Sorting Algorithms

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Topic Overview

- Issues in Sorting on Parallel Computers
- Sorting Networks
- Bubble Sort and its Variants
- Quicksort
- Bucket and Sample Sort
- Other Sorting Algorithms

Sorting: Overview

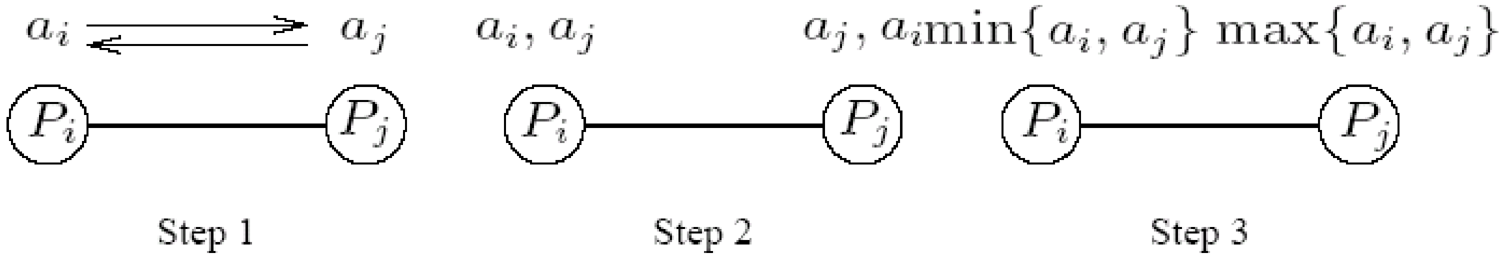
- One of the most commonly used and well-studied kernels.
- Sorting can be ***comparison-based*** or ***noncomparison-based***.
- The fundamental operation of comparison-based sorting is ***compare-exchange***.
- The lower bound on any comparison-based sort of n numbers is $\Theta(n \log n)$.
- We focus here on comparison-based sorting algorithms.

Sorting: Basics

What is a parallel sorted sequence? Where are the input and output lists stored?

- We assume that the **input and output lists are distributed**.
- The sorted list is partitioned with the property that each partitioned list is sorted and each element in processor P_i's list is less than that in P_j's list if $i < j$.

Sorting: Parallel Compare Exchange Operation



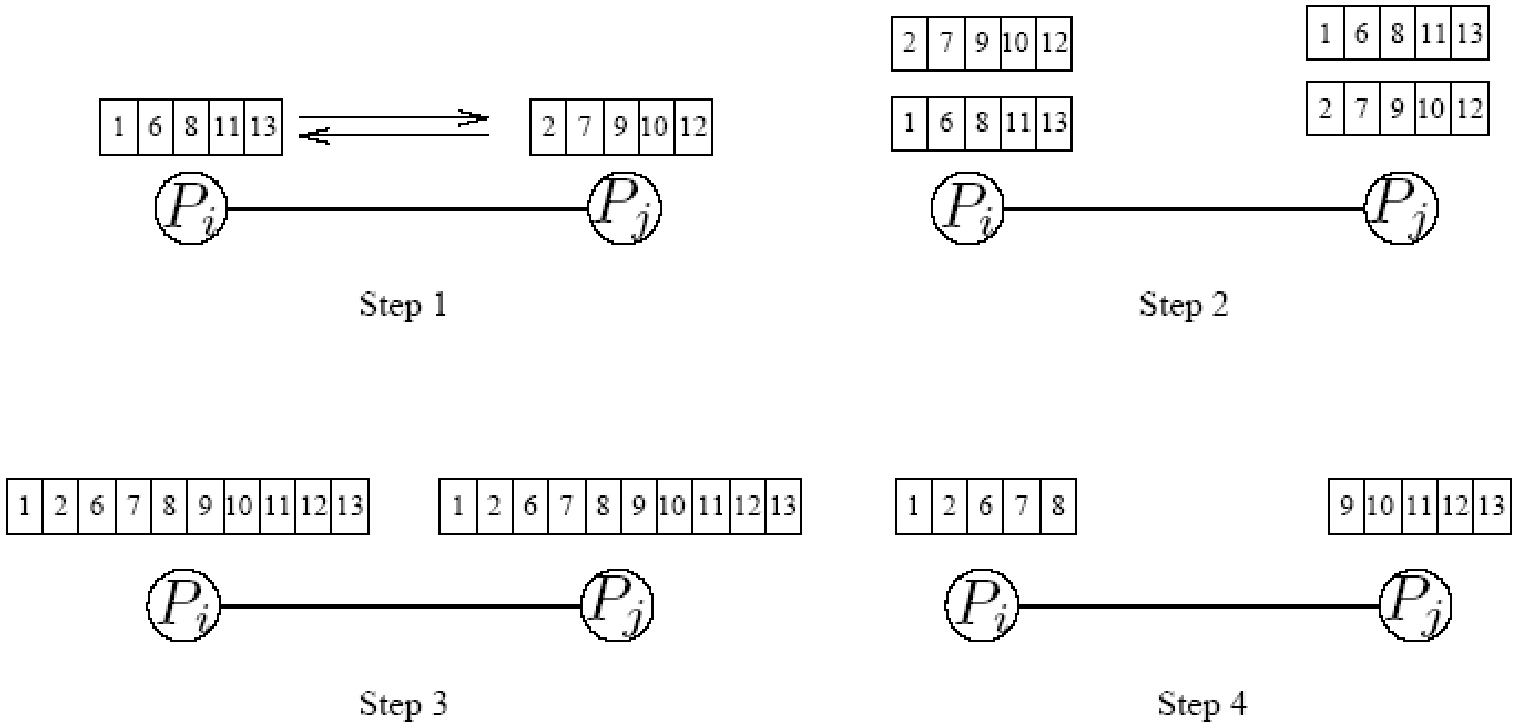
A parallel compare-exchange operation. Processes P_i and P_j send their elements to each other. Process P_i keeps $\min\{a_i,a_j\}$, and P_j keeps $\max\{a_i, a_j\}$.

