Chapter 1

Fortune's Formula

In this chapter we will introduce the world's most famous betting paradigm, Kelly criterion, often dubbed as the fortune's formula. Initially used to exploit roulette game, later blackjack, then horse racing and finally stock market.

1.1 Coin toss

Assume a simple coin toss game. We win if the coin comes up heads, if it is tails we lose. If we win, we receive o_g times the amount that we bet in addition to our original bet money. The definitions are:

- p winning probability.
- q loosing probability.
- w number of wins.
- l number of loses.
- $t = w + l$ number of all trials.
- o odds, sometimes also called dividends.
- o_g odds gain. Payoff from the winning bet in addition to the original amount.
- b is the fraction of our wealth that we decide to bet.
- W_0 is our starting wealth
- W_t is our final wealth after t trials

Next please assume we play a slightly profitable game of coin toss. Probability of winning the game is $p = 0.4$. Odds are $o = 3.0$, hence our odds gain is $o_g = o - 1 = 2.0$. We start with initial bank W_0 of 100,- CZK and we bet $b = 0.05$, 5 percent fraction of our bank every time. We play this game $t = 3$ times, we win some $w = 2$ and we lose some $l = 1$. Our final wealth W_3 can hence be calculated as follows:

$$W_3 = (1 + 2 \cdot 0.05)^2 \cdot (1 - 0.05)^1 \cdot 100 \tag{1.1}$$

We make a gain of 2.0 times our decided "bank" fraction 0.05 twice and we loose 0.05 of our bank once.

$$W_3 = 114.95 \tag{1.2}$$

What should be noted is that the order of our wins and losses does not matter as long as the respective counts of winning w and loosing games l follows the problem definition.

1.1.1 Growth rate

The example from 1.1 can be generalized in the following definition of wealth after t trials.

$$W_t = (1 + o_g \cdot b)^w \cdot (1 - b)^l \cdot W_0 \tag{1.3}$$

Next we will define the average growth rate with given fraction b over t trials to be $g(b)$:

$$g(b) = \frac{1}{t} \cdot \frac{W_t}{W_0} = \frac{1}{t} \cdot (1 + o_g \cdot b)^w \cdot (1 - b)^l \tag{1.4}$$

Clearly any racional player wishes for his final bank to be as large as possible. He should therefore wish for his wealth to grow as quickly as possible. What we will next refer to as the optimal strategy is a strategy that maximizes the growth rate defined in 1.4.

1.2 Search For Optimality

Maximization of the average growth rate can be expressed as optimization problem.

$$\underset{\boldsymbol{b}}{\text{maximize}} \quad g(b) = \frac{1}{t} \cdot (1 + o_g \cdot b)^w \cdot (1 - b)^l$$

We differentiate $g(b)$ with respect to b

$$\frac{\partial g}{\partial b} = \frac{o_g w (1 - b)^l \cdot (b o_g + 1)^{w-1}}{t} - \frac{l(1 - b)^{l-1} \cdot (b o_g + 1)^w}{t} \tag{1.5}$$

We set the derivative equal to zero and solve for b.

$$b^{\star} \text{ such that } \frac{o_g w(1-b)^l \cdot (bo_g+1)^{w-1}}{t} - \frac{l(1-b)^{l-1} \cdot (bo_g+1)^w}{t} = 0 \quad (1.6)$$

The only root that makes sense in our context is

$$b^{\star} = 1 \quad (1.7)$$

And it is indeed a maximum as proved by Edward Thorp in Thorp, 2008

To maximize the growth rate, player should be betting the whole bank in every single trial. Would that be rational way of bank management? We can easily see where the problem lies.

$$P(ruin) = 1 - p^t \quad (1.8)$$

$$\lim_{t \to \infty} (1 - p^t) = 1 \quad (1.9)$$

where *ruin* is a state where player's wealth reaches zero. $W_i = 0$ for some i.

Clearly, player would surely come to ruin with such strategy. It would only take a single lost game for it to happen.

A different approach would be to define the above problem as a minimization of risk. In that case however the optimal strategy would be to withhold all the money and never bet or to make minimum allowed bets as shown in Feller, 1968, unless the winning probability is $p = 1.0$, which would mean that player is not playing a game anymore, but receiving free money. Therefore even though such strategy minimizes risk, it unfortunately minimizes growth as well.

We conclude this section with a statement that both strategies are infeasible for us. There is however a perfect way to balance both growth rate and risk in a single strategy.

