

# Randomized Sampling-based Motion Planning Methods

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Lecture 06

**B4M36UIR – Artificial Intelligence in Robotics**

# Overview of the Lecture

- Part 1 – Randomized Sampling-based Motion Planning Methods
  - Sampling-Based Methods
  - Probabilistic Road Map (PRM)
  - Characteristics
  - Rapidly Exploring Random Tree (RRT)
- Part 2 – Optimal Sampling-based Motion Planning Methods
  - Optimal Motion Planners
  - Rapidly-exploring Random Graph (RRG)

# Part I

## Part 1 – Sampling-based Motion Planning

# (Randomized) Sampling-based Motion Planning

- It uses an explicit representation of the obstacles in  $\mathcal{C}$ -space

- A “black-box” function is used to evaluate a configuration  $q$  is a collision-free, e.g.,
- Based on geometrical models and testing collisions of the models
- In 2D or 3D shape of the robot and environment can be represented as sets of triangles, i.e., tessellated models
- Collision test – an intersection of triangles



E.g., using RAPID library <http://gamma.cs.unc.edu/OBB/>

- **Creates a discrete representation of  $\mathcal{C}_{free}$**
- Configurations in  $\mathcal{C}_{free}$  are sampled randomly and connected to a roadmap (**probabilistic roadmap**)
- Rather than full completeness they provide **probabilistic completeness** or resolution completeness

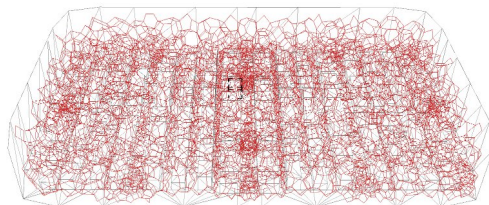
*Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)*

## Probabilistic Roadmaps

A discrete representation of the continuous  $\mathcal{C}$ -space generated by randomly sampled configurations in  $\mathcal{C}_{free}$  that are connected into a graph

- **Nodes** of the graph represent admissible configuration of the robot
- **Edges** represent a feasible path (trajectory) between the particular configurations

*Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)*



*Having the graph, the final path (trajectory) is found by a graph search technique*

# Incremental Sampling and Searching

- Single query sampling-based algorithms incrementally create a search graph (roadmap)
  1. **Initialization** –  $G(V, E)$  an undirected search graph,  $V$  may contain  $q_{start}$ ,  $q_{goal}$  and/or other points in  $\mathcal{C}_{free}$
  2. **Vertex selection method** – choose a vertex  $q_{cur} \in V$  for expansion
  3. **Local planning method** – for some  $q_{new} \in \mathcal{C}_{free}$ , attempt to construct a path  $\tau : [0, 1] \rightarrow \mathcal{C}_{free}$  such that  $\tau(0) = q_{cur}$  and  $\tau(1) = q_{new}$ ,  $\tau$  must be checked to ensure it is collision free
    - If  $\tau$  is not a collision-free, go to Step 2
  4. **Insert an edge in the graph** – Insert  $\tau$  into  $E$  as an edge from  $q_{cur}$  to  $q_{new}$  and insert  $q_{new}$  to  $V$  if  $q_{new} \notin V$
  5. **Check for a solution** – Determine if  $G$  encodes a solution, e.g., single search tree or graph search
  6. **Repeat to Step 2** – iterate unless a solution has been found or a termination condition is satisfied

LaValle, S. M.: Planning Algorithms (2006), Chapter 5.4

# Probabilistic Roadmap Strategies

## Multi-Query – roadmap based

- Generate a single roadmap that is then used for planning queries several times.
- An representative technique is **Probabilistic RoadMap (PRM)**

Kavraki, L., Svestka, P., Latombe, J.-C., Overmars, M. H.B (1996): Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces. T-RO.

## Single-Query – incremental

- For each planning problem constructs a new roadmap to characterize the subspace of  $\mathcal{C}$ -space that is relevant to the problem.
  - Rapidly-exploring Random Tree – RRT *LaValle, 1998*
  - Expansive-Space Tree – EST *Hsu et al., 1997*
  - Sampling-based Roadmap of Trees – SRT  
*(combination of multiple-query and single-query approaches)*  
*Plaku et al., 2005*

## Multi-Query Strategy

Build a roadmap (graph) representing the environment

### 1. Learning phase

1.1 Sample  $n$  points in  $C_{free}$

1.2 Connect the random configurations using a local planner

### 2. Query phase

2.1 Connect start and goal configurations with the PRM

*E.g., using a local planner*

2.2 Use the graph search to find the path



Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces

Lydia E. Kavraki and Petr Svestka and Jean-Claude Latombe and Mark H. Overmars,

IEEE Transactions on Robotics and Automation, 12(4):566–580, 1996.

