***Logical reasoning and programming*, lab session I**

(October 1, 2018)

I.1 Decide which of the following formulae are tautologies:

(a) $((p \to q) \to q) \to q$,
(b) $((p \to q) \to p) \to p$,
(c) $(p \to q) \vee (q \to p)$,
(d) $((p \to q) \wedge q) \to p$,
(e) $\neg p \to \neg(p \vee (p \wedge q))$.

Although formulae contain only two variables, try to find a better method than truth tables if possible.

I.2 Assume that $\varphi \leftrightarrow \psi$ is false, meaning $v(\varphi \leftrightarrow \psi) = 0$ for all valuations v. What can we say about the validity of the following formulae:

(a) $\varphi \wedge \psi$,
(b) $\varphi \vee \psi$,
(c) $\varphi \to \psi$.

I.3 If $\varphi \to \psi \in \texttt{TAUT}$ and $\psi \to \chi \in \texttt{TAUT}$, then $\varphi \to \chi \in \texttt{TAUT}$. Why? Does the claim still hold if we replace `TAUT` with `SAT` and why?

I.4 Let $\mathrm{Cl}(\Gamma) = \{\, \varphi \colon \Gamma \models \varphi \,\}$. Decide whether to any two sets of formulae Γ and Δ hold:

(a) $\Gamma \subseteq \mathrm{Cl}(\Gamma)$,
(b) $\mathrm{Cl}(\mathrm{Cl}(\Gamma)) = \mathrm{Cl}(\Gamma)$,
(c) $\mathrm{Cl}(\Gamma \cup \Delta) = \mathrm{Cl}(\Gamma) \cup \mathrm{Cl}(\Delta)$.

If the equality does not hold in (b) or (c), does at least one of the inclusions hold?

I.5 Recall that a Boolean function of n-variables is a function $f\colon \{0,1\}^n \to \{0,1\}$. Describe all functions of one variable. How many distinct Boolean functions of n variables exist? Try $n = 2$ first. Do you know, why are functions NAND ($\uparrow$) and NOR ($\downarrow$) interesting?[1]

I.6 If we use standard rewriting rules for producing a CNF, then we usually conclude by some simplifications—remove duplicate clauses and literals. Why can we do that? Is it correct that there is no need for a variable to occur more than once in a clause?

I.7 Produce a formula in CNF which is equivalent to $\varphi = (a \to (c \wedge d)) \vee (b \to (c \wedge e))$. Then use the Tseytin transformation to produce a formula in CNF which is equisatisfiable to φ.

[1] These connectives are also called Sheffer stroke and Peirce arrow, respectively. They are defined as $x \uparrow y := \neg(x \wedge y)$ and $x \downarrow y := \neg(x \vee y)$.