1.3 Utility

A completely different idea is to value money using utility functions. In the problem from the previous section. 1.2 we used linear utility function U. Simply:

$$U(W) = W \quad (1.10)$$

We have shown that it is insufficient for our purpose of wealth allocation. The idea of many great thinkers such as *Danielle Bernoulli* in *Exposition of a new* *theory on the measurement of risk* Bernoulli, 2011 is to use logarithmic utility function to value our money.

$$U(W) = \log(W) \tag{1.11}$$

Intuitively, it makes perfect sense. For a player who re-invests his money in every single game, who's wealth is compounding, reaching a ruin situation means complete stop to his operation. Hence it should be penalized accordingly.

$$\lim_{W \to 0^+} \log(W) = -\infty \tag{1.12}$$

Is there a situation where using linear utility function would be sensible? (Kelly Jr, 2011) Provides a great example.

> Suppose the situation were different; for example, suppose the gambler's wife allowed him to bet one dollar each week but not to reinvest his winnings. He should then maximize his expectation (expected value of capital) on each bet. He would bet all his available capital (one dollar) on the event yielding the highest expectation. With probability one he would get ahead of anyone dividing his money differently.

Such player should therefore forget about the Kelly criterion. The reason is that his winnings do not compound, they simply accumulate (Poundstone, 2010).

1.4 Kelly Criterion

The fortune's formula is based on the of idea evaluating the growth rate using logarithmic utility function.

$$G(b) = \frac{1}{t}\log(\frac{W_t}{W_0})) \tag{1.13}$$

$$G(b) = \log[(\frac{W_t}{W_0})^{\frac{1}{t}}] \tag{1.14}$$

$$G(b) = \log[(\frac{(1 + o_g \cdot b)^w \cdot (1 - b)^l \cdot W_0}{W_0})^{\frac{1}{t}}] \tag{1.15}$$

$$= \log[(1 + o_g \cdot b)^{\frac{w}{t}} \cdot (1 - b)^{\frac{l}{t}}] \tag{1.16}$$

$$G(b) = \frac{w}{t}\log(1 + o_g \cdot b) + \frac{l}{t}\log(1 - b) \tag{1.17}$$

$\frac{w}{t}$ and $\frac{l}{t}$ stand for our probabilities of winning p and loosing q. Therefore our final formula for average logarithmic growth rate looks as follows:

$$\mathbb{E}[G(b)] = p\log(1 + o_g \cdot b) + q\log(1 - b)) \tag{1.18}$$

We now repeat the same process as in 1.2

$$\underset{\boldsymbol{b}}{\text{maximize}} \quad \mathbb{E}[G(b)] = p\log(1 + o_g \cdot b) + q\log(1 - b)$$

This time differentiation should yield a very different result.

$$\frac{\partial\,\mathbb{E}[G(b)]}{\partial b} = \frac{p \cdot o_g}{1 + o_g \cdot b} - \frac{q}{1 - b} \tag{1.19}$$

$$b^\star \;\; such \;\; that \quad \frac{p \cdot o_g}{1 + o_g \cdot b} - \frac{q}{1 - b} = 0 \tag{1.20}$$

We follow through with the calculation.

$$\frac{p \cdot o_g}{1 + o_g \cdot b} = \frac{q}{1 - b} \tag{1.21}$$

$$p \cdot o_g(1 - b) = q(1 + o_g \cdot b) \tag{1.22}$$

$$po_g - q = o_g b(p + q) \tag{1.23}$$

Where $p + q = 1$ and our optimal strategy is hence defined as:

$$b^\star = \frac{po_g - q}{o_g} \tag{1.24}$$

It is indeed a maximum as shown in Latane, 2011. It is also a well known formula sometimes written as

$$\frac{edge}{odds} \tag{1.25}$$

or using different notation where b stands for $o_g = odds - 1$

$$\frac{pb - q}{b} \tag{1.26}$$

What happens if we choose optimal fraction $b^\star$ according to 1.24 in our original problem 1.1

$$b^\star = \frac{po_g - q}{o_g} = \frac{0.4 \cdot 2 - 0.6}{2} = 0.1 \tag{1.27}$$

The following experiment provides an illustration why it is not reasonable to bet higher than the Kelly optimal fraction $b^\star$.

Assume we play the game 1.1 $t = 30000$ times, $b_1 = 0.05$, our initial guess is the green player, $b^\star = 0.1$, the optimal Kelly fraction is the blue player. Red player is betting $b_h = 0.25$.