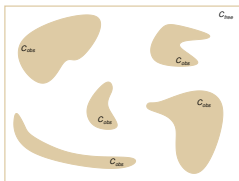
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*First planner that demonstrates ability to solve general planning problems in more than 4-5 dimensions.*

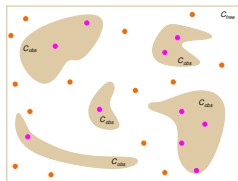


## PRM Construction

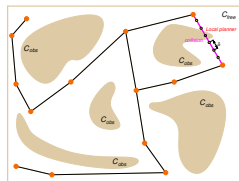
#1 Given problem domain



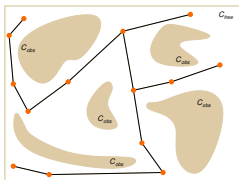
#2 Random configuration



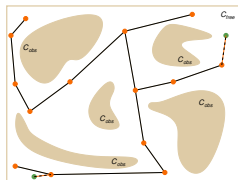
#3 Connecting samples



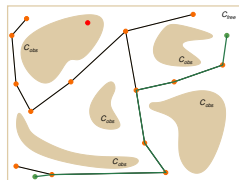
#4 Connected roadmap



#5 Query configurations

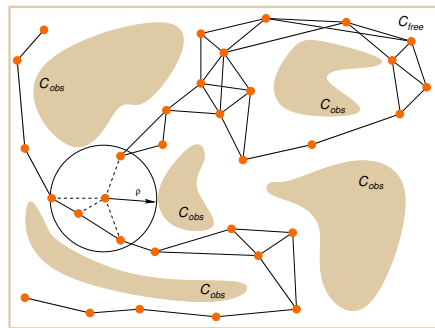


#6 Final found path



# Practical PRM

- Incremental construction
- Connect nodes in a radius  $\rho$
- Local planner tests collisions up to selected resolution  $\delta$
- Path can be found by Dijkstra's algorithm



What are the properties of the PRM algorithm?

*We need a couple of more formalisms.*

## Path Planning Problem Formulation

- Path planning problem is defined by a triplet

$$\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal}),$$

- $\mathcal{C}_{free} = \text{cl}(\mathcal{C} \setminus \mathcal{C}_{obs})$ ,  $\mathcal{C} = (0, 1)^d$ , for  $d \in \mathbb{N}$ ,  $d \geq 2$
  - $q_{init} \in \mathcal{C}_{free}$  is the initial configuration (condition)
  - $\mathcal{Q}_{goal}$  is the goal region defined as an open subspace of  $\mathcal{C}_{free}$
- Function  $\pi : [0, 1] \rightarrow \mathbb{R}^d$  of *bounded variation* is called:
  - **path** if it is continuous;
  - **collision-free path** if it is path and  $\pi(\tau) \in \mathcal{C}_{free}$  for  $\tau \in [0, 1]$ ;
  - **feasible** if it is collision-free path, and  $\pi(0) = q_{init}$  and  $\pi(1) \in \text{cl}(\mathcal{Q}_{goal})$ .
- A function  $\pi$  with the total variation  $\text{TV}(\pi) < \infty$  is said to have bounded variation, where  $\text{TV}(\pi)$  is the total variation
 
$$\text{TV}(\pi) = \sup_{\{n \in \mathbb{N}, 0 = \tau_0 < \tau_1 < \dots < \tau_n = s\}} \sum_{i=1}^n |\pi(\tau_i) - \pi(\tau_{i-1})|$$
- The total variation  $\text{TV}(\pi)$  is de facto a path length

# Path Planning Problem

## ■ Feasible path planning:

For a path planning problem  $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$

- Find a feasible path  $\pi : [0, 1] \rightarrow \mathcal{C}_{free}$  such that  $\pi(0) = q_{init}$  and  $\pi(1) \in \text{cl}(\mathcal{Q}_{goal})$ , if such path exists
- Report failure if no such path exists

## ■ Optimal path planning:

*The optimality problem asks for a feasible path with the minimum cost*

For  $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$  and a cost function  $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$

- Find a feasible path  $\pi^*$  such that  $c(\pi^*) = \min\{c(\pi) : \pi \text{ is feasible}\}$
- Report failure if no such path exists

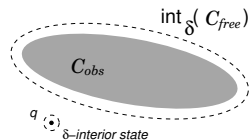
*The cost function is assumed to be monotonic and bounded, i.e., there exists  $k_c$  such that  $c(\pi) \leq k_c \text{TV}(\pi)$*

