Description Logics – Reasoning

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Outline

Inference Problems

Inference Problems in TBOX

We have introduced syntax and semantics of the language $\mathcal{ALC}$. Now, let's look on automated reasoning. Having a $\mathcal{ALC}$ theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. For TBOX $\mathcal{T}$ and concepts $C_{(i)}$, we want to decide whether

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All these tasks can be reduced to unsatisfiability checking of a single concept ...

Reducting Subsumption to Unsatisfiability

Example

These reductions are straighforward – let's show, how to reduce subsumption checking to unsatisfiability checking. Reduction of other inference problems to unsatisfiability is analogous.

$$
\begin{aligned}
(\mathcal{T} \models C_1 \sqsubseteq C_2) & & \text{iff}\\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad \mathcal{I} \models C_1 \sqsubseteq C_2) & \text{iff}\\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}) & \text{iff}\\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad C_1^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus C_2^{\mathcal{I}}) \subseteq \emptyset & \text{iff}\\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad \mathcal{I} \models C_1 \sqcap \neg C_2 \sqsubseteq \bot & \text{iff}\\
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All these tasks, as well as concept unsatisfiability checking, can be reduced to consistency checking. Under which condition and how ?

Reduction of concept unsatisfiability to theory consistency

Example

Consider an $\mathcal{ALC}$ theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a concept C and a fresh individual a_f not occuring in $\mathcal{K}$:

$$
\begin{aligned}
(\mathcal{T} \models C \sqsubseteq \bot) & \qquad \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies \mathcal{I} \models C \sqsubseteq \bot) & \qquad \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies C^{\mathcal{I}} \subseteq \emptyset) & \qquad \text{iff} \\
\neg \left[(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge C^{\mathcal{I}} \not\subseteq \emptyset) \right] & \qquad \text{iff} \\
\neg \left[(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge a_f^{\mathcal{I}} \in C^{\mathcal{I}}) \right] & \qquad \text{iff} \\
(\mathcal{T}, \{C(a_f)\}) \quad \text{is inconsistent} &
\end{aligned}
$$

Note that for more expressive description logics than $\mathcal{ALC}$, the ABOX has to be taken into account as well due to its interaction with TBOX.

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We will introduce tableau algorithms.

Tableaux Algorithms

- Tableaux Algorithms (TAs) serve for checking theory consistency in a simple manner: **“Consistency of the given ABOX $\mathcal{A}$ w.r.t. TBOX $\mathcal{T}$ (resp. consistency of theory $\mathcal{K}$) is proven if we succeed in constructing a model of $\mathcal{T} \cup \mathcal{A}$.” (resp. theory $\mathcal{K}$)**

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  - choosen *strategy* for rule application
- TAs are not new in DL – they were known for FOL as well.

Completion Graphs

completion graph is a labeled oriented graph $G = (V_G, E_G, L_G))$, where each node $x \in V_G$ is labeled with a set $L_G(x)$ of concepts and each edge $\langle x, y \rangle \in E_G$ is labeled with a set of edges $L_G(\langle x, y \rangle)$[1]

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direct clash occurs in a completion graph $G = (V_G, E_G, L_G))$, if $\{A, \neg A\} \subseteq L_G(x)$, or $\bot \in L_G(x)$, for some atomic concept A and a node $x \in V_G$

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Do not mix with notion of *complete graphs* known from graph theory.

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Tableau Algorithm for $\mathcal{ALC}$

Tableau Algorithm for $\mathcal{ALC}$ with empty TBOX

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   - Sets $V_{G_0}, E_{G_0}, L_{G_0}$ are smallest possible with these properties.

Tableau algorithm for $\mathcal{ALC}$ without TBOX (2)

. . .

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4 (Rule Application) Find a rule that is applicable to G and apply it. As a result, we obtain from the state S a new state S'. Jump to step 2.

TA for $\mathcal{ALC}$ without TBOX – Inference Rules

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then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, and $L_{G'}(a) = L_G(a) \cup \{C_1, C_2\}$ and otherwise is the same as L_G.

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if $(C_1 \sqcup C_2) \in L_G(a)$ and $\{C_1, C_2\} \cap L_G(a) = \emptyset$ for some $a \in V_G$.

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if $(\exists R \cdot C) \in L_G(a_1)$ and there exists no $a_2 \in V_G$ such that $R \in L_G(a_1, a_2)$ and at the same time $C \in L_G(a_2)$.