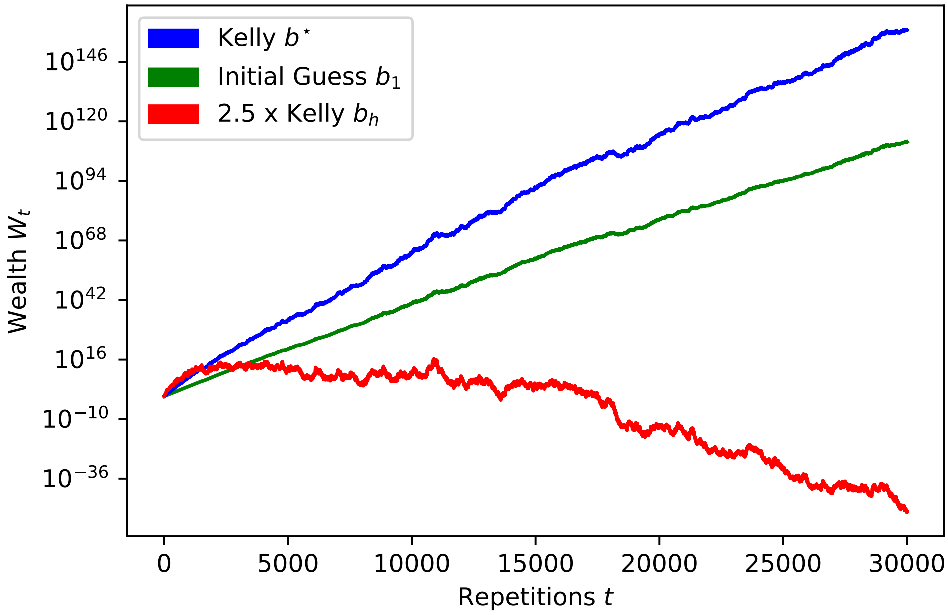

Figure 1.1: Mean trajectories of 1000 parallel histories for $b^\star$, b_1, b_h

Note that the red trajectory b_h does indeed "beat" Kelly on the first 500 or so trials, but eventually it leads to ruin and the red player leaving the casino in tears.

Both blue and green player would go home smiling with blue player being the richer of the two.

The important conclusion is that Kelly fraction $b^\star$ acts like the upper bound of the achievable wealth growth. Betting anything less than $b^\star$ leads to sub-optimal final wealth. Betting anything higher lowers returns and increases risk.

> For true long-term investors, the Kelly criterion is the boundary between aggressive and insane risk-taking. Like most boundaries, it is an invisible line. You can be standing right on it, and you

> won't see a neat dotted line painted on the ground. Nothing dramatic happens when you cross the line. Yet the situation on the ground is treacherous because the risk-taker, though heading for doom, is liable to find things getting better before they get worse. (Poundstone, 2010)

We close this section with a statement that $b^\star = \frac{p \cdot o_g - q}{o_g}$ is the optimal strategy for binary game such as coin toss. What can we do when the game is more complex remains to be answered.

1.5 Exclusive Games

In our first game 1.1, we discussed a situation where player has a binary choice of betting fraction of his wealth on a single outcome and the other option of leaving some of the money aside.

Next we assume a game with K outcomes that are exclusive, e.g. horse race where only a single horse can win.

1.5.1 Growth Rate

We have already discussed how Kelly criterion uses the idea of evaluating our money with logarithmic utility function. Given fractional strategy b and probability p the Kelly criterion is usually expressed as follows from (Cover et al., 2012).

$$W(\boldsymbol{b}, \boldsymbol{p}) = E[\log(\boldsymbol{b}^T \boldsymbol{o})] \tag{1.28}$$

It again restates the expected logarithm of growth rate. This time however it is for general K exclusive outcome game.

$$W(b, p) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \tag{1.29}$$

Please note that $1 + o_g \cdot b$ transformed into $b_i \cdot o_i$, because o stands for odds $o = o_g + 1$ as defined, hence it already includes the original bet amount.

1.5.2 Kelly Proof

In the previous section we defined Kelly criterion for exclusive games with K outcomes.

The maximization of the above defined growth rate 1.29 looks as follows from

(Cover et al., 2012).