# Probabilistic Completeness 1/2

First, we need **robustly feasible path planning problem**

$(\mathcal{C}_{free}, q_{init}, Q_{goal})$

- $q \in \mathcal{C}_{free}$  is  **$\delta$ -interior state of  $\mathcal{C}_{free}$**  if the closed ball of radius  $\delta$  centered at  $q$  lies entirely inside  $\mathcal{C}_{free}$



- **$\delta$ -interior** of  $\mathcal{C}_{free}$  is  $\text{int}_\delta(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} \mid \mathcal{B}_{q,\delta} \subseteq \mathcal{C}_{free}\}$

*A collection of all  $\delta$ -interior states*

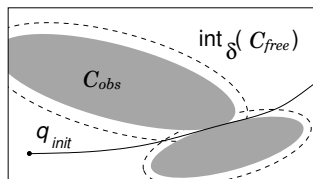
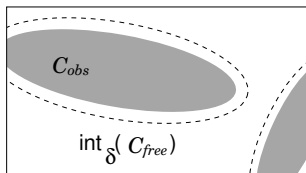
- A collision free path  $\pi$  has **strong  $\delta$ -clearance**, if  $\pi$  lies entirely inside  $\text{int}_\delta(\mathcal{C}_{free})$
- $(\mathcal{C}_{free}, q_{init}, Q_{goal})$  is **robustly feasible** if a solution exists and it is a feasible path with **strong  $\delta$ -clearance**, for  $\delta > 0$

## Probabilistic Completeness 2/2

An algorithm  $\mathcal{ALG}$  is **probabilistically complete** if, for any *robustly feasible path planning problem*  $\mathcal{P} = (C_{free}, q_{init}, Q_{goal})$

$$\lim_{n \rightarrow \infty} Pr(\mathcal{ALG} \text{ returns a solution to } \mathcal{P}) = 1.$$

- It is a “relaxed” notion of completeness
- Applicable only to problems with a **robust solution**



*We need some space, where random configurations can be sampled*

# Asymptotic Optimality 1/4

Asymptotic optimality relies on a notion of **weak  $\delta$ -clearance**

*Notice, we use strong  $\delta$ -clearance for probabilistic completeness*

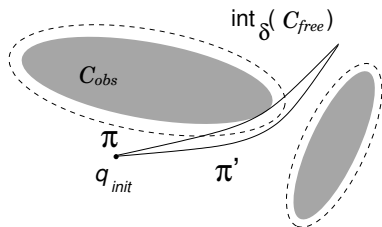
- Function  $\psi : [0, 1] \rightarrow \mathcal{C}_{free}$  is called **homotopy**, if  $\psi(0) = \pi_1$  and  $\psi(1) = \pi_2$  and  $\psi(\tau)$  is collision-free path for all  $\tau \in [0, 1]$
- A collision-free path  $\pi_1$  is **homotopic** to  $\pi_2$  if there exists homotopy function  $\psi$

*A path homotopic to  $\pi$  can be continuously transformed to  $\pi$  through  $\mathcal{C}_{free}$*

## Asymptotic Optimality 2/4

- A collision-free path  $\pi : [0, s] \rightarrow \mathcal{C}_{free}$  has **weak  $\delta$ -clearance** if there exists a path  $\pi'$  that has **strong  $\delta$ -clearance** and homotopy  $\psi$  with  $\psi(0) = \pi$ ,  $\psi(1) = \pi'$ , and for all  $\alpha \in (0, 1]$  there exists  $\delta_\alpha > 0$  such that  $\psi(\alpha)$  has strong  $\delta$ -clearance

*Weak  $\delta$ -clearance does not require points along a path to be at least a distance  $\delta$  away from obstacles*



- A path  $\pi$  with a weak  $\delta$ -clearance
- $\pi'$  lies in  $\text{int}_\delta(\mathcal{C}_{free})$  and it is the same homotopy class as  $\pi$



## Asymptotic Optimality 3/4

- It is applicable with a **robust optimal solution** that can be obtained as a limit of robust (non-optimal) solutions
- A collision-free path  $\pi^*$  is **robustly optimal solution** if it has *weak  $\delta$ -clearance* and for any sequence of collision free paths  $\{\pi_n\}_{n \in \mathbb{N}}$ ,  $\pi_n \in \mathcal{C}_{free}$  such that  $\lim_{n \rightarrow \infty} \pi_n = \pi^*$ ,

$$\lim_{n \rightarrow \infty} c(\pi_n) = c(\pi^*)$$

*There exists a path with strong  $\delta$ -clearance, and  $\pi^*$  is homotopic to such path and  $\pi^*$  is of **the lower cost**.*

- Weak  $\delta$ -clearance implies robustly feasible solution problem  
(*thus, probabilistic completeness*)