Sorting: Basics

What is the parallel counterpart to a sequential comparator?

- If each processor has one element, the **compare exchange** operation **stores the smaller element at the processor with smaller id**. This can be done in $t_s + t_w$ time.
- If we have **more than one element per processor**, we call this operation a **compare split**. Assume each of two processors have n/p elements.
- After the compare-split operation, the smaller n/p elements are at processor P_i and the larger n/p elements at P_j, where $i < j$.
- The time for a compare-split operation is $(t_s + t_w n/p)$, assuming that the **two partial lists were initially sorted.**

Sorting: Parallel Compare Split Operation

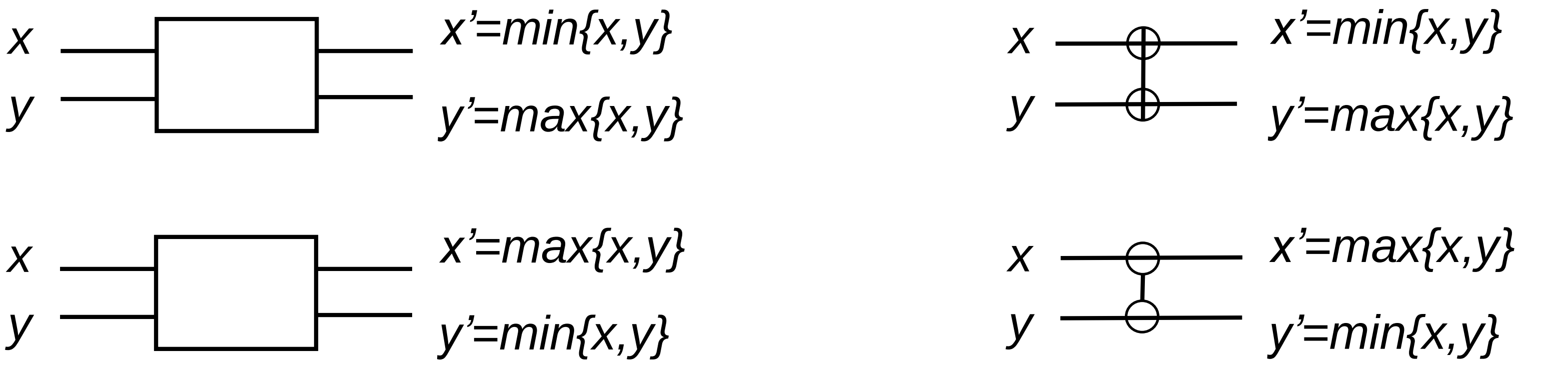


A **compare-split** operation. Each process **sends** its block of size *n/p* to the other process. Each process **merges** the received block with its own block and **retains only the appropriate half** of the merged block. In this example, process P_i retains the smaller elements and process P_i retains the larger elements.