TA for $\mathcal{ALC}$ without TBOX – Inference Rules

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TA Run Example

Example

Let's check consistency of the ontology $\mathcal{K}_2 = (\emptyset, \mathcal{A}_2)$, where $\mathcal{A}_2 = \{(\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \neg\exists maDite \cdot (Muz \sqcap Prarodic))(JAN)\}$.

- Let's transform the concept into NNF: $\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \forall maDite \cdot (\neg Muz \sqcup \neg Prarodic)$

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TA Run Example (2)

Example

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- Now, four sequences of steps 2,3,4 of the TA are performed. TA state in step 4, evolves as follows:

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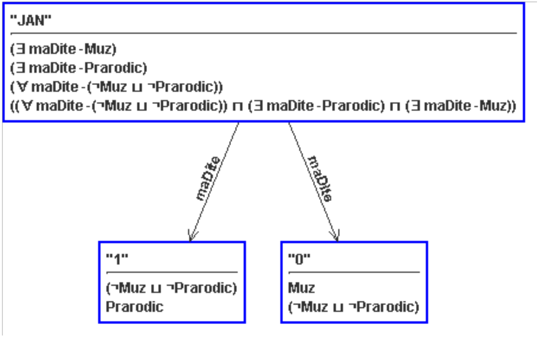
- $\{G_0\} \xrightarrow{\sqcap\text{-rule}} \{G_1\} \xrightarrow{\exists\text{-rule}} \{G_2\} \xrightarrow{\exists\text{-rule}} \{G_3\} \xrightarrow{\forall\text{-rule}} \{G_4\}$, where G_4 is



TA Run Example (3)

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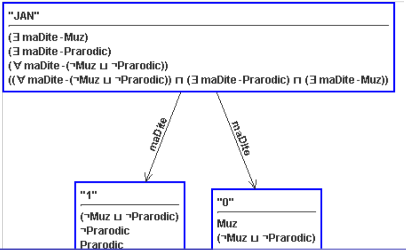
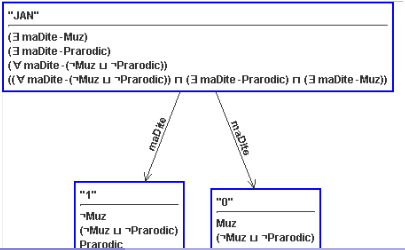
- Now, we have to apply the $\sqcup$-rule to the concept $\neg Muz \sqcup \neg Rodic$ either in the label of node "0", or in the label of node "1". Its application e.g. to node "1" we obtain the state $\{G_5, G_6\}$ (G_5 left, G_6 right)





TA Run Example (4)

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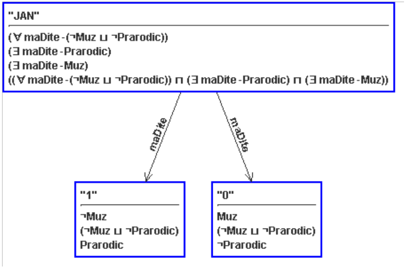
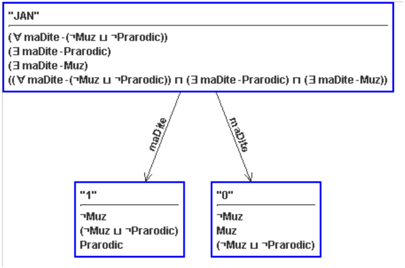
- We see that G_5 contains a direct clash in node "1". The only other option is to go through the graph G_6. By application of $\sqcup$-rule we obtain the state $\{G_5, G_7, G_8\}$, where G_7 (left), G_8 (right) are derived from G_6 :





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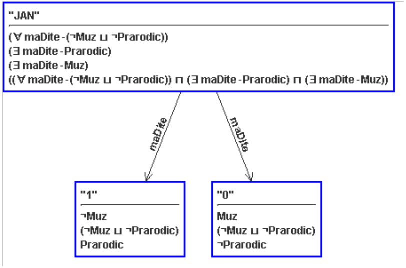

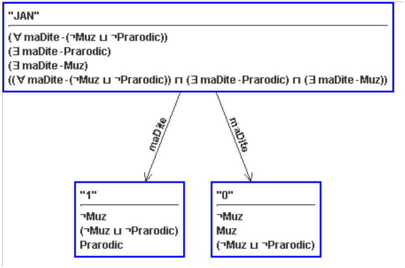

- G_7 is complete and without direct clash.