## Asymptotic Optimality 4/4

An algorithm  $\mathcal{ALG}$  is **asymptotically optimal** if, for any path planning problem  $\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$  and cost function  $c$  that admit a robust optimal solution with the finite cost  $c^*$

$$Pr \left( \left\{ \lim_{i \rightarrow \infty} Y_i^{\mathcal{ALG}} = c^* \right\} \right) = 1$$

- $Y_i^{\mathcal{ALG}}$  is the extended random variable corresponding to the minimum-cost solution included in the graph returned by  $\mathcal{ALG}$  at the end of iteration  $i$

## Properties of the PRM Algorithm

- Completeness for the standard PRM has not been provided when it was introduced
- A simplified version of the PRM (called sPRM) has been mostly studied
- sPRM is probabilistically complete

*What are the differences between PRM and sPRM?*

## PRM vs simplified PRM (sPRM)

**Algorithm 1: PRM****Vstup:**  $q_{init}$ , number of samples  $n$ , radius  $\rho$ **Výstup:** PRM –  $G = (V, E)$  $V \leftarrow \emptyset; E \leftarrow \emptyset;$ **for**  $i = 0, \dots, n$  **do**     $q_{rand} \leftarrow \text{SampleFree};$      $U \leftarrow \text{Near}(G = (V, E), q_{rand}, \rho);$      $V \leftarrow V \cup \{q_{rand}\};$     **foreach**  $u \in U$ , with increasing     $\|u - q_r\|$  **do**        **if**  $q_{rand}$  and  $u$  are not in the  
        same connected component of         $G = (V, E)$  **then**            **if**  $\text{CollisionFree}(q_{rand}, u)$             **then**                 $E \leftarrow E \cup$                  $\{(q_{rand}, u), (u, q_{rand})\};$ **return**  $G = (V, E);$ **Algorithm 2: sPRM****Vstup:**  $q_{init}$ , number of samples  $n$ ,radius  $\rho$ **Výstup:** PRM –  $G = (V, E)$  $V \leftarrow \{q_{init}\} \cup$  $\{\text{SampleFree}_i\}_{i=1, \dots, n-1}; E \leftarrow \emptyset;$ **foreach**  $v \in V$  **do**     $U \leftarrow \text{Near}(G = (V, E), v, \rho) \setminus \{v\};$     **foreach**  $u \in U$  **do**        **if**  $\text{CollisionFree}(v, u)$  **then**             $E \leftarrow E \cup \{(v, u), (u, v)\};$ **return**  $G = (V, E);$ There are several ways for the set  $U$  of vertices to connect them

- $k$ -nearest neighbors to  $v$
- variable connection radius  $\rho$  as a function of  $n$

## PRM – Properties

- **sPRM** (simplified PRM)
  - **Probabilistically complete and asymptotically optimal**
  - Processing complexity  $O(n^2)$
  - Query complexity  $O(n^2)$
  - Space complexity  $O(n^2)$
- Heuristics practically used are usually not probabilistic complete
  - $k$ -nearest sPRM is not probabilistically complete
  - variable radius sPRM is not probabilistically complete

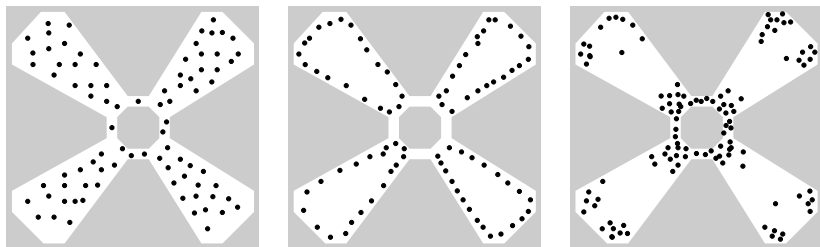
*Based on analysis of Karaman and Frazzoli*

### PRM algorithm:

- + Has very simple implementation
- + Completeness (for sPRM)
- Differential constraints (car-like vehicles) are not straightforward

## Comments about Random Sampling 1/2

- Different sampling strategies (distributions) may be applied



- Notice, one of the main issues of the randomized sampling-based approaches is the narrow passage
- Several modifications of sampling based strategies have been proposed in the last decades

## Comments about Random Sampling 2/2

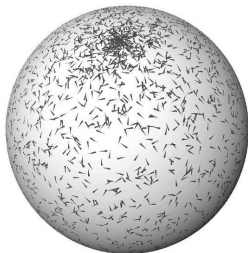
- A solution can be found using only a few samples.

