Solving Normal-Form Games

Karel Horák
(based on slides of Branislav Bošanský)

Artificial Intelligence Center,
Department of Computer Science,
Faculty of Electrical Engineering,
Czech Technical University in Prague

karel.horak@agents.fel.cvut.cz

October 30, 2018

Previously ... on multi-agent systems.

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
    - $\mathcal{N}$ – a set of players

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - u – outcome for each combination of actions

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - u – outcome for each combination of actions
2. Pure strategies

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - u – outcome for each combination of actions
2. Pure strategies
3. Dominance of strategies

Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - u – outcome for each combination of actions
2. Pure strategies
3. Dominance of strategies
4. Nash equilibrium

... and now we continue ...

Rock Paper Scissors

- 2 players...

Rock Paper Scissors

- 2 players...
- What are the actions of the players? (A_1, A_2)

Rock Paper Scissors

- 2 players...
- What are the actions of the players? (A_1, A_2)
- What are their pure strategies?

Rock Paper Scissors

- 2 players...
- What are the actions of the players? (A_1, A_2)
- What are their pure strategies?

Here pure strategies coincide with actions.
That will change soon – next week :-)

Rock Paper Scissors

- 2 players...
- What are the actions of the players? (A_1, A_2)
- What are their pure strategies?

Here pure strategies coincide with actions.
That will change soon – next week :-)

- What are the possible outcomes?

Rock Paper Scissors

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

Rock Paper Scissors

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

What is the best strategy to play in Rock-Paper-Scissors?

Rock Paper Scissors

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

Rock Paper Scissors

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

The concept of pure strategies is not sufficient.

Mixed Strategies

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We use $\mathcal{S}_{-i}$ to denote strategies of all other players except player i.

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We use $\mathcal{S}_{-i}$ to denote strategies of all other players except player i.

We extend the utility function to correspond to *expected utility*:

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We use $\mathcal{S}_{-i}$ to denote strategies of all other players except player i.

We extend the utility function to correspond to *expected utility*:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We use $\mathcal{S}_{-i}$ to denote strategies of all other players except player i.

We extend the utility function to correspond to *expected utility*:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

We can extend existing concepts (dominance, best response, ...) to mixed strategies.

Dominance

Dominance

Definition (Strong Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *strongly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Dominance

Definition (Strong Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *strongly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition (Weak Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *weakly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in \mathcal{S}_{-i}$ such that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Dominance

Definition (Strong Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *strongly dominates* s_i' if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

Definition (Weak Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *weakly dominates* s_i' if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $\exists s_{-i} \in \mathcal{S}_{-i}$ such that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

Definition (Very Weak Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *very weakly dominates* s_i' if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$.

Best Response and Equilibria

Best Response and Equilibria

Definition (Best Response)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that $s_i^* \in BR_i(s_{-i})$ iff $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

Best Response and Equilibria

Definition (Best Response)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that $s_i^* \in BR_i(s_{-i})$ iff $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

Definition (Nash Equilibrium)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Strategy profile $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$.

Existence of Nash equilibria?

Existence of Nash equilibria?

	C	**D**
C	$(-1, -1)$	$(-5, 0)$
D	$(0, -5)$	$(-3, -3)$

Existence of Nash equilibria?

	C	**D**
C	$(-1,-1)$	$(-5,0)$
D	$(0,-5)$	$(-3,-3)$

	R	**P**	**S**
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$

Existence of Nash equilibria?

	C	**D**
C	$(-1,-1)$	$(-5,0)$
D	$(0,-5)$	$(-3,-3)$

	R	**P**	**S**
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$

Theorem (Nash)

Every game with a finite number of players and action profiles has at least one Nash equilibrium in mixed strategies.

Support of Nash Equilibria

Support of Nash Equilibria

Definition (Support)

The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\mathrm{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}$.

Support of Nash Equilibria

Definition (Support)

The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\mathrm{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}$.

Question

Assume Nash equilibrium (s_i, s_{-i}) and let $a_i \in \mathrm{Supp}(s_i)$ be an (arbitrary) pure strategy from the support of s_i. Which of the following possibilities can hold?