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \end{aligned}$$

Using the method of lagrange multipliers we expand the above problem into

$$\ell(b) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) + \lambda \cdot \sum_{i=1}^{K} b_i \tag{1.30}$$

We differentiate with respect to b

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{K} \frac{p_i}{b_i \cdot o_i} + \lambda \tag{1.31}$$

Bookie's estimated probability distribution sums up to 1.0.

$$\sum_{i=1}^{K} \frac{1}{o_i} = 1.0 \tag{1.32}$$

We simplify our derivation into

$$\frac{\partial \ell}{\partial b} = \frac{p_i}{b_i} + \lambda \qquad i = 1, 2, ..., K \tag{1.33}$$

To find the optimum of b_i we set the above equation equal to 0

$$b_i = -\frac{p_i}{\lambda} \tag{1.34}$$

Now we substitue back into the constraint $\sum_{i=1}^{K} b_i = 1.0$ we find that

$$\lambda = -1 \qquad b_i^{\star} = p_i \tag{1.35}$$

This only tells us that the $b_i = p_i$ is only a stationary point. In the next section we will prove that it is indeed a maximum.

1.5.3 Maximum

We will prove that probability proportional strategy $b_i = p_i$ is the maximum of the above defined problem. Before we proceed, a few additional definitions are necessary.

First we recall the formula for entropy:

$$H(P) = -\sum_{i=1}^{n} p_i \log(p_i) \tag{1.36}$$

Next we define Kullback–Leibler divergence from definition:

$$D(P||Q) = -\sum_{i=0}^{n} p_i \log \frac{q_i}{p_i} \tag{1.37}$$

We start with a standard formula for growth rate.

$$W(b,p) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \tag{1.38}$$

In this step we use a convenient trick from (Cover et al., 2012).

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i} p_i \cdot o_i) \tag{1.39}$$

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i}) + \sum_{i=1}^{K} p_i \log(p_i) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.40}$$

We can now transform our formula into:

$$W(b,p) = -D(p||b) - H(p) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.41}$$

KL-divergence being non negative we can safely conclude:

$$W(b,p) \leq -H(p) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.42}$$

And we can only achieve equality,(maximum growth rate) if distance,("KL-divergence") is 0.

$$D(p||b) = 0 \tag{1.43}$$

that holds for:

$$b^\star = p \tag{1.44}$$

Probability proportional gambling achieves maximum growth rate when bookie's probabilities $\frac{1}{o_i}$ sum up to 1.0.

Interesting finding is that in such a case, odds are completely ignored by the growth optimal strategy. All that matters is the probability. In other texts this strategy is often referred to as "betting your beliefs".

What should be noted from this section is that Kelly growth optimal betting is closely linked to Kullback–Leibler divergence, which is a fact we will later investigate in chapter 2.

1.5.4 Dividends

In the previous section we proved that probability proportional strategy is optimal when bookie's probabilities sum up to 1. This happens only if the dividends are fair. From the perspective of dividends,("odds") we can distinguish three cases. Fair odds, super-fair and sub-fair odds, (Cover et al., 2012).

1.5.4.1 Fair

The dividend implied probabilities $\frac{1}{o_i}$ sum up to 1.0

$$\sum_{i=1}^{K} \frac{1}{o_i} = 1.0 \tag{1.45}$$

Optimal strategy is probability proportional. Intuitively, if odds are fair, they do not provide any more information.

1.5.4.2 Super-Fair

In this case odds are even better than fair. It is an arbitrage situation, that in real life happens very rarely, if so, it is only by a mistake of bookie.

$$\sum_{i=1}^{K} \frac{1}{o_i} \leq 1.0 \tag{1.46}$$

1.5.4.3 Sub-Fair

This is the case we will be investigating in this text. It represents most of the betting situations in real life. Subfair odds usually lowered by some margin or "track-take".

$$(1 - tt) \cdot o_i \qquad tt \in (0, 1) \tag{1.47}$$

where tt stands for track-take. Hence

$$\sum_{i=1}^{K} \frac{1}{o_i} \geq 1.0 \tag{1.48}$$

probability proportional gambling is no longer growth optimal. For answer we will have to look into Kelly Jr, 2011 and the legendary problem of a *Gambler with a private wire.*

In (Kelly Jr, 2011) story begins with a gambler who owns a "private-wire", through which he receives insider tips on which horse will win the race. Received tips are not 100% reliable, though gambler always knows how "unreliable" the tips are. He knows the true probability distribution.

1.5.5 Life Is Not Fair

What can we do if odds are not fair? It is the most common real life situation, when the odds implied probabilities sum up to over 1.0.

$$\sum_{i=1}^{K} \frac{1}{o_i} \geq 1.0 \tag{1.49}$$

Clearly, probability proportional strategy is no longer optimal. Because of the existing "track-take", betting on all the horses (outcomes) is no longer sensible.

