Multiagent Systems (BE4M36MAS)

Mechanism Design and Auctions

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Previously ... on multi-agent systems.

1. Games
2. Social Choice
3. Auctions and Resource Allocations

Motivation

We want to design rules for the game.

Consider a bribe.

	L	R
U	(3,3)	(6,4)
D	(4,6)	(2,2)

	L	R
U	(**13**,3)	(6,4)
D	(4,6)	(2,**12**)

Consider a mediator

	C	D
c	(4,4)	(0,6)
d	(6,0)	(1,1)

	M	C	D
m	(4,4)	(6,0)	(1,1)
c	(0,6)	(4,4)	(0,6)
d	(1,1)	(6,0)	(1,1)

Bayesian Games

Definition (Bayesian Game)

A **Bayesian game** is a tuple $\langle \mathcal{N}, \mathcal{O}, \Theta, \rho, u \rangle$ where

- $\mathcal{N} = \{1, \ldots, n\}$ is the set of players
- $\mathcal{O}$ is a set of outcomes
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$, Θ_i is the type space of player i
- $\rho : \Theta \to [0, 1]$ is a common prior over types Θ
- $u = u_1, \ldots, u_n$, where $u_i : \Theta \times \mathcal{O} \to \mathbb{R}$ is the utility function of player i

Bayes-Nash equilibrium (BNE): rational, risk-neutral players are seeking to maximize their expected payoff, given their beliefs about the other players' types.

Mechanism

Definition (Mechanism)

A **mechanism** (for a Bayesian game setting $\langle \mathcal{N}, \mathcal{O}, \Theta, \rho, u \rangle$) is a pair (A, M), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent $i \in \mathcal{N}$; and
- $M : A \to \prod(\mathcal{O})$ maps each action profile to a distribution over outcomes.

A mechanism is **deterministic** if for every $a \in A$, there exists $o \in \mathcal{O}$ such that $M(a)(o) = 1$; in this case we write simply $M(a) = o$.

Implementation of Strategies in a Mechanism

Definition (Implementation in dominant strategies)

Given a Bayesian game setting $\langle \mathcal{N}, \mathcal{O}, \Theta, \rho, u \rangle$ a mechanism (A, M) is an implementation in dominant strategies of a social choice function C (over $\mathcal{N}$ and $\mathcal{O}$) if for any vector of utility functions u, the game has an equilibrium in dominant strategies, and in any such equilibrium a^* we have $M(a^*) = C(u)$.

We can have other implementations (e.g., a **Bayes-Nash equilibrium**).

Truthful Mechanisms

Definition (Truthfulness)

A mechanism is called **truthful** when agents truthfully disclose their preferences to the mechanism in an equilibrium.

We can achieve such mechanism by simply asking the agents for their type (e.g., their true valuations).

Such mechanisms are called **direct mechanisms**; in these mechanisms, the only action available to each agent is to announce his private information. In a Bayesian game an agents private information is his type; hence, direct mechanisms have $A_i = \Theta_i$.

Revelation Principle

Theorem (Revelation Principle)

If there exists any mechanism that implements a social choice function C in dominant strategies then there exists a direct mechanism that implements C in dominant strategies and is truthful.

Theorem (Gibbard-Satterthwaite)

Consider any social choice function C of $\mathcal{N}$ and $\mathcal{O}$. If:

- $|\mathcal{O}| \geq 3$ *(there are at least three outcomes);*
- C *is onto; that is, for every* $o \in \mathcal{O}$ *there is a preference profile* $[\succ]$ *such that* $C([\succ]) = o$*, and*
- C *is dominant-strategy truthful,*

then C *is dictatorial.*

Quasilinear utility function

We may restrict some assumptions to go around the impossibility theorem.

Definition (Quasilinear utility function)

Agents have **quasilinear utility functions** in an n-player Bayesian game when the set of outcomes is $\mathcal{O} = X \times \mathbb{R}^n$ for a finite set X, and the utility of an agent i given joint type Θ is given by $u_i(o, \Theta) = u_i(x, \Theta) - f_i(p_i)$, where $o = (x, p)$ is an element of $\mathcal{O}$, $u_i : X \times \Theta \to \mathbb{R}$ is an arbitrary function and $f_i : \mathbb{R} \to \mathbb{R}$ is a strictly monotonically increasing function.

(for example, x is an allocation of the item(s) in an auction, p is the payment)

Quasilinear utility function

Definition (Quasilinear mechanism)

A mechanism in the quasilinear setting (for a Bayesian game setting $\langle \mathcal{N}, \mathcal{O} = X \times \mathbb{R}^n, \Theta, \rho, u \rangle$) is a triple (A, χ, p), where:

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent $i \in \mathcal{N}$,
- $\chi : A \to \prod(X)$ maps each action profile to a distribution over choices, and
- $p : A \to \mathbb{R}^n$ maps each action profile to a payment for each agent.

