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AE4M33RZN, Fuzzy logic: Fuzzy relations

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Organizational:

- Next week, there will be a short test (max 5 points).
- This week we are having the last **theoretical lecture**.

Fuzzy implication

We already know *fuzzy negation* $\underset{\circ}{\neg}$, *fuzzy conjunction* $\underset{\circ}{\wedge}$ and *fuzzy discjunction* $\overset{\circ}{\vee}$. What about other operators?

Definition

Fuzzy implication is any function

$$\underset{\circ}{\overset{\circ}{\Rightarrow}} : [0,1]^2 \to [0,1] \tag{1}$$

which overlaps with the boolean implication on $x, y \in \{0,1\}$:

$$(x \underset{\circ}{\overset{\circ}{\Rightarrow}} y) = (x \Rightarrow y) \,. \tag{2}$$

Residue implication

Despite the lack of a uniform definition of fuzzy implication, there is a useful class of implications:

Defintion

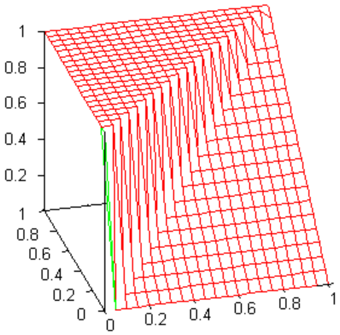
The ***R-implication*** (residuum, „***reziduovaná implikace***“) is a function obtained from a fuzzy T-norm:

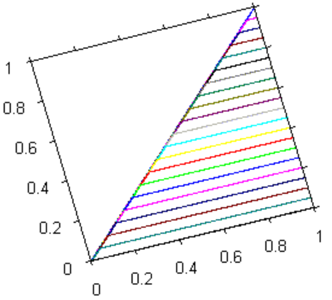
$$\alpha \overset{\mathrm{R}}{\underset{\circ}{\Rightarrow}} \beta = \mathbf{sup}\{\gamma \mid \alpha \underset{\circ}{\wedge} \gamma \leqslant \beta\} \tag{RI}$$

R-implication: Examples (1)

Standard implication (Gödel) is derived from (RI) using the standard cojunction $\underset{S}{\wedge}$:

$$\alpha \overset{R}{\underset{S}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ \beta & \text{otherwise} \end{cases} \tag{3}$$

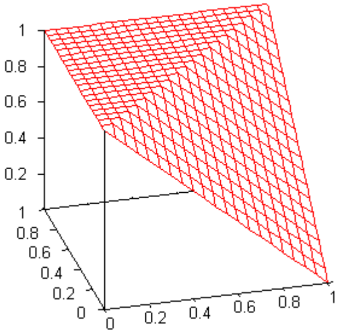


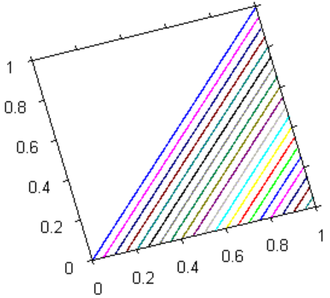


R-implication: Examples (2)

Łukasiewicz implication is derived from (RI) using the Łukasiewicz cojunction $\underset{\mathrm{L}}{\wedge}$:

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ 1 - \alpha + \beta & \text{otherwise} \end{cases} \tag{4}$$

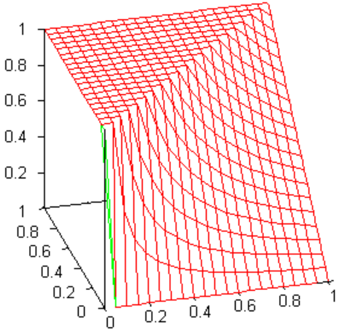


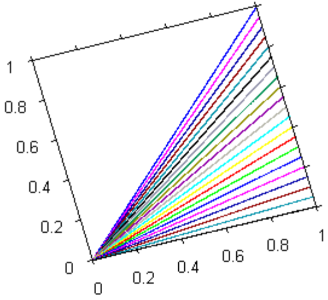


R-implication: Examples (3)

Algebraic implication (Gougen, Gaines) is derived from (RI) using the algebraic cojunction $\underset{\mathrm{A}}{\wedge}$:

$$\alpha \underset{\mathrm{A}}{\overset{\mathrm{R}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases} \quad (5)$$





R-implication: Properties

Theorem 207.