*Do you know the Oracleum? (from Alice in Wonderland)*

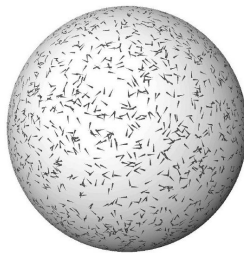
- Sampling strategies are important

- Near obstacles
- Narrow passages
- Grid-based
- Uniform sampling must be carefully considered

*James J. Kuffner (2004): Effective Sampling and Distance Metrics for 3D Rigid Body Path Planning. ICRA.*



Naïve sampling



Uniform sampling of  $SO(3)$  using Euler angles

# Rapidly Exploring Random Tree (RRT)

## Single-Query algorithm

- It incrementally builds a graph (tree) towards the goal area.

*It does not guarantee precise path to the goal configuration.*

1. Start with the initial configuration  $q_0$ , which is a root of the constructed graph (tree)

2. Generate a new random configuration  $q_{new}$  in  $\mathcal{C}_{free}$

3. Find the closest node  $q_{near}$  to  $q_{new}$  in the tree

*E.g., using KD-tree implementation like ANN or FLANN libraries*

4. Extend  $q_{near}$  towards  $q_{new}$

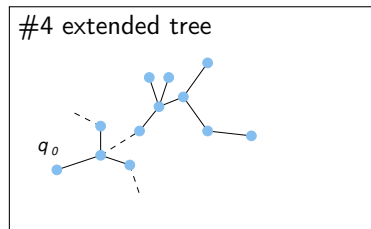
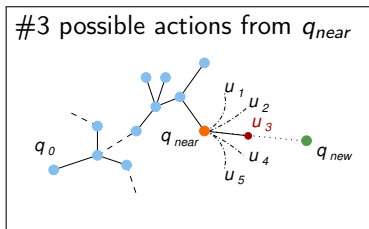
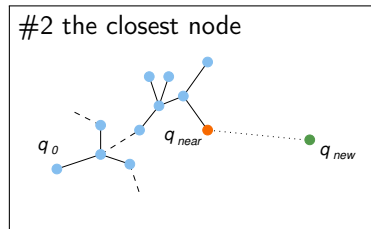
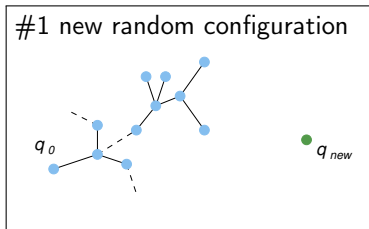
*Extend the tree by a small step, but often a direct control  $u \in \mathcal{U}$  that will move robot the position closest to  $q_{new}$  is selected (applied for  $\delta t$ )*

5. Go to Step 2, until the tree is within a sufficient distance from the goal configuration

*Or terminates after dedicated running time*



## RRT Construction



## RRT Algorithm

- Motivation is a single query and *control-based* path finding
- It incrementally builds a graph (tree) towards the goal area

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### Algorithm 3: Rapidly Exploring Random Tree (RRT)

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**Vstup:**  $q_{init}$ , number of samples  $n$

**Výstup:** Roadmap  $G = (V, E)$

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$V \leftarrow \{q_{init}\}; E \leftarrow \emptyset;$

**for**  $i = 1, \dots, n$  **do**

$q_{rand} \leftarrow \text{SampleFree};$

$q_{nearest} \leftarrow \text{Nearest}(G = (V, E), q_{rand});$

$q_{new} \leftarrow \text{Steer}(q_{nearest}, q_{rand});$

**if**  $\text{CollisionFree}(q_{nearest}, q_{new})$  **then**

$V \leftarrow V \cup \{x_{new}\}; E \leftarrow E \cup \{(x_{nearest}, x_{new})\};$

**return**  $G = (V, E);$

---

*Extend the tree by a small step, but often a direct control  $u \in \mathcal{U}$  that will move robot to the position closest to  $q_{new}$  is selected (applied for  $dt$ )*

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Rapidly-exploring random trees: A new tool for path planning

S. M. LaValle,

Technical Report 98-11, Computer Science Dept., Iowa State University, 1998

## Properties of RRT Algorithms

- Rapidly explores the space

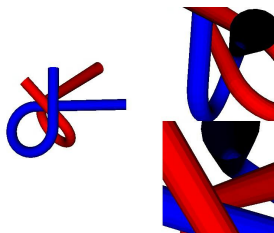
*$q_{new}$  will more likely be generated in large not yet covered parts*

- Allows considering kinodynamic/dynamic constraints (during the expansion)
- Can provide trajectory or a sequence of direct control commands for robot controllers
- A collision detection test is usually used as a “black-box”