- $u_i(a_i, s_{-i}) < u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) = u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})$

Support of Nash Equilibria

Corollary

Let $s \in \mathcal{S}$ *be a Nash equilibrium and* $a_i, a'_i \in \mathcal{A}_i$ *are actions from the support of* s_i*. Now,* $u_i(a_i, s_{-i}) = u_i(a'_i, s_{-i})$*.*

Support of Nash Equilibria

Corollary

Let $s \in \mathcal{S}$ be a Nash equilibrium and $a_i, a_i' \in \mathcal{A}_i$ are actions from the support of s_i. Now, $u_i(a_i, s_{-i}) = u_i(a_i', s_{-i})$.

Can we exploit this fact to find a Nash equilibrium?

Finding Nash Equilibria

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Finding Nash Equilibria

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Column player (player 2) plays $\mathbf{L}$ with probability p and $\mathbf{R}$ with probability $(1 - p)$.

Finding Nash Equilibria

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Column player (player 2) plays $\mathbf{L}$ with probability p and $\mathbf{R}$ with probability $(1-p)$. In NE it holds

$$\mathbb{E}u_1(\mathbf{U}) = \mathbb{E}u_1(\mathbf{D})$$

$$2p + 0(1-p) = 0p + 1(1-p)$$

$$p = \frac{1}{3}$$

Finding Nash Equilibria

	$\mathbf{L}$	$\mathbf{R}$
$\mathbf{U}$	$(2,1)$	$(0,0)$
$\mathbf{D}$	$(0,0)$	$(1,2)$

Column player (player 2) plays $\mathbf{L}$ with probability p and $\mathbf{R}$ with probability $(1-p)$. In NE it holds

$$\mathbb{E}u_1(\mathbf{U}) = \mathbb{E}u_1(\mathbf{D})$$
$$2p + 0(1-p) = 0p + 1(1-p)$$
$$p = \frac{1}{3}$$

Similarly, we can compute the strategy for player 1 arriving at $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$ as Nash equilibrium.

Finding Nash Equilibria

Can we use the same approach here?

	L	**C**	**R**
U	$(2, 1)$	$(0, 0)$	$(0, 0)$
M	$(0, 0)$	$(1, 2)$	$(0, 0)$
D	$(0, 0)$	$(0, 0)$	$(-1, -1)$

Finding Nash Equilibria

Can we use the same approach here?

	L	**C**	**R**
U	$(2, 1)$	$(0, 0)$	$(0, 0)$
M	$(0, 0)$	$(1, 2)$	$(0, 0)$
D	$(0, 0)$	$(0, 0)$	$(-1, -1)$

Not really... No strategy s_i of the row player ensures $u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R)$:-(

Finding Nash Equilibria

Can we use the same approach here?

	L	**C**	**R**
U	$(2,1)$	$(0,0)$	$(0,0)$
M	$(0,0)$	$(1,2)$	$(0,0)$
D	$(0,0)$	$(0,0)$	$(-1,-1)$

Not really... No strategy s_i of the row player ensures $u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R)$:-(

Can something help us?

Finding Nash Equilibria

Can we use the same approach here?

	L	C	R
U	$(2,1)$	$(0,0)$	$(0,0)$
M	$(0,0)$	$(1,2)$	$(0,0)$
D	$(0,0)$	$(0,0)$	$(-1,-1)$

Not really... No strategy s_i of the row player ensures
$u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R)$:-(

Can something help us?

Iterated removal of dominated strategies...

Maxmin

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Maxmin

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Recall that there are multiple Nash equilibria in this game.

Maxmin

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Maxmin

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff.

Maxmin

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

Maxmin

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Maxmin and Minmax

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Maxmin and Minmax

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Definition (Minmax, two-player)

In a two-player game, the *minmax strategy* for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the *minmax value* for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Maxmin and Minmax

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Definition (Minmax, two-player)

In a two-player game, the *minmax strategy* for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the *minmax value* for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Maxmin strategies are conservative strategies against a worst-case opponent.

Maxmin and Minmax

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Definition (Minmax, two-player)

In a two-player game, the *minmax strategy* for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the *minmax value* for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Maxmin strategies are conservative strategies against a worst-case opponent.

Minmax strategies represent punishment strategies for player $-i$.

Maxmin

What is the maxmin strategy for the row player in this game?

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Zero-sum case

What about zero-sum case? How do

- player i's maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
- player i's minmax, $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

relate?

Zero-sum case

What about zero-sum case? How do

- player i's maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
- player i's minmax, $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

relate?

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

Zero-sum case

What about zero-sum case? How do

- player i's maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
- player i's minmax, $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

relate?

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

... but we can prove something stronger ...

