Logical reasoning and programming, lab session I

(October 1, 2018)

I.1 Decide which of the following formulae are tautologies:

(a) $((p \to q) \to q) \to q$,

(b) $((p \to q) \to p) \to p$,

(c) $(p \to q) \lor (q \to p)$,

(d) $((p \to q) \land q) \to p$,

(e) $\neg p \to \neg(p \lor (p \land q))$.

Although formulae contain only two variables, try to find a better method than truth tables if possible.

I.2 Assume that $\varphi \leftrightarrow \psi$ is false, meaning $v(\varphi \leftrightarrow \psi) = 0$ for all valuations v. What can we say about the validity of the following formulae:

(a) $\varphi \land \psi$,

(b) $\varphi \lor \psi$,

(c) $\varphi \to \psi$.

I.3 If $\varphi \to \psi \in \mathtt{TAUT}$ and $\psi \to \chi \in \mathtt{TAUT}$, then $\varphi \to \chi \in \mathtt{TAUT}$. Why? Does the claim still hold if we replace TAUT with SAT and why?

I.4 Let $\mathrm{Cl}(\Gamma) = \{\, \varphi \colon \Gamma \models \varphi \,\}$. Decide whether to any two sets of formulae Γ and Δ hold:

(a) $\Gamma \subseteq \mathrm{Cl}(\Gamma)$,

(b) $\mathrm{Cl}(\mathrm{Cl}(\Gamma)) = \mathrm{Cl}(\Gamma)$,

(c) $\mathrm{Cl}(\Gamma \cup \Delta) = \mathrm{Cl}(\Gamma) \cup \mathrm{Cl}(\Delta)$.

If the equality does not hold in (b) or (c), does at least one of the inclusions hold?

I.5 Recall that a Boolean function of n-variables is a function $f \colon \{0,1\}^n \to \{0,1\}$. Describe all functions of one variable. How many distinct Boolean functions of n variables exist? Try $n = 2$ first. Do you know, why are functions NAND ($\uparrow$) and NOR ($\downarrow$) interesting?[1]

I.6 Produce a formula in CNF which is equivalent to $\varphi = (a \to (c \land d)) \lor (b \to (c \land e))$. Then use the Tseytin transformation to produce a formula in CNF which is equisatisfiable to φ.

[1] These connectives are also called Sheffer stroke and Peirce arrow, respectively. They are defined as $x \uparrow y := \neg(x \land y)$ and $x \downarrow y := \neg(x \lor y)$.