Thankfully, in Kelly Jr, 2011 a waterfall algorithm is presented for such a case. It is later rediscovered in Smoczynski et al., 2010.

Algorithm 1 Kelly exclusive algorithm

procedure KELLY-EXCLUSIVE($\boldsymbol{i}, \boldsymbol{p}, \boldsymbol{o}$)
    $\boldsymbol{chosen} = [\ ]$
    $\boldsymbol{fractions} = [0, 0, ..., 0]$
    $R = 1.0$
    $\boldsymbol{ev} = \boldsymbol{p} \cdot \boldsymbol{o}$
    $(\tilde{\boldsymbol{i}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{o}}) = \text{order\_descending}((\boldsymbol{i}, \boldsymbol{p}, \boldsymbol{o}), \text{by}=\boldsymbol{ev})$
    for (i, p_i, o_i) `in` $(\tilde{\boldsymbol{i}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{o}})$ **do**
        if $p_i \cdot o_i > R$ **then**
            $\boldsymbol{chosen}.\text{add}((i, p_i, o_i))$
            $R = \frac{1 - sum(\boldsymbol{chosen}.p)}{1 - sum(\frac{1}{\boldsymbol{chosen}.o})}$
        end if
    end for

    for (i, p_i, o_i) `in` $\boldsymbol{chosen}$ **do**
        $\boldsymbol{fractions}[i] = p_i - \frac{R}{o_i}$
    end for

    return $\boldsymbol{fractions}$
end procedure

Where R stands for reserved rate. $\boldsymbol{i}$ is the vector of event identifiers $\boldsymbol{p}$ is the vector of probabilities, $\boldsymbol{o}$ is the vector of odds. In simple terms, the algorithm can be explained as:

1. Order all of the possible bets from most to least profitable (highest to lowest ev).

2. For each event, see if the ev for that event exceeds the "reserve rate" for your existing set of bets, (The reserve rate is initially 1.0 when your set

of planned bets is empty). If the ev is higher, then add that bet to your chosen set of bets.

3. After each addition of a bet, update the reserved rate according to. R = (1 – (sum of each probability bet on)) / (1 – (sum of each odds implied probabilities))

4. Once the optimal set of outcomes is discovered. The fractions are calculated as

$$b_i^\star = p_i - \frac{R}{o_i} \tag{1.50}$$

Next we define the expected gain to be.

$$\mu = p \cdot o - 1 = ev - 1 \tag{1.51}$$

The important finding, that can be looked at as counter intuitive is that Kelly may decide to bet on an outcome with negative expected gain μ if certain conditions are met. The reason behind this is that such diversified betting portfolio has higher geometric mean return than non diversified.

Assume a horse race of 3 horse with the following definitions.

$$\boldsymbol{p} = [0.08, 0.5, 0.42] \tag{1.52}$$

$$\boldsymbol{o} = [19, 1.99, 1.3] \tag{1.53}$$

Hence, μ from definition:

$$\boldsymbol{\mu} = [0.52, -0.005, -0.454] \tag{1.54}$$

Using Kelly exclusive algorithm on this problem yields the following optimal fractions $\boldsymbol{b}^\star$

$$\boldsymbol{b}^\star = [0.03030916, 0.0255648, 0.0] \tag{1.55}$$

The second horse,(betting opportunity) is of interest to us. Clearly it has negative expected gain μ, why exactly did Kelly exclusive algorithm decided to bet on this outcome?

The following experiment gives a clear answer. Blue player plays according to Kelly suggested strategy, red player decided to bet according to Kelly but only where he can expect positive gain. Very sound decision, one could say. We repeat the game 10000 times for 10000 parallel histories and we take the mean history for each one.