Let $\underset{\circ}{\wedge}$ be a continuous fuzzy conjunction. Then R-implication satisfies:

$$\alpha \underset{\circ}{\overset{\mathrm{R}}{\Rightarrow}} \beta = 1 \text{ iff } \alpha \leqslant \beta \tag{I1}$$

$$1 \underset{\circ}{\overset{\mathrm{R}}{\Rightarrow}} \beta = \beta \tag{I2}$$

$$\alpha \underset{\circ}{\overset{\mathrm{R}}{\Rightarrow}} \beta \text{ is not increasing in } \alpha \text{ and not decreasing in } \beta \tag{I3}$$

R-implication: Properties

Proof of theorem 207.

Let's denote $\{\gamma \mid \alpha \underset{\circ}{\wedge} \gamma \leqslant \beta\} = \gamma$.

- Proving (I3) uses monotonicity: Increasing α can only shrink γ and increasing β can only enlarge γ.
- Proving (I2) is easy: $1 \underset{\circ}{\overset{R}{\Rightarrow}} \beta = \sup\{\gamma \mid 1 \underset{\circ}{\wedge} \gamma \leqslant \beta\}$. From definition of $\underset{\circ}{\wedge}$, we write $1 \underset{\circ}{\overset{R}{\Rightarrow}} \beta = \sup\{\gamma \mid \gamma \leqslant \beta\} = \beta$.

R-implication: Properties

Proof of theorem 207 (contd.).

- For (I1) one needs to check 2 cases:
  - If $\alpha \leqslant \beta$, then $1 \in \gamma$, because $\alpha \underset{\circ}{\wedge} 1 = \alpha \leqslant \beta$ and therefore the condition $\alpha \underset{\circ}{\wedge} \gamma \leqslant \beta$ is true for all possible values of γ.
  - If $\alpha > \beta$, then $1 \notin \gamma$, because $\alpha \underset{\circ}{\wedge} 1 = \alpha > \beta$ and therefore the condition $\alpha \underset{\circ}{\wedge} \gamma \leqslant \beta$ is false for $\gamma = 1$.

S-implication

Defintion

The ***S-implication*** is a function obtained from a fuzzy disjunction $\overset{\circ}{\vee}$:

$$\alpha \overset{S}{\underset{\circ}{\Rightarrow}} \beta = \underset{S}{\neg}\,\alpha \overset{\circ}{\vee} \beta \tag{SI}$$

Example

Kleene-Dienes implication from $\overset{S}{\vee}$

$$\alpha \overset{S}{\underset{S}{\Rightarrow}} \beta = \mathbf{max}(1 - \alpha, \beta) \tag{6}$$

Generalized fuzzy inclusion

Previously, we used the logical negation $\underset{\circ}{\neg}$ to define the set complement, the conjunction $\underset{\circ}{\wedge}$ to define the set intersection, etc.

Can we use the implication $\overset{\circ}{\underset{\circ}{\Rightarrow}}$ to define the the fuzzy inclusion?

Definition

The *generalized fuzzy inclusion* $\overset{\circ}{\underset{\circ}{\subseteq}}$ is a function that assigns a degree to the the inclusion of set $A \in \mathbb{F}(\Delta)$ in set $B \in \mathbb{F}(\Delta)$:

$$A \overset{\circ}{\underset{\circ}{\subseteq}} B = \inf\{A(x) \overset{\circ}{\underset{\circ}{\Rightarrow}} B(x) \mid x \in \Delta\} \tag{7}$$

Generalized fuzzy inclusion: Example

Fuzzy inclusion (non-generalized)

Definition

The *fuzzy inclusion* $\subseteq$ is a predicate (assigns a true/false value) which hold for two fuzzy sets $A, B \in \mathbb{F}(\Delta)$ iff

$$\mu_A(x) \leqslant \mu_B(x) \text{ for all } x \in \Delta . \tag{8}$$

Fuzzy inclusion (non-generalized)

In vertical representation, the definition has a straightforward equivalent:

$$\mu_A \leqslant \mu_B \tag{9}$$

In horizontal representation, there is a theorem:

Theorem 214.

Let $A, B \in \mathbb{F}(\Delta)$ if and only if

$$\mathrm{R}_A(\alpha) \subseteq \mathrm{R}_B(\alpha) \text{ for all } \alpha \in [0,1] \ . \tag{10}$$

Fuzzy inclusion (non-generalized)

Proof of theorem 214.

$\Rightarrow$ Assume $A \subseteq B$ and $x \in \mathrm{R}_A(\alpha)$ for some value α. If $\alpha \leqslant A(x)$, then $A(x) \leqslant B(x)$ (from the definition of $A \subseteq B$) and therefore $x \in \mathrm{R}_B(\alpha)$ and $\mathrm{R}_A(\alpha) \subseteq \mathrm{R}_B(\alpha)$.

