Normal-Form Games

October 24, 2017

Outline

- Scope of MAS in Game Theory
- Game representations
  - Normal-form games
- What are the problems compared to, e.g., planning
- Analysis of a game
- Properties and computation of Nash equilibrium
- Game modeling

Noncooperative Game Theory

- **Single round games**
  - **Normal-form games**
  - **Extensive-form games**
  - **MAIDS, Congestion games**
- Multiple round games
  - Repeated games
  - Stochastic games

Types of games

- Two-player vs n-player
- Zero-sum games vs general-sum games
- Sequential vs one-shot
- Perfect-information vs imperfect-information
- Finite vs infinite

Types of games

- **Two-player** vs n-player
- **Zero-sum games** vs general-sum games
- **Sequential** vs **one-shot**
- **Perfect-information** vs **imperfect-information**
- **Finite** vs infinite

Normal-form games

- Players set $\mathcal{N} = \{1, ..., n\}$
- Actions set $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_n$
- Utility functions $u = \langle u_1, ... u_n \rangle$, where $u_i : \mathcal{A} \to \mathbb{R}$

Normal-form games

- Represented as n-dimensional matrix
- Every entry is n-dimensional tuple of utilities for every player

Strategies

- A pure strategy a_i in normal-form games represents the choice of specific action $a_i \in \mathcal{A}_i$ for player i
- A mixed strategy s_i is a strategy distribution over pure strategies
- Strategy profile a/s is a set of pure/mixed strategies, one for every player
- Overloading of utility function $u(a_i, a_{-i})$, $u(s_i, s_{-i})$, $u(s)$

Why GT

- Why do we need Game Theory?

Approaches for reasoning about games

- Studying game structure/properties
  - Social welfare optimality
  - Pareto optimality
- Stable strategies (solution concepts)
  - Maxmin
  - Minmax
  - Nash equilibrium
  - Stackelberg equilibrium
  - Correlated equilibrium
- Computation helpers
  - Dominance

Social welfare

- Defined as

$$WF = \sum_{i \in \mathcal{N}} u_i(s) \quad (1)$$

- Not stable against deviations
- Cooperative players

Pareto optimality

- Reasoning about outocomes
- Outcome o pareto dominates outcome o' iff

$$\forall i \in \mathcal{N} o_i \geq o'_i \; and \; \exists i \in \mathcal{N} o_i > o'_i \tag{2}$$

- Outcome o is pareto optimal if it is not pareto dominated by any other outcome o'

Dominance

- Strict dominance
  - Strategy a_i strictly dominates a_i' iff

$$\forall a_{-i} \in \mathcal{A}_{-i} : u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \tag{3}$$

- Weak dominance
  - Strategy a_i weakly dominates a_i' iff

$$\forall a_{-i} \in \mathcal{A}_{-i} : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \; and \tag{4}$$

$$\exists a_{-i} \in \mathcal{A}_{-i} : u_i(a_i, a_{-i}) > u(a_i', a_{-i}) \tag{5}$$

- Very weak dominance
  - Strategy a_i very weakly dominates a_i' iff

$$\forall a_{-i} \in \mathcal{A}_{-i} : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \tag{6}$$

Nash equilibrium

- A strategy s_i^* is the best response to strategies s_{-i}, written as $s_i^* \in BR(s_{-i})$ iff

$$\forall s_i \in \mathcal{S}_i u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \tag{7}$$

- Nash equilibrium
  - Strategy profile $s = \{s_1, ..., s_n\}$ is a Nash equilibrium iff

$$\forall i \in \mathcal{N} s_i \in BR(s_{-i}) \tag{8}$$

- Stable against deviations of players as every player plays his best response to the strategies of the rest
- Assumes self-interested rational players
- Every finite game has a non-empty set of Nash equilibria
- Examples

Properties of NE

- Values in NE might differ
- Strategies not interchangeable
- Mistake of the opponent might hurt me

Properties of NE in zero-sum games

- All NE have the same value for i (value of the game)
- The value is guaranteed (mistakes of the opponent only increase my expected outcome)
- Strategies are interchangeable between NE
- minmax = maxmin = NE = SE