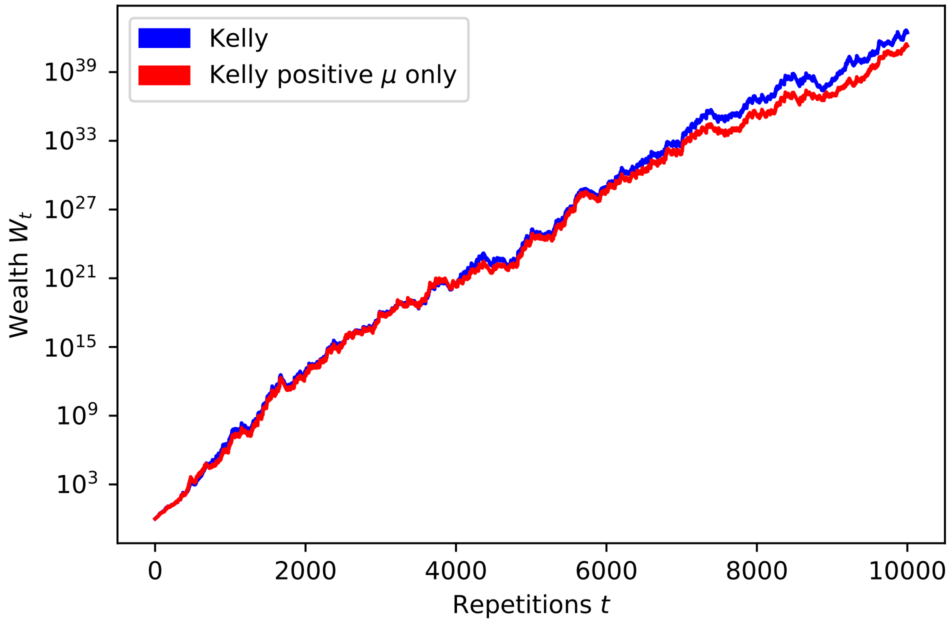

Figure 1.2: Blue-Kelly player, Red-Positive μ Kelly player, mean trajectory of 10000 parallel histories over 10000 games

The important idea to remember from this section is that it sometimes pays to bet on negative expectation bets in combination with other positive expectation ones. The second important thing is that modifying Kelly fractions in any way results in somewhat sub-optimal strategy.

1.6 K-outcome Games

Question that we also need to answer is: What can we do if the game is non-exclusive? A game where multiple bets pay off after a single outcome. Assume football outcome 5:1 for team A, then bets defined as: team A will score over 2 goals, team A will score over 4 goals, both pay off.

To be able to proceed with a solution, first we need be able to formulate such games. Taking inspiration from (Busseti et al., 2016) we define a return matrix $\boldsymbol{R}$ such that columns represent different assets available to us and rows represent different probabilities of our world. Each "box" hence represents a single payoff from a single asset.

We include additional asset to our representation. The risk-free cash asset which allows our strategy to put money aside. In addition it also allows us to model that leaving a large money aside can cost us small amounts of money in every betting turn ("inflation") or possibility to keep our "cash" in a bank with some interest rate.

All in all, our model allocates wealth among $n+1$ assets, n risky assets and 1 risk-free cash asset. Our "world" has K possible probabilistic outcomes.

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \dots & \boldsymbol{a_{n-1}} & \boldsymbol{c} \end{bmatrix} \tag{1.56}$$

$r_{i,j}$ stands for a single return in our matrix $\boldsymbol{R}$. Each asset column vector $\boldsymbol{a_i}$ is defined as follows.

$$\boldsymbol{a_i} = \begin{bmatrix} r_{i,1} \\ r_{i,2} \\ \dots \\ r_{i,K} \end{bmatrix} \tag{1.57}$$

$\boldsymbol{b}$ stands for chosen bet fractions.

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_c \end{bmatrix} \tag{1.58}$$

and of course probability vector $\boldsymbol{p}$

$$\boldsymbol{p} = \begin{bmatrix} p_1 & p_2 & \dots & p_K \end{bmatrix} \tag{1.59}$$

1.6.1 2-asset game

The most basic case of a game we divide our bank between a single asset and cash. We redefine our previously used example of a fair coin toss that either pays off $o = 3.0$ or nothing.

$$\boldsymbol{R} = \begin{bmatrix} 3.0 & 1.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{1.60}$$

$$\boldsymbol{p} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \tag{1.61}$$

1.6.2 3-asset game

A great example is basketball, because interstingly basketball has no draw state. For two teams A,B our 3 assets are $WINA$, $WINB$, $CASH$. Assume a slightly less profitable game where probability of winning for team A is

$p_A = 0.6$ and payoff is $o_a = 1.8$, probability of winning for team B is $p_b = 0.4$ and payoff is $o_b = 2.01$.

$$\boldsymbol{R} = \begin{bmatrix} 1.8 & 0.0 & 1.0 \\ 0.0 & 2.1 & 1.0 \end{bmatrix} \tag{1.62}$$

$$\boldsymbol{p} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \tag{1.63}$$

This format allows us to formulate a much more complex game. Imagine you are faced with a choice of allocating your money between 3 wheel's of fortune. Taken from (Poundstone, 2010).