TA Run Example (5)

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- $``JAN''^{\mathcal{I}_2} = Jan, ``0''^{\mathcal{I}_2} = i_2, ``1''^{\mathcal{I}_2} = i_1$,

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Finiteness of the TA is an easy consequence of the following:

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- for each completion graph $G = (V_G, E_G, L_G)$ it holds that number of nodes in V_G is less or equal to the number of individuals in $\mathcal{A}$ plus number of existential quantifiers in $\mathcal{A}$.
- after application of any of the following rules $\rightarrow_\sqcap, \rightarrow_\exists, \rightarrow_\forall$ graph G is either enriched with a new node, new edge, or labeling of an existing node/edge is enriched. All these operations are finite.

Soundness

- Soundness of the TA can be verified as follows. For any $\mathcal{I} \models \mathcal{A}_{G_i}$, it must hold that $\mathcal{I} \models \mathcal{A}_{G_{i+1}}$. We have to show that application of each rule preserves consistency. As an example, let's take the $\rightarrow_\exists$ rule:

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- For other rules, the soundness is shown in a similar way.

Completeness

- To prove completeness of the TA, it is necessary to construct a model for each complete completion graph G that doesn't contain a direct clash. Canonical model $\mathcal{I}$ can be constructed as follows:
  - the domain $\Delta^{\mathcal{I}}$ will consist of all nodes of G.
- Observe that $\mathcal{I}$ is a model of $\mathcal{A}_G$. A backward induction can be used to show that $\mathcal{I}$ must be also a model of each previous step and thus also $\mathcal{A}$.

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- What about complexity of the algorithm ?
  - P-SPACE (between NP and EXP-TIME).

General Inclusions

We have presented the tableau algorithm for consistency checking of $\mathcal{K} = (\emptyset, \mathcal{A})$. How the situation changes when $\mathcal{T} \neq \emptyset$?

- consider $\mathcal{T}$ containing axioms of the form $C_i \sqsubseteq D_i$ for $1 \leq i \leq n$. Such $\mathcal{T}$ can be transformed into a single axiom

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  where $\top_C$ denotes a concept $(\neg C_1 \sqcup D_1) \sqcap \ldots \sqcap (\neg C_n \sqcup D_n)$
- for each model $\mathcal{I}$ of the theory $\mathcal{K}$, each element of $\Delta^{\mathcal{I}}$ must belong to $\top_C^{\mathcal{I}}$. How to achieve this ?

General Inclusions (2)

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then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, a $L_{G'}(a) = L_G(a) \cup \{\top_C\}$ and otherwise is the same as L_G.

General Inclusions (2)

What about this ?

$\rightarrow_{\sqsubseteq}$ rule

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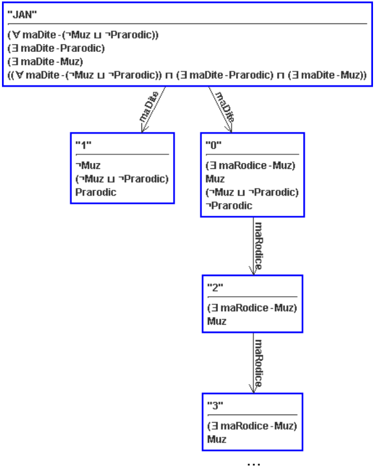
Example

Consider $\mathcal{K}_3 = (\{Muz \sqsubseteq \exists maRodice \cdot Muz\}, \mathcal{A}_2)$. Then $\top_C$ is $\neg Muz \sqcup \exists maRodice \cdot Muz$. Let's use the introduced TA enriched by $\rightarrow_{\sqsubseteq}$ rule. Repeating several times the application of rules $\rightarrow_{\sqsubseteq}$, $\rightarrow_{\sqcup}$, $\rightarrow_{\exists}$ to G_7 (that is not complete w.r.t. to $\rightarrow_{\sqsubseteq}$ rule) from the previous example we get ...

General Inclusions (3)

Example



... this algorithm doesn't necessarily terminate ☹.

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- *exists*− rule is only applicable if the node a_1 in its definition is not blocked by another node.

Blocking in TA (2)

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- **Introduced TA with subset blocking is sound, complete and finite decision procedure for $\mathcal{ALC}$.**

Let's play . . .

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