$\Leftarrow$ Assume $\mathrm{R}_A(\alpha) \subseteq \mathrm{R}_B(\alpha)$. Firstly recall the horizontal-vertical translation formula: $\mu_A(x) = \sup\{\alpha \in [0, 1] \mid x \in \mathrm{R}_A(\alpha)\}$. Since $\{\alpha \mid x \in \mathrm{R}_A(x)\} \subseteq \{\alpha \mid x \in \mathrm{R}_B(x)\}$, the inequality $A(x) \leqslant \sup\{\alpha \mid x \in \mathrm{R}_B(x)\} \leqslant B(x)$ holds.

Cutworhiness

We ended up with 2 equal definitions of set inclusion: using vertical and horizontal representation. Can we generalize this?

Cutworhiness

Let P be a predicate (returns true/false) over fuzzy sets. P is called *cutworthy* („řezově dědičná vlastnost“) if the implication holds:

$$P(A_1, \ldots, A_n) \Rightarrow P(\mathrm{R}_{A_1}(\alpha), \ldots, \mathrm{R}_{A_n}(\alpha)) \text{ for all } \alpha \in [0, 1] \qquad (11)$$

There is a related notion: We define P as *cut-consistent* („řezově konzistentní“) using the same definition, but replacing $\Rightarrow$ with $\Leftrightarrow$.

Cutworhiness: Examples

- The theorem 214 can be stated as: “Set inclusion is cut-consistent.”

Brain teasers

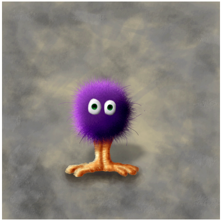
- *Strong normality* of A is defined as $A(x) = 1$ for some $x \in \Delta$.

  A is strongly-normal Strong normality is **cut-consistent**:
  iff every cut strongly normal. **iff** every its cut is non-empty

- *Being crisp* is

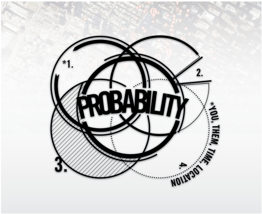
  cutworthy, but not cut-consistent:
  Every cut is crisp by definition, therefore cutworthiness.
  But even **non-crisp sets** have crisp cuts,
  therefore the property is not not cut-consistent.

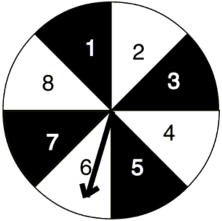
Google: “fuzzy”



Sources: M. Taylor's Weblog, M. Taylor's Weblog, Eddie's Trick Shop.

Google: "probability"





Sources: Life123, WhatWeKnowSoFar, Probability Problems.

Fuzzy vs. probability

- *Vagueness* vs. *uncertainty*.

- Fuzzy logic is *functional*.

Crisp relations

Definition

A *binary crisp relation* R from X onto Y is a subset of the cartesian product $X \times Y$:

$$R \in \mathbb{P}(X \times Y) \tag{12}$$

Definition

The *inverse relation* R^{-1} to R is a relation from Y to X s.t.

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\} \tag{13}$$

Crisp relations: Inverse

Definition

Let X, Y, Z be sets. Then the *compound* of relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$ is the relation

$$R \bigcirc S = \{(x, z) \in X \times Z \,|\, (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y\} \tag{14}$$

Crisp relations: Properties

The *equality* relation on Δ is $E = \{(x,x) \mid x \in \Delta\}$.

Then the relation $R \subseteq \Delta \times \Delta$ is called

property	**using logical connectives**	**using set axioms**
reflexive	$\forall x.\, (x,x) \in R$	$E \subseteq R$
symmetric	$(x,y) \in R \Rightarrow (y,x) \in R$	$R = R^{-1}$
anti-symmetric	$(x,y) \in R \wedge (y,z) \in R \Rightarrow y = z$	$R \cap R^{-1} \subseteq E$
transitive	$(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$	$R \bigcirc R \subseteq R$
partial order	reflexive, transitive and anti-symmetric	
equivalence	reflexive, transitive and symmetric	

Fuzzy relations

Definition

A *binary fuzzy relation* R from X onto Y is a fuzzy subset on the universe $X \times Y$.

$$R \in \mathbb{F}(X \times Y) \tag{15}$$

Definition

The *fuzzy inverse* relation $R^{-1} \in \mathbb{F}(Y \times X)$ to $R \in \mathbb{F}(X \times Y)$, s.t.