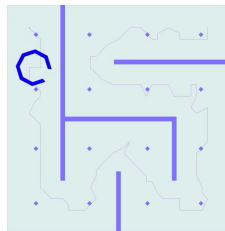
*E.g., RAPID, Bullet libraries*

- Similarly to PRM, RRT algorithms have poor performance in narrow passage problems
- RRT algorithms provides feasible paths  
*It can be relatively far from optimal solution, e.g., according to the length of the path*
- Many variants of RRT have been proposed

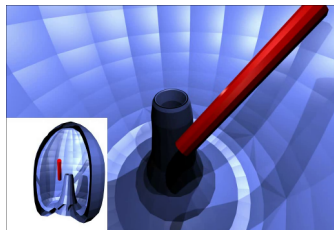
## RRT – Examples 1/2



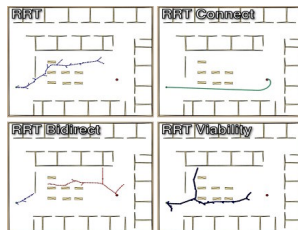
Alpha puzzle benchmark



Apply rotations to reach the goal



Bugtrap benchmark

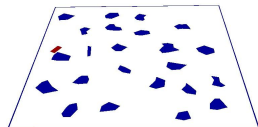


Variants of RRT algorithms

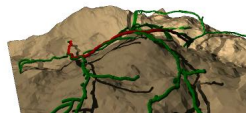
*Courtesy of V. Vonásek and P. Vaněk*

## RRT – Examples 2/2

- Planning for a car-like robot

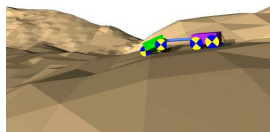


- Planning on a 3D surface



- Planning with dynamics

*(friction forces)*



*Courtesy of V. Vonásek and P. Vaněk*

# Car-Like Robot

## ■ Configuration

$$\vec{x} = \begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$$

*position and orientation*

## ■ Controls

$$\vec{u} = \begin{pmatrix} v \\ \varphi \end{pmatrix}$$

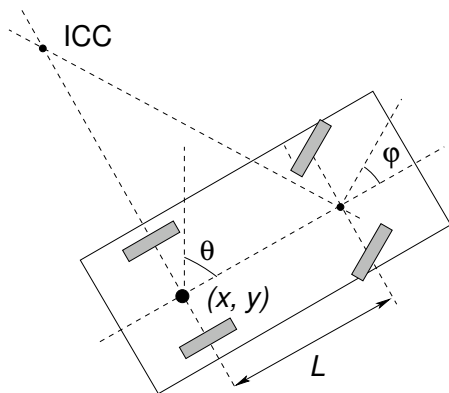
*forward velocity, steering angle*

## ■ System equation

$$\dot{x} = v \cos \phi$$

$$\dot{y} = v \sin \phi$$

$$\dot{\phi} = \frac{v}{L} \tan \varphi$$



*Kinematic constraints*  $\dim(\vec{u}) < \dim(\vec{x})$

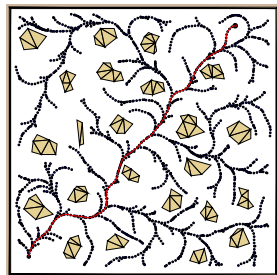
*Differential constraints on possible  $\dot{q}$ :*

$$\dot{x} \sin(\phi) - \dot{y} \cos(\phi) = 0$$

## Control-Based Sampling

- Select a configuration  $q$  from the tree  $T$  of the current configurations
- Pick a control input  $\vec{u} = (v, \phi)$  and integrate system (motion) equation over a short period

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \varphi \end{pmatrix} = \int_t^{t+\Delta t} \begin{pmatrix} v \cos \phi \\ v \sin \phi \\ \frac{v}{L} \tan \phi \end{pmatrix} dt$$



- If the motion is collision-free, add the endpoint to the tree

*E.g., considering  $k$  configurations for  $k\delta t = dt$*

## Part II

# Part 2 – Optimal Sampling-based Motion Planning Methods



# Sampling-Based Motion Planning

- PRM and RRT are theoretically probabilistic complete
- They provide a feasible solution without quality guarantee

*Despite that, they are successfully used in many practical applications*

- In 2011, a systematical study of the asymptotic behavior of randomized sampling-based planners has been published

*It shows, that in some cases, they converge to a non-optimal value with a probability 1*

- Based on the study, new algorithms have been proposed: **RRG** and optimal RRT (**RRT\***)

Karaman, S., Frazzoli, E. (2011): Sampling-based algorithms for optimal motion planning. IJRR.



