▶Three-Point Exterior Orientation Problem (P3P)

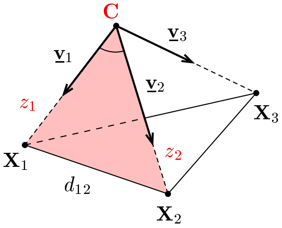
Calibrated camera rotation and translation from Perspective images of 3 reference Points.

Problem: Given $\mathbf{K}$ and three corresponding pairs $\left\{(m_i,\, X_i)\right\}_{i=1}^{3}$, find $\mathbf{R}$, $\mathbf{C}$ by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}\,(\mathbf{X}_i - \mathbf{C}), \qquad i = 1, 2, 3$$

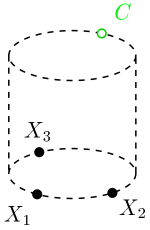
1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then
$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}\,(\mathbf{X}_i - \mathbf{C}). \tag{9}$$
2. Eliminate $\mathbf{R}$ by taking rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$
$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \tag{10}$$
3. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3,\ i \neq j$
$$d_{ij}^2 = z_i^2 + z_j^2 - 2\, z_i\, z_j\, c_{ij},$$
$$z_i = \|\mathbf{X}_i - \mathbf{C}\|,\ d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|,\ c_{ij} = \cos(\angle \underline{\mathbf{v}}_i\, \underline{\mathbf{v}}_j)$$
4. Solve system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]
there may be no real root; there are up to 4 solutions that cannot be ignored
(verify on additional points)
5. Compute $\mathbf{C}$ by trilateration (3-sphere intersection) from $\mathbf{X}_i$ and z_i, λ_i from (10) and $\mathbf{R}$ from (9)

configuration w/o rotation



Similar problems (P4P with unknown f) at http://cmp.felk.cvut.cz/minimal/ (with code)

Degenerate (Critical) Configurations for Exterior Orientation



unstable solution

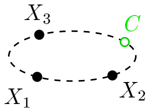
- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3)

unstable: a small change of X_i results in a large change of C

can be detected by error propagation



no solution

1. C cocyclic with (X_1, X_2, X_3)

- additional critical configurations depend on the method to solve the quadratic equations

[Haralick et al. IJCV 1994]

►Populating A Little ZOO of Minimal Geometric Problems in CV

problem	**given**	**unknown**	slide
resectioning	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{6}$	$\mathbf{P}$	65
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{C}$	69

- resectioning and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- more problems to come

Part IV

Computing with a Camera Pair

4. Camera Motions Inducing Epipolar Geometry
5. Estimating Fundamental Matrix from 7 Correspondences
6. Estimating Essential Matrix from 5 Correspondences
7. Triangulation: 3D Point Position from a Pair of Corresponding Points
8. Camera Motions Inducing Homographies
9. Estimating Relative Homography from Correspondences

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

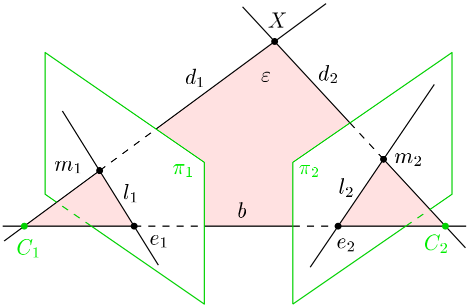
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: d_2, b, d_1 are coplanar

a necessary condition, see also Slide 87

▶Cross Products and Maps by Antisymmetric 3×3 Matrices

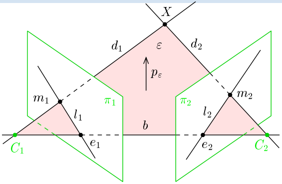
- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 antisymmetric matrix

$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F^2 = \sum_{i,j} |a_{ij}|^2$)
3. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$
4. $\operatorname{rank}[\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
5. if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}\,[\mathbf{b}]_\times \mathbf{R}^\top$
6. $[\mathbf{B}\mathbf{z}]_\times \simeq \mathbf{B}^{-\top}[\mathbf{z}]_\times \mathbf{B}^{-1}$ — in general, $[\mathbf{A}^{-1}\mathbf{t}]_\times \cdot \det \mathbf{A} = \mathbf{A}^\top [\mathbf{t}]_\times \mathbf{A}$
7. if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $[\mathbf{R}_b \mathbf{b}]_\times = [\mathbf{b}]_\times$

▶Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix}, \quad i = 1,2$$

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2\mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\mathbf{t}_{21} = \mathbf{R}_{21}\mathbf{t}_1 - \mathbf{t}_2 = \mathbf{R}_2\mathbf{b}$

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top\mathbf{t}$ (Slides 30 and 32)

$$0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \underline{\mathbf{l}}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top(\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \underbrace{\left(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times\right)}_{\text{fundamental matrix } \mathbf{F}} \underline{\mathbf{m}}_1$$

Epipolar constraint $\underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1 = 0$ is a point-line incidence constraint

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2$
- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top\underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1\mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1\mathbf{C}_2 - \mathbf{Q}_1\mathbf{C}_1 = \mathbf{K}_1\mathbf{R}_1\mathbf{b}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\mathbf{K}_1\mathbf{R}_1\mathbf{b}]_\times = \overset{\circledast 1}{\cdots} = \mathbf{K}_2^{-\top} \underbrace{[\mathbf{t}_{21}]_\times \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}} \mathbf{K}_1^{-1}$$ Slide 74

$$\mathbf{E} = [\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2\mathbf{b}]_\times}_{\text{baseline in Cam 1}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1\mathbf{b}]_\times}_{\text{baseline in Cam 2}}$$