$$R(y, x) = R^{-1}(x, y) \tag{16}$$

Projection

Defintion

Let $R \in \mathbb{F}(X \times Y)$ be a fuzzy binary relation. The *first* and second projection of R is

$$R^{(1)}(x) = \bigvee_{y \in Y}^{S} R(x, y) \tag{17}$$

$$R^{(2)}(y) = \bigvee_{x \in X}^{S} R(x, y) \tag{18}$$

Projection: Example

R	y_1	y_2	y_3	y_4	y_5	y_6	$R^{(1)}(x)$
x_1	0.1	0.2	0.4	0.8	1	0.8	1
x_2	0.2	0.4	0.8	1	0.8	0.6	1
x_3	0.4	0.8	1	0.8	0.4	0.2	1
$R^{(2)}(y)$	0.4	0.8	1	0.8	0.4	0.2	

Sometimes there is a *total projection* defined as $R^{(T)} = \bigvee_{x \in X} \bigvee_{y \in Y} R(x, y)$. But we already know this notion as $\mathrm{Height}(R)$.

Cylindrical extension

Can we reconstruct a fuzzy relation from its projections? There is an unique largest relation with prescribed projections:

Definition

Let $A \in \mathbb{F}(X)$ and $B \in \mathbb{F}(Y)$ be fuzzy sets. The *cylindrical extension* („cylindrické rozšíření“, „kartézský součin fuzzy množin“) is defined as

$$A \times B(x, y) = A(x) \underset{S}{\wedge} B(y) \tag{19}$$

Brain teaser

Why can't there be a relation Q bigger than $A \times B$, whose projections are $Q^{(1)} = A$ and $Q^{(2)} = B$?

Cylindrical extension: Drawing

$$A(x) = \begin{cases} x - 1 & x \in [1, 2] \\ 3 - x & x \in [2, 3] \\ 0 & \text{otherwise} \end{cases} \qquad B(x) = \begin{cases} x - 3 & x \in [3, 4] \\ 5 - x & x \in [4, 5] \\ 0 & \text{otherwise} \end{cases}$$

Composition of fuzzy relations

Definition

Let X, Y, Z be crisp sets. $R \in \mathbb{F}(X \times Y)$, $S \in \mathbb{F}(Y \times Z)$ and $\underset{\circ}{\wedge}$ some fuzzy conjunction. Then the $\underset{\circ}{\bigcirc}$-composition („$\underset{\circ}{\bigcirc}$-složená relace“) is

$$R \underset{\circ}{\bigcirc} S(x, z) = \bigvee_{y \in Y}^{S} R(x, y) \underset{\circ}{\wedge} S(y, z) \tag{20}$$

1. For infinite domains, $\bigvee^{S}$ is computed using the sup instead of max.
2. Instead of the “ for some y” in *crisp relations*, the disjunction “finds such a y” that maximizes the conjunction.

Example of a fuzzy relation

$$R(x, y) = \begin{cases} x + y & x, y \in \left[0, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases} \qquad S(x, y) = \begin{cases} x \cdot y & x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Properties of fuzzy relations

Then the relation $R \subseteq \Delta \times \Delta$ is called

property	using set axioms
reflexive	$E \subseteq R$
symmetric	$R = R^{-1}$
$\circ$-anti-symmetric	$R \underset{\circ}{\cap} R^{-1} \subseteq E$
$\circ$-transitive	$R \underset{\circ}{\bigcirc} R \subseteq R$
$\circ$-partial order	reflexive, $\circ$-transitive and $\circ$-anti-symmetric
$\circ$-equivalence	reflexive, $\circ$-transitive and $\circ$-symmetric

Properties in a finite domain

If the universe Δ is finite, the relation can be written as a matrix. Their properties are reflected in the relation's matrix:

- **Reflexivity:** Cells on the main diagonal are 1.
- **Symmetricity:** Cells symmetric over the main diagonal are equal.
- **Anti-symmetricity:** Cells symmetric over the main diagonal have conjunction equal to zero.
  - For $\mathbb{S}$- and $\mathbb{A}$-anti-symmetricity, one of the elements must be zero.
  - For $\mathbb{L}$-anti-symmetricity, their sum must be less or equal to 1.
- **Transitivity:** More difficult (see example on the next slide).

Example on fuzzy relation properties

Let $\Delta = \{A, B, C, D\}$ and $R \in \mathbb{F}(\Delta \times \Delta)$.