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Consequences:

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Consequences:

1. $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Consequences:

1. $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
2. we can safely play Nash strategies in zero-sum games

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Consequences:

1. $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
2. we can safely play Nash strategies in zero-sum games
3. all Nash equilibria have the have the same payoff (by convention, the maxmin value for player 1 is called *value of the game*).

Computing NE in Zero-Sum Games

Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

$$\max_{s,U} \quad U \tag{1}$$

$$\text{s.t.} \quad \sum_{a_1 \in \mathcal{A}_1} s(a_1) u_1(a_1, a_2) \geq U \qquad \forall a_2 \in \mathcal{A}_2 \tag{2}$$

$$\sum_{a_1 \in \mathcal{A}_1} s(a_1) = 1 \tag{3}$$

$$s(a_1) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1 \tag{4}$$

Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

$$\max_{s,U} \quad U \tag{1}$$

$$\text{s.t.} \quad \sum_{a_1 \in \mathcal{A}_1} s(a_1) u_1(a_1, a_2) \geq U \qquad \forall a_2 \in \mathcal{A}_2 \tag{2}$$

$$\sum_{a_1 \in \mathcal{A}_1} s(a_1) = 1 \tag{3}$$

$$s(a_1) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1 \tag{4}$$

Computing a Nash equilibrium in zero-sum normal-form games can be done in polynomial time.

Computing NE in General-Sum Games

Computing NE in General-Sum Games

The problem is more complex for general-sum games (LCP program):

$$\sum_{a_2 \in \mathcal{A}_2} u_1(a_1, a_2) s_2(a_2) + q(a_1) = U_1 \qquad \forall a_1 \in \mathcal{A}_1$$

$$\sum_{a_1 \in \mathcal{A}_1} u_2(a_1, a_2) s_1(a_1) + w(a_2) = U_2 \qquad \forall a_2 \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1 \qquad \sum_{a_2 \in \mathcal{A}_2} s_2(a_2) = 1$$

$$q(a_1) \geq 0,\ w(a_2) \geq 0,\ s_1(a_1) \geq 0,\ s_2(a_2) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

$$s_1(a_1) \cdot q(a_1) = 0,\ s_2(a_2) \cdot w(a_2) = 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

Computing NE in General-Sum Games

The problem is more complex for general-sum games (LCP program):

$$\sum_{a_2 \in \mathcal{A}_2} u_1(a_1, a_2) s_2(a_2) + q(a_1) = U_1 \qquad \forall a_1 \in \mathcal{A}_1$$

$$\sum_{a_1 \in \mathcal{A}_1} u_2(a_1, a_2) s_1(a_1) + w(a_2) = U_2 \qquad \forall a_2 \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1 \qquad \sum_{a_2 \in \mathcal{A}_2} s_2(a_2) = 1$$

$$q(a_1) \geq 0,\ w(a_2) \geq 0,\ s_1(a_1) \geq 0,\ s_2(a_2) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

$$s_1(a_1) \cdot q(a_1) = 0,\ s_2(a_2) \cdot w(a_2) = 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

Computing a Nash equilibrium in two-player general-sum normal-form game is a PPAD-complete problem.

Computing NE in General-Sum Games

The problem is more complex for general-sum games (LCP program):

$$\sum_{a_2 \in \mathcal{A}_2} u_1(a_1, a_2) s_2(a_2) + q(a_1) = U_1 \qquad \forall a_1 \in \mathcal{A}_1$$
$$\sum_{a_1 \in \mathcal{A}_1} u_2(a_1, a_2) s_1(a_1) + w(a_2) = U_2 \qquad \forall a_2 \in \mathcal{A}_2$$
$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1 \qquad \sum_{a_2 \in \mathcal{A}_2} s_2(a_2) = 1$$
$$q(a_1) \geq 0,\ w(a_2) \geq 0,\ s_1(a_1) \geq 0,\ s_2(a_2) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$
$$s_1(a_1) \cdot q(a_1) = 0,\ s_2(a_2) \cdot w(a_2) = 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

Computing a Nash equilibrium in two-player general-sum normal-form game is a PPAD-complete problem. The problem gets even more complex (FIXP-hard) when moving to $n \geq 3$ players.

Regret

Regret

The concept of regret is useful when the other players are not completely malicious.

Regret

The concept of regret is useful when the other players are not completely malicious.

	L	**R**
U	$(100, a)$	$(1 - \varepsilon, b)$
D	$(2, c)$	$(1, d)$

Regret

The concept of regret is useful when the other players are not completely malicious.