We simplify the notation and denote v_i as a true valuation for the item (type of the player), and $\hat{v}_i$ the action (bid) of the agent.

Quasilinear utility function

We can define several desirable properties, such as individual rationality (ex interim, ex post).

Definition (Efficiency)

A quasilinear mechanism is strictly **Pareto efficient**, or just **efficient**, if in equilibrium it selects a choice x such that $\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x')$.

Definition (Revenue maximization)

A quasilinear mechanism is **revenue maximizing** when, among the set of functions χ and p that satisfy the other constraints, the mechanism selects the χ and p that maximize $\mathbb{E}_v\left[\sum_i p_i(s(v))\right]$, where $s(v)$ denotes the agents' equilibrium strategy profile.

Groves mechanisms

Definition (Groves mechanisms)

Groves mechanisms are direct quasilinear mechanisms (χ, p), for which:

- $\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$
- $p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$

Theorem

Truth telling is a dominant strategy under any Groves mechanism.

Vickrey-Clark-Groves (VCG) mechanism

Definition (Clarke Tax)

The **Clarke tax** sets the h_i term in a Groves mechanism as

$$h_i(\hat{v}_i) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i}))$$

where χ is the Groves mechanism allocation function.

Definition (Vickrey-Clark-Groves (VCG) mechanism)

The VCG mechanism is a direct quasilinear mechanism (χ, p), where

- $\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$
- $p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$

Second-Price as VCG mechanism

Sealed-bid second-price auction is a direct application of VCG mechanism in a symmetric auction.

Choices (x_i represents that item is assigned to agent i):

$$X = \{x_i | i \in \mathcal{N}\}$$

Valuations:

$$V_i = \{v_i \mid v_i(x_i) \geq 0, \forall j \neq i, v_i(x_j) = 0\}$$

Revenue equivalence

Consider a sealed-bid auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from the interval $[0, 1]$. What the is the expected revenue in auctions when players follow equilibrium strategies?

1. first-price sealed-bid auction
2. second-price sealed-bid auction

Recall that the equilibrium strategy for the first-price auction is $\left(\frac{v_1}{2}, \frac{v_2}{2}\right)$ and (v_1, v_2) for the second-price.

1. $\mathbb{E}\left[\max\left\{\frac{v_1}{2}, \frac{v_2}{2}\right\}\right] = \int_0^1 z^2 dz = \frac{1}{3}$
2. $\mathbb{E}\left[\min\left\{v_1, v_2\right\}\right] = \frac{1}{3}$

Revenue equivalence

To some extent, the expected revenue is equivalent under different auctions.

Theorem

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. Then any **efficient** *auction mechanism in which any agent with valuation $\underline{v}$ has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation v_i making the same expected payment.*

Can we do even better? What if we relax the efficiency assumption and decide not to sell unless there is a reasonable price?

Towards Optimal Auctions

Recall a sealed-bid auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from the interval $[0, 1]$. What the is the expected revenue in auctions when players follow equilibrium strategies in a second-price sealed-bid auction if there is a reserve price of R?

- no sale if both bids are below R (happens with probability R^2)
- sale at price R if one bid is above the reserve price and the second one is below (happens with probability $2(1 - R)R$)
- sale at second highest price if both bods are above R (happens with probability $(1 - R)^2$)

Expected revenue $\frac{1+3R^2-4R^3}{3}$ is in our example maximized for $R = \frac{1}{2}$, with value $\frac{5}{12} > \frac{1}{3}$.

Towards Optimal Mechanisms

Assume that the valuations of the agents, $v_1, \ldots, v_n$, are drawn independently at random from known (but not necessarily identical) continuous probability distributions.

We denote by F_i the cumulative distribution function from which bidder i's valuation, v_i , is drawn and by f_i its density function.

We assume that $v_i \in [0, h]$ for all i.

Definition

The **virtual valuation** of agent i with valuation v_i is

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

Optimal Mechanisms

Theorem (Myerson (1981))

The optimal (single-good) auction in terms of a direct mechanism: The good is sold to the agent $i = \arg\max_i \phi_i(\hat{v}_i)$, *as long as* $\hat{v}_i \geq r_i^*$. *If the good is sold, the winning agent* i *is charged the smallest valuation that he could have declared while still remaining the winner:*

$$\inf\{v_i^* \ : \ \phi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \phi_i(v_i^*) \geq \phi_j(\hat{v}_j)\}$$

Corollary

In a symmetric setting, the optimal (single-good) auction is a second price auction with a reserve price of $r^* = \frac{1-F_i(r^*)}{f_i(r^*)}$.