Sorting Networks

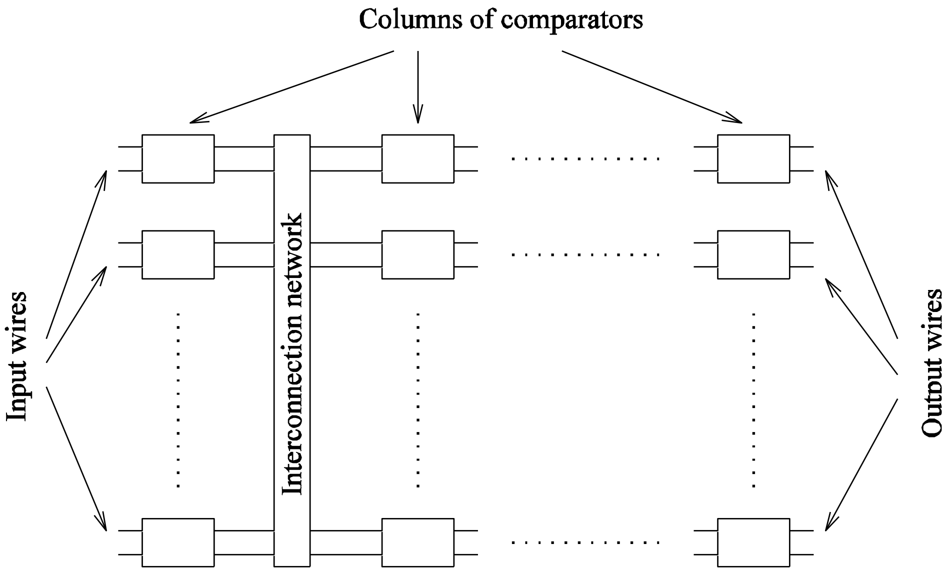
- Networks of comparators designed specifically for sorting.
- A comparator is a device with two inputs x and y and two outputs x' and y'. For an ***increasing comparator***, $x' = \min\{x,y\}$ and $y' = \max\{x,y\}$; and vice-versa.
- We denote an **increasing comparator** by $\oplus$ and a **decreasing comparator** by $\ominus$.
- The **speed** of the network is **proportional to its depth**.

Sorting Networks: Comparators



A schematic representation of comparators: (a) an increasing comparator, and (b) a decreasing comparator.

Sorting Networks



A typical sorting network. Every sorting network is made up of a **series of columns**, and each column contains a number of comparators connected in parallel.

Sorting Networks: Bitonic Sort

- A **bitonic sorting network** sorts n elements in $\Theta(\log^2 n)$ time.
- A bitonic sequence has two tones - **increasing** and **decreasing**, or vice versa. Any **cyclic rotation** of such sequence is also considered bitonic.
- $\langle 1,2,4,7,6,0 \rangle$ is a bitonic sequence, because it first increases and then decreases. $\langle 8,9,2,1,0,4 \rangle$ is another bitonic sequence, because it is a cyclic shift of $\langle 0,4,8,9,2,1 \rangle$.
- The kernel of the network is the **rearrangement of a bitonic sequence into a sorted sequence**.

Sorting Networks: Bitonic Sort

- Let $s = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be a bitonic sequence such that $a_0 \leq a_1 \leq \cdots \leq a_{n/2-1}$ and $a_{n/2} \geq a_{n/2+1} \geq \cdots \geq a_{n-1}$.

- Consider the following subsequences of s:

$$s_1 = \langle \min\{a_0, a_{n/2}\}, \min\{a_1, a_{n/2+1}\}, \ldots, \min\{a_{n/2-1}, a_{n-1}\} \rangle$$
$$s_2 = \langle \max\{a_0, a_{n/2}\}, \max\{a_1, a_{n/2+1}\}, \ldots, \max\{a_{n/2-1}, a_{n-1}\} \rangle \quad (1)$$

- Note that s_1 and s_2 are **both bitonic** and each element of s_1 is **less than every element** in s_2.