	L	**R**
U	$(100, a)$	$(1-\varepsilon, b)$
D	$(2, c)$	$(1, d)$

Definition (Regret)

A player i's *regret* for playing an action a_i if the other agents adopt action profile a_{-i} is defined as

$$\left[\max_{a_i' \in \mathcal{A}_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i})$$

Regret

Regret

Definition (MaxRegret)

A player is *maximum regret* for playing an action a_i is defined as

$$\max_{a_{-i} \in \mathcal{A}_{-i}} \left(\left[\max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

Regret

Definition (MaxRegret)

A player is *maximum regret* for playing an action a_i is defined as

$$\max_{a_{-i} \in \mathcal{A}_{-i}} \left(\left[\max_{a_i' \in \mathcal{A}_i} u_i(a_i', a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

Definition (MinimaxRegret)

Minimax regret actions for player i are defined as

$$\arg\min_{a_i \in \mathcal{A}_i} \max_{a_{-i} \in \mathcal{A}_{-i}} \left(\left[\max_{a_i' \in \mathcal{A}_i} u_i(a_i', a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

Correlated Equilibrium

Consider again the following game:

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Correlated Equilibrium

Consider again the following game:

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination?

Correlated Equilibrium

Consider again the following game:

	L	**R**
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

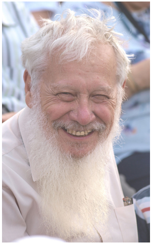
Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination? We can use *a correlation device*—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.

Correlated Equilibrium

Consider again the following game:

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination? We can use *a correlation device*—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.



Robert Aumann

Correlated Equilibrium

Correlated Equilibrium

Definition (Correlated Equilibrium (simplified))

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let σ be a probability distribution over joint pure strategy profiles $\sigma \in \Delta(\mathcal{A})$. We say that σ is a correlated equilibrium if for every player i, every signal $a_i \in \mathcal{A}_i$ and every possible action $a'_i \in \mathcal{A}_i$ it holds

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a'_i, a_{-i})$$

Correlated Equilibrium

Definition (Correlated Equilibrium (simplified))

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let σ be a probability distribution over joint pure strategy profiles $\sigma \in \Delta(\mathcal{A})$. We say that σ is a correlated equilibrium if for every player i, every signal $a_i \in \mathcal{A}_i$ and every possible action $a_i' \in \mathcal{A}_i$ it holds

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i})$$

Corollary

For every Nash equilibrium there exists a corresponding Correlated Equilibrium.

Computing Correlated Equilibrium

Computing Correlated Equilibrium

Computing a Correlated equilibrium is easier compared to Nash and can be found by linear programming even in general-sum case:

Computing Correlated Equilibrium

Computing a Correlated equilibrium is easier compared to Nash and can be found by linear programming even in general-sum case:

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i}) \qquad \forall i \in \mathcal{N}, \forall a_i, a_i' \in \mathcal{A}_i$$

$$\sum_{a \in \mathcal{A}} \sigma(a) = 1 \qquad \sigma(a) \geq 0 \qquad \forall a \in \mathcal{A}$$

Stackelberg Equilibrium

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

$$\arg\max_{s \in \mathcal{S}; \forall i \in \mathcal{N} \setminus \{1\} s_i \in BR_i(s_{-i})} u_1(s)$$

Stackelberg Equilibrium

Consider the following game:

	L	**R**
U	$(4, 2)$	$(6, 1)$
D	$(3, 1)$	$(5, 2)$

Stackelberg Equilibrium

Consider the following game:

	$\mathbf{L}$	$\mathbf{R}$
$\mathbf{U}$	$(4,2)$	$(6,1)$
$\mathbf{D}$	$(3,1)$	$(5,2)$

$(\mathbf{U},\mathbf{L})$ is a Nash equilibrium.

Stackelberg Equilibrium

Consider the following game:

	$\mathbf{L}$	$\mathbf{R}$
$\mathbf{U}$	$(4,2)$	$(6,1)$
$\mathbf{D}$	$(3,1)$	$(5,2)$

$(\mathbf{U}, \mathbf{L})$ is a Nash equilibrium.

What happens when the row player commits to play strategy $\mathbf{D}$ with probability 1? Can the row player get even more?

