbf{R}^*$; then

$$(\mathbf{Q}_1^*\mathbf{Q}_2^{*\,-1})^\top [\underline{\mathbf{e}}_1^*]_\times = (\mathbf{K}_1^*\mathbf{R}^{*\top}\mathbf{K}_2^{*\,-1})^\top \mathbf{F}^*$$

- we look for $\mathbf{R}^*$, $\mathbf{K}_1^*$, $\mathbf{K}_2^*$ compatible with

$$(\mathbf{K}_1^*\mathbf{R}^{*\top}\mathbf{K}_2^{*\,-1})^\top \mathbf{F}^* = \lambda \mathbf{F}^*, \qquad \mathbf{R}^*\mathbf{R}^{*\top} = \mathbf{I}, \qquad \mathbf{K}_1^*, \mathbf{K}_2^* \text{ upper triangular}$$

- we also want $\mathbf{b}^*$ from $\underline{\mathbf{e}}_1^* \simeq \mathbf{P}_1^*\underline{\mathbf{C}}_2^* = \mathbf{K}_1^*\mathbf{b}^*$ — $\mathbf{b}^*$ in cam. 1 frame
- result:

$$\mathbf{R}^* = \mathbf{I}, \quad \mathbf{b}^* = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_1^* = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2^* = \begin{bmatrix} k_{21} & k_{22} & k_{23} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

- rectified cameras are in canonical position with respect to each other

  not rotated, canonical baseline

- rectified calibration matrices can differ in the first row only
- when $\mathbf{K}_1^* = \mathbf{K}_2^*$ then the rectified pair is called the standard stereo pair and the homographies standard rectification homographies

Thank You