R	A	B	C	D
A		0.5		0.1
B			0.2	
C				
D		0.2		

Fill the missing cells in the table to make R

a) $\mathbb{S}$-equivalence

b) $\mathbb{A}$-equivalence

Theorem 234.

Let R, S and T be relations (defined over sets that "make sense") The following equations hold:

$$R \underset{\circ}{\bigcirc} E = R, \quad E \underset{\circ}{\bigcirc} R = R \tag{21}$$

$$(R \underset{\circ}{\bigcirc} S)^{-1} = S^{-1} \underset{\circ}{\bigcirc} R^{-1} \tag{22}$$

$$R \underset{\circ}{\bigcirc} (S \underset{\circ}{\bigcirc} T) = (R \underset{\circ}{\bigcirc} S) \underset{\circ}{\bigcirc} T \tag{23}$$

$$(R \overset{S}{\bigcap} S) \underset{\circ}{\bigcirc} T = (R \underset{\circ}{\bigcirc} T) \underset{\circ}{\bigcirc} (S \underset{\circ}{\bigcirc} T) \tag{24}$$

$$R \underset{\circ}{\bigcirc} (S \overset{S}{\bigcap} T) = (R \underset{\circ}{\bigcirc} S) \underset{\circ}{\bigcirc} (R \underset{\circ}{\bigcirc} T) \tag{25}$$

(21) describes the *identity element*, (22) the *inverse of composition*, (23) is the *asociativity*, (24) and (25) the *right-* and *left-distributivity*.

Proof of 234.

Proving (21) and (22) is trivial.

$$"R \underset{\circ}{\bigcirc} (S \underset{\circ}{\bigcirc} T)"(x, w) = \bigvee_{y}^{S} R(x, y) \underset{\circ}{\wedge} "S \underset{\circ}{\bigcirc} T"(y, w) \tag{26}$$

$$= \bigvee_{y}^{S} R(x, y) \underset{\circ}{\wedge} \left(\bigvee_{z}^{S} S(y, z) \underset{\circ}{\wedge} T(z, w) \right) \tag{27}$$

$$= \bigvee_{y}^{S} \bigvee_{z}^{S} R(x, y) \underset{\circ}{\wedge} S(y, z) \underset{\circ}{\wedge} T(z, w) \tag{28}$$

$$= \bigvee_{z}^{S} \bigvee_{y}^{S} R(x, y) \underset{\circ}{\wedge} S(y, z) \underset{\circ}{\wedge} T(z, w) \tag{29}$$

Proof of 234 (contd.).

$$= \bigvee_z^S \bigvee_y^S R(x,y) \underset{\circ}{\wedge} S(y,z) \underset{\circ}{\wedge} T(z,w) \tag{30}$$

$$= \bigvee_z^S \left(\bigvee_y^S R(x,y) \underset{\circ}{\wedge} S(y,z) \right) \underset{\circ}{\wedge} T(z,w) \tag{31}$$

$$= \bigvee_z^S {}''R \underset{\circ}{\bigcirc} S''(x,z) \underset{\circ}{\wedge} T(z,w) \tag{32}$$

$$= {}''R \underset{\circ}{\bigcirc} S \underset{\circ}{\bigcirc} T''(x,w) \tag{33}$$

Proof of (24) and (25) is similar (uses the distributivity law), only shorter. See [Navara and Olšák, 2001] for details.

Extensions: Sometimes it is useful to consider...

- ...a *ε-reflective* relation

$$R(x, x) \geqslant \varepsilon \tag{34}$$

- ...a *weakly reflexive* relation

$$R(x, y) \leqslant R(x, x) \text{ and } R(y, x) \leqslant R(x, x) \text{ for all } x, y \tag{35}$$

- Relation is 1-reflective **iff** reflexive.
- **If** a relation is reflexive, **then** it is weakly reflexive.

Extensions: Sometimes it is useful to consider...

- ...a *non-involutive negation* by refusing (N2)

$$\underset{\circ}{\neg}\,\underset{\circ}{\neg}\,\alpha \neq \alpha$$

and adopting a weaker axiom

$$\underset{\circ}{\neg}\,\underset{\circ}{\neg}\,0 = 1 \text{ and } \underset{\circ}{\neg}\,\underset{\circ}{\neg}\,1 = 0 \tag{N0}$$

Example

Gödel negation

$$\underset{\mathrm{G}}{\neg}\,\alpha = \begin{cases} 1 & \alpha = 0 \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

Bibliography

Navara, M. and Olšák, P. (2001).
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