Epipole is the Image of the Other Camera

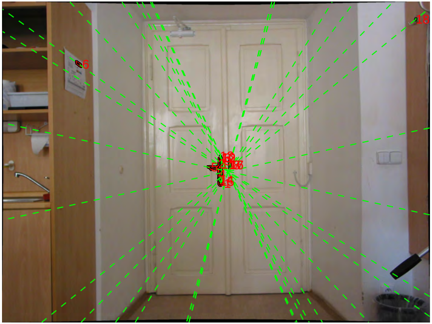


image 1

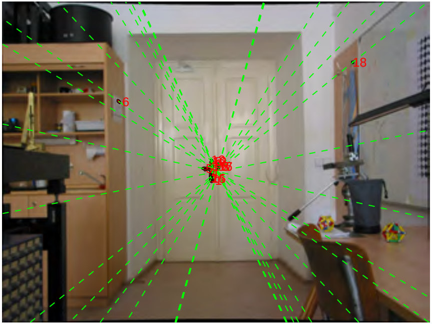
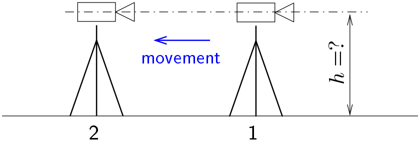
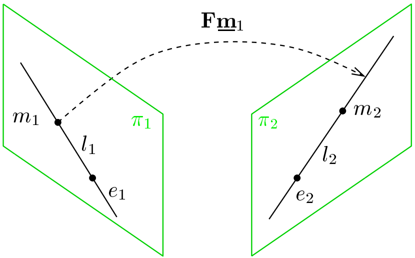


image 2

Camera moved horizontally: How high is it above floor?



►A Summary of the Epipolar Constraint



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

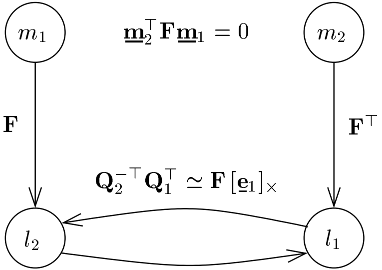
$$\mathbf{F} \simeq \mathbf{K}_2^{-\top} \mathbf{E}\, \mathbf{K}_1^{-1}$$

$$\mathbf{E} \simeq [\mathbf{t}_{21}]_\times \mathbf{R}_{21} = [\mathbf{R}_2 \mathbf{b}]_\times \mathbf{R}_{21} = \mathbf{R}_{21} [\mathbf{R}_1 \mathbf{b}]_\times$$

$$\underline{\mathbf{e}}_1 \simeq \mathrm{null}(\mathbf{F}), \quad \underline{\mathbf{e}}_2 \simeq \mathrm{null}(\mathbf{F}^\top)$$

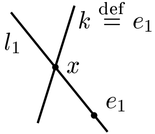
- $\mathbf{E}$ captures the relative pose
- the translation length $\mathbf{t}_{21}$ is lost

$\mathbf{E}$ is homogeneous



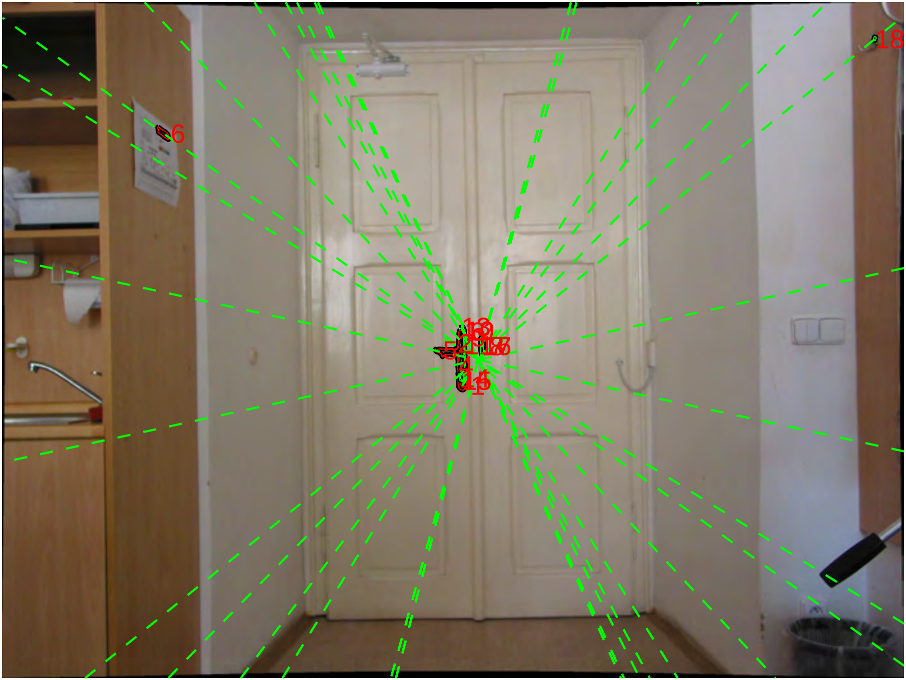
proof of $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\, [\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1$: line/point transmutation

$$\underline{\mathbf{l}}_2 \simeq \mathbf{F}\, \underline{\mathbf{x}} \simeq \mathbf{F}(\underline{\mathbf{k}} \times \underline{\mathbf{l}}_1) = \mathbf{F}\, [\underline{\mathbf{k}}]_\times \underline{\mathbf{l}}_1 = \mathbf{F}\, [\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1$$



$x \neq e_1$, $e_1 \notin k$: $\underline{\mathbf{k}}^\top \underline{\mathbf{e}}_1 = \|\underline{\mathbf{e}}_1\|^2 \neq 0$

Thank You


6
18