13. Approximation algorithms for (Min,+)-problems

We still consider the task

$$s^* \in \operatorname*{argmin}_{s \in K^V} U(s) = \operatorname*{argmin}_{s \in K^V} \Big[\sum_{i \in V}' u_i(s) + \sum_{ij \in E}' u_{ij}(s_i, s_j) \Big]$$

A. Iterated descent

- define a family of neighbourhoods $N_m(s) \subset K^V$, $m = 1, \dots, M$
- repeatedly solve the restricted problem

$$s^{(t+1)} \in \operatorname*{argmin}_{s \in N_m(s^{(t)})} U(s)$$

until no further improvement is possible, i.e.

$$s^{(t)} \in \operatorname*{argmin}_{s \in N_m(s^{(t)})} U(s) \quad \forall\, m = 1, \dots, M.$$

Iterated conditional modes (ICM)

Choose a very simple neighbourhood $N(s) \subset K^V$

$$N(s) = \{ s' \in K^V \mid d_H(s, s') \leq 1 \},$$

where d_H is the Hamming distance. The task $\operatorname*{argmin}_{s' \in N(s)} U(s')$ is easy to solve, however the neighbourhood $N(s)$ is very small: $|N(s)| = |V| + 1$

α-Expansions (Boykov et al. 2001)

For each label $\alpha \in K$ define the neighbourhood

$$N_\alpha(s) = \{ s' \in K^V \mid s'_i = \alpha \text{ if } s'_i \neq s_i,\ \forall i \in V \}$$

The size of them is exponential, i.e. $|N_\alpha(s)| \sim 2^{|V|}$

Is the task $\arg\min_{s' \in N_\alpha(s)} U(s')$ solvable in polynomial time?

Yes, if $\boxed{u_{ij}(k,k') + u_{ij}(\alpha,\alpha) \leq u_{ij}(k,\alpha) + u_{ij}(\alpha,k')}$

holds for $\forall \{ij\} \in E$ and $\forall\ k, k' \in K \setminus \alpha$.

This can be seen by constructing a binary (Min,+) problem that is equivalent to the considered reduced opt. task

$$V' = \{ i \in V \mid s_i \neq \alpha \}, \quad E' = \{ \{i,j\} \in E \mid i,j \in V' \}$$

$y_i = 0,1$ encode the labelling $s' \in N_\alpha(s)$, i.e.

$s'_i = s_i \Leftrightarrow y_i = 0$ and $s'_i = \alpha \Leftrightarrow y_i = 1$

The pairwise functions of this equivalent problem are submodular if the condition given above holds.

__Example 1__ Consider the Potts model $u_{ij}(k,k') = a_{ij}(1-\delta_{kk'})$, $a_{ij} > 0$. It is not submodular if $|K| > 2$. However, it fulfills the above conditions.

__Theorem 1__ (w/o proof)

Let $\bar{s}$ be a fixpoint of α-expansions $\forall \alpha \in K$. Then $U(\bar{s}) \leq 2c \min_{s \in K^V} U(s)$, where c is defined by

$$c = \max_{ij \in E} \frac{\max_{k \neq k'} u_{ij}(k,k')}{\min_{k \neq k'} u_{ij}(k,k')} . \qquad \blacksquare$$

__$\alpha\beta$-Swaps__ (Boykov et al. 2001)

Define neighbourhoods $N_{\alpha\beta}$ for each pair of labels

$$N_{\alpha\beta}(s) = \left\{ s' \in K^V \;\middle|\; s'_i = \begin{cases} s_i & \text{if } s_i \neq \alpha, \beta \\ \alpha, \beta & \text{otherwise} \end{cases} \quad \forall i \in V \right\}$$

The reduced task $\operatorname{argmin}_{s' \in N_{\alpha\beta}(s)} U(s)$ is tractable if the restriction of every $u_{ij}: K^2 \to \mathbb{R}$ to $\{\alpha, \beta\}^2 \subset K^2$ is submodular for every pair $\alpha, \beta \in K$ of labels.

Example 2 Consider the truncated metric on $K \subset \mathbb{Z}$

$$u_{ij}(k,k') = a_{ij} \min(c, |k-k'|), \quad a_{ij} > 0.$$

It is not submodular. It allows α-β swaps but does not allow α-expansions.

B. Algorithms based on equivalent transformations and LP relaxations

- Loopy belief propagation (aka message passing): Apply equivalent transformations which resemble dynamic programming on trees, iteratively until convergence. Popular, but not well grounded. See next section.

More principled approaches start from LP relaxations of the discrete optimisation problem

$$U(s) = \sum_{i \in V} u_i(s_i) + \sum_{ij \in E} u_{ij}(s_i, s_j) \to \min_{s \in K^V}$$

A lower bound is given by

$$\sum_{i \in V} \min_{k \in K} u_i(k) + \sum_{ij \in E} \min_{k,k' \in K} u_{ij}(k,k') \leq \min_{s \in K^V} U(s)$$

Combine it with equivalent transf. and maximise the lower bound w.r.t. them

$$B(\varphi) = \sum_{i \in V} \min_k \Big[u_i(k) - \sum_{j \in V_i} \varphi_{ij}(k) \Big] +$$

$$+ \sum_{ij \in E} \min_{k,k'} \Big[\varphi_{ij}(k) + u_{ij}(k,k') + \varphi_{ji}(k') \Big] \to \max_{\varphi}$$

This can be expressed as a <u>linear optimisation</u> task by introducing additional variables

$$\sum_{i\in V} c_i + \sum_{ij\in E} c_{ij} \rightarrow \max_{c,\varphi}$$

s.t. $$c_i + \sum_{j\in N_i} \varphi_{ij}(k) \leq u_i(k) \quad \forall i\in V,\ \forall k\in K$$

$$c_{ij} - \varphi_{ij}(k) - \varphi_{ji}(k') \leq u_{ij}(k,k') \quad \forall ij\in E,\ \forall k,k'\in K$$

It is important to notice, that this LP-task is <u>dual</u> to the following direct relaxation of the discrete optimisation task. Encode the label $s_i \in K$ by 1-out of-K encoding with components denoted as $\lambda_i(k)=0,1$ and similarly for edges, by $\lambda_{ij}(k,k')=0,1$

$$\sum_{i\in V}\sum_{k\in K} \lambda_i(k)\, u_i(k) + \sum_{ij\in E}\sum_{k,k'\in K} \lambda_{ij}(k,k')\, u_{ij}(k,k') \rightarrow \min_{\lambda\geq 0}$$

s.t. $$\lambda_i(k) = \sum_{k'\in K} \lambda_{ij}(k,k') \quad \forall\ i,j\in E,\ \forall k\in K$$

$$\sum_{k\in K} \lambda_i(k) = 1 \quad \forall i\in V$$

$$\sum_{k,k'} \lambda_{ij}(k,k') = 1 \quad \forall ij\in E$$

Relaxing the integrality constraints $\lambda_i(k)=0,1$, $\lambda_{ij}(k,k')=0,1$ makes this an LP task.

Besides general LP solvers, there are several algorithms tailored for such LP relaxations, which try to solve the primal or the dual LP-task (or both simultaneously).

- tree reweighted message passing (Kolmogorov, 2006)
- (min,+) diffusion

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