Normal-Form Games

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Noncooperative Game Theory

- **Single round games**
  - **Normal-form games**
  - **Extensive-form games**
  - **MAIDS, Congestion games**
- Multiple round games
  - Repeated games
  - Stochastic games

Types of games

- Two-player vs n-player
- Zero-sum games vs general-sum games
- Sequential vs one-shot
- Perfect-information vs imperfect-information
- Finite vs infinite

Types of games

- **Two-player** vs n-player
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- **Finite** vs infinite

Normal-form games

- Players set $P = \{1, ..., n\}$
- Actions set $A = A_1 \times ... \times A_i$
- Utility functions $u = \langle u_1, ...u_n \rangle$, where $u_i : A \to \mathbb{R}$

Normal-form games

- Represented as n-dimensional matrix
- Every entry is n-dimensional tuple of utilities for every player

Strategies

- A pure strategy s_i in normal-form games represents the choice of specific action $a \in A_i$ for player i
- A mixed strategy m_i is a strategy distribution over pure strategies
- Strategy profile s/m is a set of pure/mixed strategies, one for every player
- Overloading of utility function $u(s_i, s_{-i})$, $u(m_i, m_{-i})$, $u(m)$

Why GT

- Why do we need Game Theory?

Approaches for reasoning about games

- Studying game structure/properties
  - Social welfare optimality
  - Pareto optimality
- Stable strategies (solution concepts)
  - Maxmin
  - Minmax
  - Nash equilibrium
  - Stackelberg equilibrium
  - Correlated equilibrium
- Computation helpers
  - Dominance

Social welfare

- Defined as

$$WF = \sum_{i \in P} u_i(m) \tag{1}$$

- Not stable against deviations
- Cooperative players

Pareto optimality

- Reasoning about outocomes
- Outcome o pareto dominates outcome o' iff

$$\forall i \in P : o_i \geq o'_i \ and \ \exists i \in P : o_i > o'_i \tag{2}$$

- Outcome o is pareto optimal if it is not pareto dominated by any other outcome o'

Dominance

- Strict dominance
  - Strategy s_i strictly dominates s_i' iff

$$\forall s_{-i} \in S_{-i} : u(s_i, s_{-i}) > u(s_i', s_{-i}) \tag{3}$$

- Weak dominance
  - Strategy s_i weakly dominates s_i' iff

$$\forall s_{-i} \in S_{-i} : u(s_i, s_{-i}) \geq u(s_i', s_{-i}) \; and \tag{4}$$
$$\exists s_{-i} \in S_{-i} : u(s_i, s_{-i}) > u(s_i', s_{-i}) \tag{5}$$

- Very weak dominance
  - Strategy s_i very weakly dominates s_i' iff

$$\forall s_{-i} \in S_{-i} : u(s_i, s_{-i}) \geq u(s_i', s_{-i}) \tag{6}$$

Nash equilibrium

- A strategy m_i^* is the best response to strategies m_{-i}, written as $m_i^* \in BR(m_{-i})$ iff

$$\forall m_i \in \mathcal{M} u_i(m_i^*, m_{-i}) \geq u_i(m_i, m_{-i}) \tag{7}$$

- Nash equilibrium
  - Strategy profile $m = \{m_1, ..., m_n\}$ is a Nash equilibrium iff

$$\forall i \in P : m_i \in BR(m_{-i}) \tag{8}$$

- Stable against deviations of players as every player plays his best response to the strategies of the rest
- Assumes self-interested rational players
- Every finite game has a non-empty set of Nash equilibria
- Examples

Properties of NE

- Values in NE might differ
- Strategies not interchangeable
- Mistake of the opponent might hurt me

Properties of NE in zero-sum games

- All NE have the same value for i (value of the game)
- The value is guaranteed (mistakes of the opponent only increase my expected outcome)
- Strategies are interchangeable between NE
- minmax = maxmin = NE = SE

LP for solving zero-sum NFG

$$max_{U_i, m_i(a)} \; U_i \tag{9}$$

$$s.t. \sum_{s_i \in S_i} u_i(s_i, s_{-i}) m_i(s_i) \geq U_i, \quad \forall s_{-i} \in S_{-i} \tag{10}$$

$$\sum_{s_i \in S_i} m_i(s_i) = 1 \tag{11}$$

$$m_i(s_i) \geq 0, \quad \forall s_i \in S_i \tag{12}$$

- All NE are feasible solutions of this LP