<http://sertac.scripts.mit.edu/rrtstar>

## RRT and Quality of Solution 1/2

- Let  $Y_i^{RRT}$  be the cost of the best path in the RRT at the end of iteration  $i$
- $Y_i^{RRT}$  converges to a random variable

$$\lim_{i \rightarrow \infty} Y_i^{RRT} = Y_{\infty}^{RRT}$$

- The random variable  $Y_{\infty}^{RRT}$  is sampled from a distribution with zero mass at the optimum, and

$$Pr[Y_{\infty}^{RRT} > c^*] = 1$$

*Karaman and Frazzoli, 2011*

- The best path in the RRT converges to a sub-optimal solution almost surely

## RRT and Quality of Solution 2/2

- RRT does not satisfy a necessary condition for the asymptotic optimality
  - For  $0 < R < \inf_{q \in Q_{goal}} \|q - q_{init}\|$ , the event  $\{\lim_{n \rightarrow \infty} Y_n^{RRT} = c^*\}$  occurs only if the  $k$ -th branch of the RRT contains vertices outside the  $R$ -ball centered at  $q_{init}$  for infinitely many  $k$

*See Appendix B in Karaman&Frazzoli, 2011*

- It is required the root node will have infinitely many subtrees that extend at least a distance  $\epsilon$  away from  $q_{init}$ 

*The sub-optimality is caused by disallowing new better paths to be discovered*

## Rapidly-exploring Random Graph (RRG)

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**Algorithm 4:** Rapidly-exploring Random Graph (RRG)
 

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**Vstup:**  $q_{init}$ , number of samples  $n$ **Výstup:**  $G = (V, E)$  $V \leftarrow \emptyset; E \leftarrow \emptyset;$ **for**  $i = 0, \dots, n$  **do**     $q_{rand} \leftarrow \text{SampleFree};$      $q_{nearest} \leftarrow \text{Nearest}(G = (V, E), q_{rand});$      $q_{new} \leftarrow \text{Steer}(q_{nearest}, q_{rand});$     **if**  $\text{CollisionFree}(q_{nearest}, q_{new})$  **then**         $Q_{near} \leftarrow \text{Near}(G =$              $(V, E), q_{new}, \min\{\gamma_{RRG}(\log(\text{card}(V))/\text{card}(V))^{1/d}, \eta\});$          $V \leftarrow V \cup \{q_{new}\};$          $E \leftarrow E \cup \{(q_{nearest}, q_{new}), (q_{new}, q_{nearest})\};$         **foreach**  $q_{near} \in Q_{near}$  **do**            **if**  $\text{CollisionFree}(q_{near}, q_{new})$  **then**                 $E \leftarrow E \cup \{(q_{rand}, u), (u, q_{rand})\};$ 


---

**return**  $G = (V, E);$ 


---

*Proposed by Karaman and Frazzoli (2011). Theoretical results are related to properties of [Random Geometric Graphs \(RGG\)](#) introduced by Gilbert (1961) and further studied by Penrose (1999).*

## RRG Expansions

- At each iteration, RRG tries to connect new sample to all vertices in the  $r_n$  ball centered at it.
- The ball of radius

$$r(\text{card}(V)) = \min \left\{ \gamma_{RRG} \left( \frac{\log(\text{card}(V))}{\text{card}(V)} \right)^{1/d}, \eta \right\}$$

where

- $\eta$  is the constant of the local steering function
- $\gamma_{RRG} > \gamma_{RRG}^* = 2(1 + 1/d)^{1/d} (\mu(C_{free})/\xi_d)^{1/d}$ 
  - $d$  – dimension of the space;
  - $\mu(C_{free})$  – Lebesgue measure of the obstacle-free space;
  - $\xi_d$  – volume of the unit ball in  $d$ -dimensional Euclidean space.
- The connection radius decreases with  $n$
- The rate of decay  $\approx$  the average number of connections attempted is proportional to  $\log(n)$

# RRG Properties

- Probabilistically complete
- Asymptotically optimal
- Complexity is  $O(\log n)$   
*(per one sample)*
- Computational efficiency and optimality
  - Attempt connection to  $\Theta(\log n)$  nodes at each iteration;  
*in average*
    - Reduce volume of the “connection” ball as  $\log(n)/n$ ;
    - Increase the number of connections as  $\log(n)$

## Other Variants of the Optimal Motion Planning

- **PRM\*** – it follows standard PRM algorithm where connections are attempted between roadmap vertices that are within connection radius  $r$  as a function of  $n$