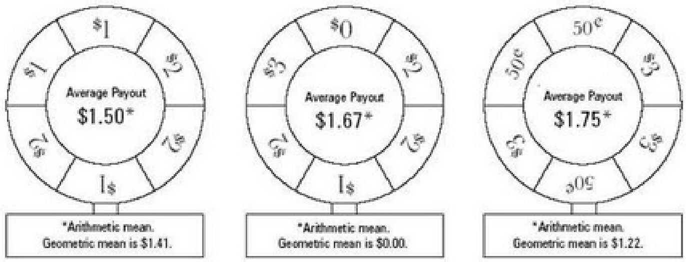

Figure 1.3: 3 wheels of fortune

We can easily represent such problem using our matrix $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & \frac{1}{2} \\ 2 & 2 & 3 \end{bmatrix} \tag{1.64}$$

Note that in this particular problem we are not using cash asset.

$$\boldsymbol{p} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \dots & \frac{1}{6} \end{bmatrix} \tag{1.65}$$

1.6.3 N-asset game

Assume horse race with 16 running horses. Bet type quinella denoted $QNL(i, j)$ pays off if pair of horses (i, j) win the race. Order does not matter. There are hence 120 different pairs, 121 different assets including cash asset and 120 probabilities in the vector $\boldsymbol{p}$. $o_{i,j}$ denotes posted odds for given $QNL(i, j)$.

$$\boldsymbol{R} = \begin{bmatrix} o_{1,1} & 0 & 0 & ... & 1 \\ 0 & o_{1,2} & 0 & ... & 1 \\ 0 & 0 & o_{1,3} & ... & 1 \\ ... & ... & ... & ... & 1 \end{bmatrix} \tag{1.66}$$

This is a bet on an exclusive outcome, hence R matrix is almost completely made up of zeros and odds diagonally.

$$\boldsymbol{p} = \left[p_{1,1}, p_{1,2}, ..., p_{15,16} \right] \tag{1.67}$$

One may argue that it would be wiser to solve such problem using Kelly exclusive algorithm and he would be correct. This example is here to display that using our $\boldsymbol{R}$ matrix in combination with probability vector $\boldsymbol{p}$, we are able to express any real world complex game.

1.6.4 General definition

Taking the matrix $\boldsymbol{R}$ and probability distribution $\boldsymbol{p}$. We proceed with general definition of the Kelly strategy.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[U(\mathbf{R} \cdot \mathbf{b})] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0, \; b_i \geq 0 \end{aligned}$$

where U is general non-decreasing utility function(more money is always at least as good as less money). In this text we will focus on the logarithmic utility function. Mainly for it's mathematical properties which we discussed above.

$$U(W) = \log(W) \tag{1.68}$$

1.7 Modern Portfolio Theory

The idea behind Modern Portfolio Theory is that portfolio $\boldsymbol{b_1}$ is superior to $\boldsymbol{b_2}$ if the expected gain $\mathbb{E}[\boldsymbol{b}]$ is at least as great.

$$\mathbb{E}[\boldsymbol{b_1}] \geq \mathbb{E}[\boldsymbol{b_2}] \tag{1.69}$$

and the risk, here general risk measure denoted r is no greater, (Markowitz, 1952).

$$r(\boldsymbol{b_1}) \leq r(\boldsymbol{b_2}) \tag{1.70}$$

This creates a partial ordering on the set of all available portfolios. Taking the portfolios that no portfolio is superior gives us the set of efficient portfolios Θ.

Markowitz, 1952 proposes measures of dispersion,(risk measures) that can possibly be used such as variance Var, standard deviation σ and “coefficient of variation“ CV.

$$Var[\boldsymbol{b}] \tag{1.71}$$

$$\sigma(\boldsymbol{b}) = \sqrt{Var[\boldsymbol{b}]} \tag{1.72}$$

$$CV(\boldsymbol{b}) = \frac{\sigma(\boldsymbol{b})}{\mathbb{E}[\boldsymbol{b}]} \tag{1.73}$$

In our case, portfolio $\boldsymbol{b}$ is actually a wealth allocation across different betting opportunities.

1.7.1 Definition

MPT can be expressed as a maximization problem:

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \boldsymbol{\mu}^T\boldsymbol{b} - \gamma\boldsymbol{b}^T\Sigma\boldsymbol{b} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

where $\boldsymbol{b}$ is fraction vector, γ is risk aversion parameter and $\boldsymbol{\mu}$ is the expected values vector of offered opportunities. In layman terms we maximize the following:

$$return - \gamma \cdot risk \tag{1.74}$$

In the most general set up risk is defined as variance Σ.

