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GRAPHICAL MARKOV MODELS (WS2011)
1. SEMINAR

Assignment 1. Consider the following *homogeneous* Markov chain with two states $K = \{0, 1\}$.[1] The matrix of transition probabilities is defined by

$$\begin{aligned} p(k = 0 \mid k' = 0) &= 1 - \alpha \\ p(k = 1 \mid k' = 0) &= \alpha \\ p(k = 0 \mid k' = 1) &= 1 \\ p(k = 1 \mid k' = 1) &= 0 \end{aligned}$$

The p.d. for the first state in the chain is $p_1(k = 0) = \beta$ and $p_1(k = 1) = 1 - \beta$.
a) Calculate the probability $p_i(k = 0)$ for the time step i, given the corresponding probability $p_{i-1}(k = 0)$ for the preceding time step. Describe the mapping $p_{i-1}(k = 0) \mapsto p_i(k = 0)$ explicitly.
b) Deduce a formula expressing $p_i(k = 0)$ in terms of the value $p_1(k = 0) = \beta$ for the first time step. Show that the probability $p_i(k = 0)$ becomes independent of β for $i \to \infty$

Assignment 2. (*Segmentation*) Consider the following language. Each word of the language has length n, consists of characters a and b only and has the form

$$\underbrace{\overbrace{a \dots a}^{n_a} b \dots b}_{n}$$

where $0 \leqslant n_a \leqslant n$. Informally said, this model describes all segmentations of a sequence into two consecutive intervals.
a) Construct a Markov chain assigning a strictly positive probability to all admissible words and at the same time zero probability to all inadmissible words.
b) Construct a Markov chain such that all admissible words are equiprobable, whereas all other words have probability zero.
Hint: Determine the p.d. for the first state in the chain and all transition probabilities (the latter may depend on the position).

Assignment 3. A tetrahedron with differently coloured facets is laying with the blue side down on a table. The tetrahedron is tilted n times over a randomly chosen edge (At each time instance there are three edges incident with the table to choose from). What is the probability to have the blue side down at the end?

Assignment 4. A homogeneous Markov chain is said to be *irreducible* if from each state it is possible to get to each other state: that is, for each pair of states k, k', there exists an $m > 0$ for which $p(s_{i+m} = k \mid s_i = k') > 0$. For each state $k \in K$ we define the *period* of k (denoted d_k)

[1] A Markov chain is homogeneous if the transition probabilities $p(s_i = k \mid s_{i-1} = k')$ do not depend on the position i in in the sequence.

to be the greatest common divisor of the numbers $m > 0$ for which $p(s_{i+m} = k \mid s_i = k) > 0$. If $d_k = 1$, the state is called aperiodic.

Consider the Markov chains from assignments 1 and 3 as well as the chain with two states $K = \{0, 1\}$ and the transition probabilities

$$p(k = 0 \mid k' = 0) = 0$$
$$p(k = 1 \mid k' = 0) = 1$$
$$p(k = 0 \mid k' = 1) = 1$$
$$p(k = 1 \mid k' = 1) = 0.$$

Are they irreducible? Analyse the periods of their states.

Assignment 5. (*Galton-Watson-Process – a population model*)
Individuals of a certain population can have $n = 0, 1, 2 \ldots$ offspring at the end of their life. The corresponding probabilities are $c_0, c_1, c_2, \ldots$. Let s_i denote the size of the population in the i-th generation.
a) Model the process as a Markov chain. Deduce a formula for the transition probabilities.
b****)** Calculate the extinction probability ρ_k, i.e. the probability that the population will extinct after a finite number of generations if it starts with k individuals in the first generation.
Hints: Express ρ_k in terms of $\rho := \rho_1$. Try to find a functional relationship for ρ and the probabilities c_k, $k = 0, 1, 2, \ldots$.

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