GRAPHICAL MARKOV MODELS (WS2017)
1. SEMINAR

Assignment 1. Consider the following homogeneous Markov chain with two states $K = \{0, 1\}$. The matrix of transition probabilities is defined by

$$p(k = 0 \mid k' = 0) = 1 - \alpha \qquad p(k = 1 \mid k' = 0) = \alpha$$
$$p(k = 0 \mid k' = 1) = 1 \qquad p(k = 1 \mid k' = 1) = 0$$

The p.d. for the first state in the chain is $p_1(k = 0) = \beta$ and $p_1(k = 1) = 1 - \beta$.
a) Calculate the probability $p_i(k = 0)$ for the time step i, given the corresponding probability $p_{i-1}(k = 0)$ for the preceding time step. Describe the mapping $p_{i-1}(k = 0) \mapsto p_i(k = 0)$ explicitly.
b) Deduce a formula expressing $p_i(k = 0)$ in terms of the value $p_1(k = 0) = \beta$ for the first time step. Show that the probability $p_i(k = 0)$ becomes independent of β for $i \to \infty$

Assignment 2. (*Segmentation*) Consider the finite language in $\{a, b\}^n$ containing all words of the form

$$\underbrace{\overbrace{a \dots a}^{n_a} b \dots b}_{n}$$

where $0 \leqslant n_a \leqslant n$. Informally said, this model describes all segmentations of a sequence into two consecutive intervals.
a) Construct a Markov chain assigning a strictly positive probability to all admissible words and zero probability to all inadmissible words.
b) Construct a Markov chain such that all admissible words are equiprobable, whereas all other words have probability zero.
Hint: Determine the p.d. for the first state in the chain and all transition probabilities (the latter may depend on the position).

Assignment 3. A tetrahedron with differently coloured facets lies with the blue side down on a table. The tetrahedron is tilted n times over a randomly chosen edge (At each time instance there are three edges incident with the table to choose from). What is the probability to have the blue side down at the end?

Assignment 4. (*Gambler's ruin*) Consider a random walk on the set $L = \{0, 1, 2, \dots, a\}$ starting in some point $x \in L$. The position jumps by either ± 1 in each time period (with equal probabilities). The walk ends if either of the boundary states $0, a$ is hit. Compute the probability $u(x)$ to finish in state a if the process starts in state x.
Hints:

(1) What are the values of $u(0)$ and of $u(a)$?
(2) Find a difference equation for $u(x)$, $0 < x < a$ by relating it with $u(x - 1)$ and $u(x + 1)$.
(3) Translate the difference equation into a relation between the successive differences $u(x + 1) - u(x)$ and $u(x) - u(x - 1)$.

(4) Deduce that the solution is a linear function of x and find its coefficients from the boundary conditions $u(0)$ and $u(a)$.