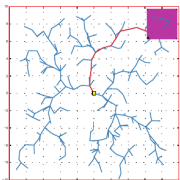
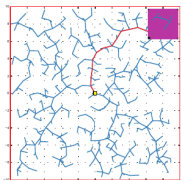
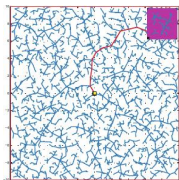
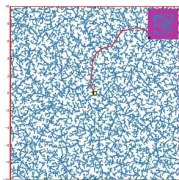
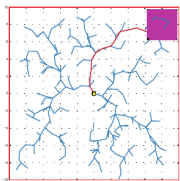
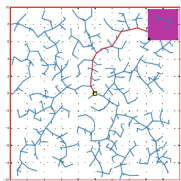
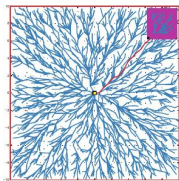
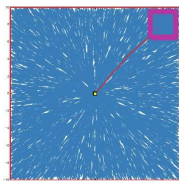
$$r(n) = \gamma_{PRM}(\log(n)/n)^{1/d}$$

- **RRT\*** – a modification of the RRG, where cycles are avoided

*A tree version of the RRG*

- A tree roadmap allows to consider non-holonomic dynamics and kinodynamic constraints
- It is basically RRG with “rerouting” the tree when a better path is discovered

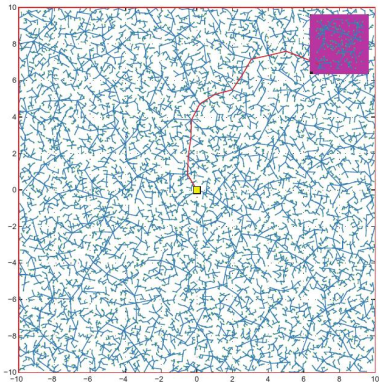
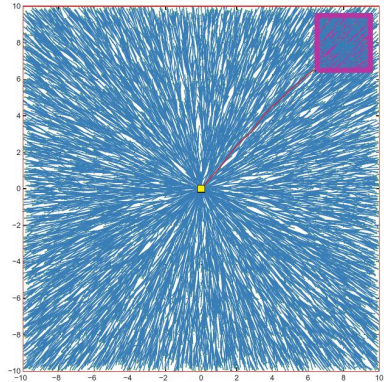
## Example of Solution 1/3

RRT,  $n=250$ RRT,  $n=500$ RRT,  $n=2500$ RRT,  $n=10000$ RRT\*,  $n=250$ RRT\*,  $n=500$ RRT\*,  $n=2500$ RRT\*,  $n=10000$ 

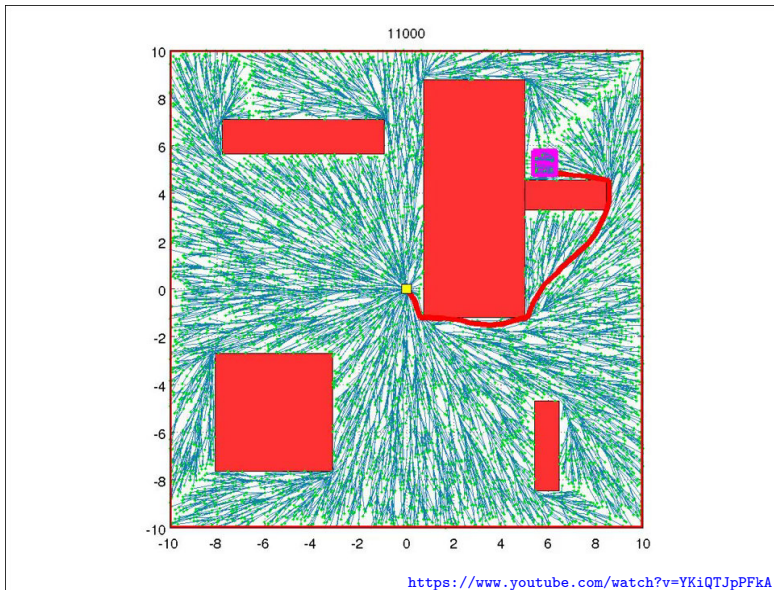
*Karaman & Frazzoli, 2011*



## Example of Solution 2/3

RRT,  $n=20000$ RRT\*,  $n=20000$

## Example of Solution 3/3



# Overview of Randomized Sampling-based Algorithms

Algorithm	Probabilistic Completeness	Asymptotic Optimality
sPRM	✓	✗
k-nearest sPRM	✗	✗
RRT	✓	✗
RRG	✓	✓
PRM*	✓	✓
RRT*	✓	✓

*Notice, k-nearest variants of RRG, PRM\*, and RRT\* are complete and optimal as well*

# Summary of the Lecture

## Topics Discussed

- Randomized Sampling-based Methods
- Probabilistic Road Map (PRM)
- Characteristics of path planning problems
- Random sampling
- Rapidly Exploring Random Tree (RRT)
- Optimal sampling-based motion planning
- Rapidly-exploring Random Graph (RRG)
  
- **Next: Multi-Goal Motion Planning and Multi-Goal Path Planning**