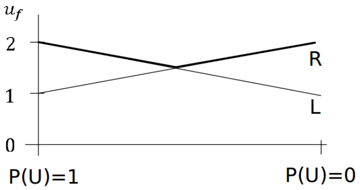
Stackelberg Equilibrium

Consider the following game:

	L	**R**
U	$(4, 2)$	$(6, 1)$
D	$(3, 1)$	$(5, 2)$

$(\mathbf{U}, \mathbf{L})$ is a Nash equilibrium.

What happens when the row player commits to play strategy $\mathbf{D}$ with probability 1? Can the row player get even more?



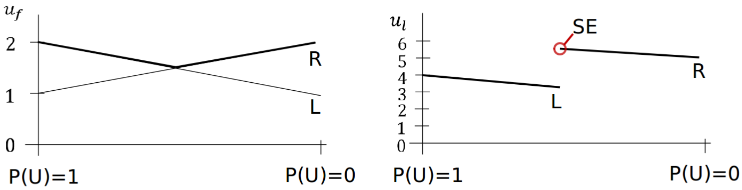
Stackelberg Equilibrium

Consider the following game:

	L	**R**
U	$(4, 2)$	$(6, 1)$
D	$(3, 1)$	$(5, 2)$

$(\mathbf{U}, \mathbf{L})$ is a Nash equilibrium.

What happens when the row player commits to play strategy $\mathbf{D}$ with probability 1? Can the row player get even more?



There may be Multiple Nash Equilibria

There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

- arbitrary but fixed tie breaking rule

There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),

There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),
- *Weak SE* – the followers select such NE that minimizes the outcome of the leader.

There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),
- *Weak SE* – the followers select such NE that minimizes the outcome of the leader.

Exact Weak Stackelberg equilibrium does not have to exist.

Different Stackelberg Equilibria

Exact Weak Stackelberg equilibrium does not have to exist.

Different Stackelberg Equilibria

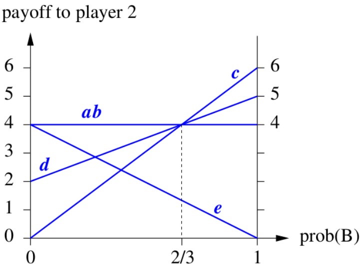
Exact Weak Stackelberg equilibrium does not have to exist.

$1 \setminus 2$	a	b	c	d	e
T	$(2,4)$	$(6,4)$	$(9,0)$	$(1,2)$	$(7,4)$
B	$(8,4)$	$(0,4)$	$(3,6)$	$(1,5)$	$(0,0)$

Different Stackelberg Equilibria

Exact Weak Stackelberg equilibrium does not have to exist.

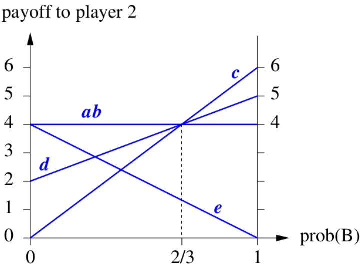
$1 \setminus 2$	a	b	c	d	e
T	$(2,4)$	$(6,4)$	$(9,0)$	$(1,2)$	$(7,4)$
B	$(8,4)$	$(0,4)$	$(3,6)$	$(1,5)$	$(0,0)$

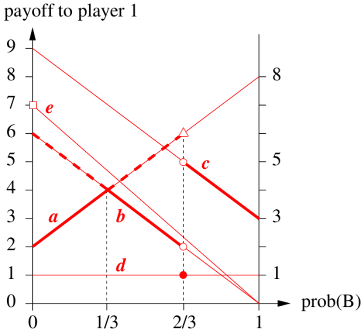


Different Stackelberg Equilibria

Exact Weak Stackelberg equilibrium does not have to exist.

$1 \setminus 2$	a	b	c	d	e
T	$(2,4)$	$(6,4)$	$(9,0)$	$(1,2)$	$(7,4)$
B	$(8,4)$	$(0,4)$	$(3,6)$	$(1,5)$	$(0,0)$





Computing a Stackelberg equilibrium in NFGs

Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower.

Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower.

Baseline polynomial algorithm requires solving $|\mathcal{A}_2|$ linear programs:

Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower.

Baseline polynomial algorithm requires solving $|\mathcal{A}_2|$ linear programs:

$$\max_{s_1 \in \mathcal{S}_1} \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_1(a_1, a_2)$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a_2) \geq \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a'_2) \quad \forall a'_2 \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1$$

one for each $a_2 \in \mathcal{A}_2$ assuming a_2 is the best response of the follower.