1.7.2 MPT and Kelly

To understand the difference between MPT and Kelly, please recall our example of the three fortune wheels in 1.3. We had three wheels $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}$ with uniform probability distribution and the following returns.

$$\boldsymbol{a_1}^T = \begin{bmatrix} 1.0 & 1.0 & 2.0 & 2.0 & 1.0 & 2.0 \end{bmatrix} \tag{1.75}$$

$$\boldsymbol{a_2}^T = \begin{bmatrix} 3.0 & 0.0 & 2.0 & 2.0 & 1.0 & 2.0 \end{bmatrix} \tag{1.76}$$

$$\boldsymbol{a_3}^T = \begin{bmatrix} 0.5 & 0.5 & 3.0 & 3.0 & 0.5 & 3.0 \end{bmatrix} \tag{1.77}$$

$$\boldsymbol{p} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \tag{1.78}$$

Assume that we are faced with a decision of choosing a single wheel and letting all of our wealth run on a chosen wheel, instead of allocating across all three as we did in 1.3. Which wheel should we choose, which one should we choose according to the Kelly criterion and which one according to the MPT?

Additional information includes arithmetic mean for each:

$$A(\boldsymbol{a_1}) = 1.50 \quad A(\boldsymbol{a_2}) = 1.67 \quad A(\boldsymbol{a_3}) = 1.75 \tag{1.79}$$

and geometric mean for each:

$$GM(\boldsymbol{a_1}) = 1.41 \quad GM(\boldsymbol{a_2}) = 0.00 \quad GM(\boldsymbol{a_3}) = 1.22 \tag{1.80}$$

Kelly criterion says we should never choose wheel $\boldsymbol{a_2}$ for such an investment. Because running all of our money through such a wheel, we would definitely be loosing everything in the long run, Poundstone, 2010. According to the Kelly strategy we should choose $\boldsymbol{a_1}$, the wheel with the highest geometric mean. This wheel would yield us the maximum compound return.

MPT on the other hand would resort from choosing a specific wheel. All three wheels are valid choices for different risk parameter γ. Using variance as a risk measure, asset $\boldsymbol{a_1}$ is perfect for people with low risk preference, $\boldsymbol{a_3}$ for people with desire for high returns and $\boldsymbol{a_2}$ for people somewhere in the middle.

Interestingly wheel $\boldsymbol{a_2}$ has lower risk than $\boldsymbol{a_3}$ even though there is a chance of loosing everything. Which hints at the imperfection of variance as a risk measure Poundstone, 2010. On the other hand, goal of the Kelly strategy is to avoid the chance of ruin, however small it is.

Obviously, this is quite a specific example. However, more complex definitions and such wheels can easily represents assets on the stock market.

MPT can be understood as a framework. Note that the following idea is very modular:

$$Maximize \quad return - risk\_parameter * risk \tag{1.81}$$

It is exactly this modularity that we will find useful in the next chapters.

We conclude this section with a statement that MPT is one of the frameworks we will be using in the next chapters. We will show a direct connection between MPT and the Kelly strategy in 3.1.

1.8 The Flat Stake

In this section we will define the flat strategy and compare it to the reinvestment startegy using the wheels example, (Poundstone, 2010).

Assume a player who bets using flat strategy meaning that for a year he gets \$1 every week from his wife. He would do best by choosing the wheel with the highest arithmetic mean $\boldsymbol{a_3}$. After 52 weeks his expected earnings look as follows:

$$52 \cdot 1.75 = 87 \tag{1.82}$$

Next assume a player who starts with \$1 and reinvests his winnings every week. Here's a comparison of how he would fare choosing each wheel.

52 weeks using Kelly advised wheel $\boldsymbol{a_1}$.

$$1.41^{52} = 67,108,864 \tag{1.83}$$

Now for the second wheel $\boldsymbol{a_2}$

$$0^{52} = 0 \tag{1.84}$$

and finally for the third wheel $\boldsymbol{a_3}$

$$1.22^{52} = 37,877 \tag{1.85}$$

In case of reinvestment we see a big difference between Kelly suggested wheel and any other wheel when player reinvests his winnings.

We conclude this chapter with a few key notes. Kelly criterion is nothing more than an upper bound of how much of gambler's wealth are the presented betting opportunities worth.

Latane, 2011 used the name "geometric mean policy" instead. In short, geometric mean policy assumes gambler can not predict what the future will bring and the best thing for him to do "right-now" is to choose a portfolio with the highest geometric mean.