3. Recognising the generating model – computing the emission probability for a feature sequence

Let a, b index two HMMs on $F^n \times K^n$

a: $p_a(s_1)$, $p_a(s_i \mid s_{i-1})$, $p_a(x_i \mid s_i)$

b: $p_b(s_1)$, $p_b(s_i \mid s_{i-1})$, $p_b(x_i \mid s_i)$

Given: observed sequence $x \in F^n$, $x = (x_1, \dots, x_n)$

Question: which model has generated x?

Any reasonable answer is based on $p_a(x), p_b(x) \Rightarrow$

Task: Compute

$$p(x) = \sum_{s \in K^n} p(x, s)$$

$$= \sum_{s_1 \in K} \dots \sum_{s_n \in K} p(s_1)\, p(x_1 \mid s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})\, p(x_i \mid s_i)$$

Denote:

$$P_i(s_{i-1}, s_i) = p(s_i \mid s_{i-1})\, p(x_i \mid s_i)$$

$$\varphi_1(s_1) = p(s_1)\, p(x_1 \mid s_1)$$

$$\psi_n(s_n) \equiv 1$$

Expression in the sum reads

$$\varphi_1(s_1) \cdot P_2(s_1, s_2) \cdot \ldots \cdot P_n(s_{n-1}, s_n) \cdot \psi_n(s_n)$$

Summation can be performed iteratively from right to left

$$\psi_{i-1}(s_{i-1}) = \sum_{s_i \in K} P_i(s_{i-1}, s_i)\, \psi_i(s_i)$$

$$p(x) = \sum_{s_1 \in K} \varphi_1(s_1)\, \psi_1(s_1)$$

or from left to right

$$\varphi_i(s_i) = \sum_{s_{i-1} \in K} \varphi_{i-1}(s_{i-1})\, P_i(s_{i-1}, s_i)$$

$$p(x) = \sum_{s_n \in K} \varphi_n(s_n)\, \psi_n(s_n)$$

In matrix-vector form we have

$$p(x) = \vec{\varphi}_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_n \cdot \vec{\psi}_n$$

Complexity: $\mathcal{O}(n|K|^2)$

Remark: The intermediate results have the following statistical meaning

$$\psi_i(s_i) = p(x_{i+1}, \ldots, x_n \,|\, s_i)$$

$$\varphi_i(s_i) = p(x_1, \ldots, x_i, s_i)$$

4. Recognising the most probable sequence of hidden states

We observe $x = (x_1, \ldots, x_n)$ for a known HMM

Question: Which sequence $s = (s_1, \ldots, s_n)$ has generated x?

If the answer is $s^* \in \underset{s \in K^n}{\operatorname{argmax}}\, p(x,s)$, we have to solve

$$s^* \in \underset{s \in K^n}{\operatorname{argmax}} \log p(x,s)$$

$$= \underset{s \in K^n}{\operatorname{argmax}} \left\{ \log\left[p(s_1)\, p(x_1|s_1)\right] + \sum_{i=2}^{n} \log\left[p(s_i|s_{i-1})\, p(x_i|s_i)\right] \right\}$$

Denote:
$$M_i(s_{i-1}, s_i) = \log\left[p(s_i \mid s_{i-1})\, p(x_i \mid s_i)\right]$$
$$\varphi_1(s_1) = \log\left[p(s_1)\, p(x_1 \mid s_1)\right]$$
$$\psi_n(s_n) \equiv 0$$

Now, translate the algorithm from Sec. 3 by replacing operators $\times \mapsto +$, $+ \mapsto \max$. E.g. maximising dynamically from right to left

$$\psi_{i-1}(s_{i-1}) = \max_{s_i \in K}\left[M_i(s_{i-1}, s_i) + \psi_i(s_i)\right]$$

$$\max_{s \in K^n} \log p(x,s) = \max_{s_1}\left[\varphi_1(s_1) + \psi_1(s_1)\right]$$

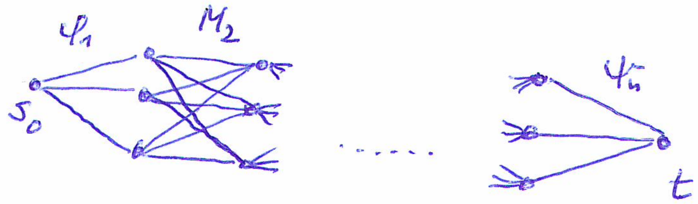
Since we are looking for $s^* \in \arg\max_{s \in K^n} p(x,s) \Rightarrow$ introduce pointers $pt_i : K \to K$ for $i = 1, \dots, n-1$

$$pt_{i-1}(s_{i-1}) \in \arg\max_{s_i \in K}\left[M_i(s_{i-1}, s_i) + \psi_i(s_i)\right]$$

and backtrack <u>an</u> optimiser s^* starting from

$$s_1^* \in \arg\max_{s_1 \in K}\left[\varphi_1(s_1) + \psi_1(s_1)\right]$$

Similarly, we can maximise from left to right. Both variants search the best path in the graph



<u>Complexity</u>: $\mathcal{O}(n|K|^2)$

5. Recognising the sequence of most probable hidden states

Is $s^* \in \arg\max_{s \in K^n} p(x,s)$ always the best answer to the question posed in Sec. 4?

The answer depends on $p(x,s)$ and the loss $\ell(s,s')$

(a) If $\ell(s',s) = \mathbb{1}\{s' \neq s\}$ then $s^* \in \arg\max_{s \in K^n} p(x,s)$

(b) If $\ell(s',s) = \sum_{i=1}^{n} \mathbb{1}\{s'_i \neq s_i\}$, i.e. Hamming distance

$$s^* \in \arg\min_{s \in K^n} \sum_{s' \in K^n} p(x,s) \sum_{i=1}^{n} [1 - \delta(s'_i, s_i)]$$

$$\hookrightarrow \quad s_i^* \in \arg\max_{s_i \in K} p(x, s_i) \qquad i = 1,..,n$$

Problem: Compute prob's $p(x, s_i = k)$ $\forall i = 1,..,n$, $\forall k \in K$

Remember quantities φ, ψ from Sec. 3

$$\varphi_i(s_i) = p(x_1,..,x_i,s_i)$$
$$\psi_i(s_i) = p(x_{i+1},..,x_n \mid s_i)$$

Since $p(x \mid s_i) \overset{!}{=} p(x_1,..,x_i \mid s_i)\, p(x_{i+1},..,x_n \mid s_i)$ holds for HMMs, we have

$$p(x, s_i) = \varphi_i(s_i)\, \psi_i(s_i)$$

Complexity: $\mathcal{O}(2n|K|^2)$