- We can **apply the procedure recursively** on s_1 and s_2 to get the sorted sequence.

Sorting Networks: Bitonic Sort

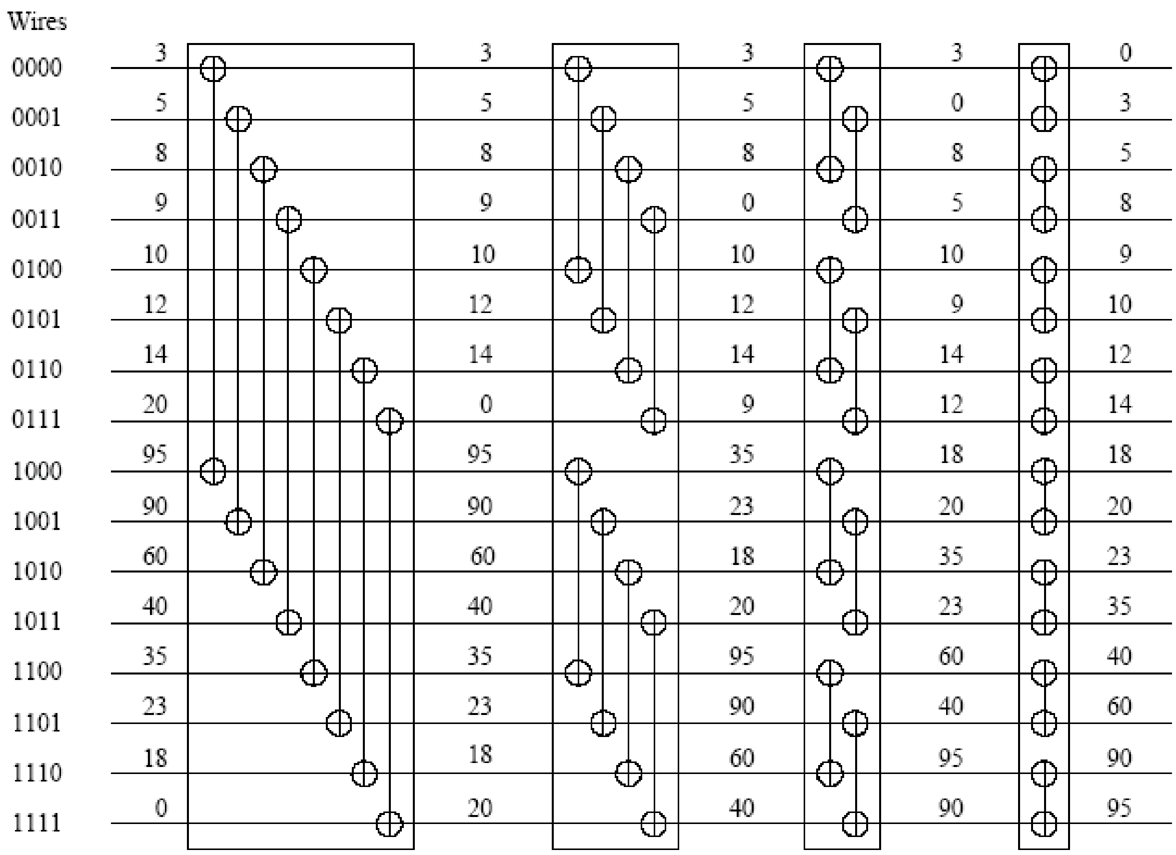
Original sequence	3	5	8	9	10	12	14	20	95	90	60	40	35	23	18	0
1st Split	3	5	8	9	10	12	14	0	95	90	60	40	35	23	18	20
2nd Split	3	5	8	0	10	12	14	9	35	23	18	20	95	90	60	40
3rd Split	3	0	8	5	10	9	14	12	18	20	35	23	60	40	95	90
4th Split	0	3	5	8	9	10	12	14	18	20	23	35	40	60	90	95

Merging a 16-element bitonic sequence through a series of $\log 16$ bitonic splits.

Sorting Networks: Bitonic Sort

- We can easily build a sorting network to implement this bitonic merge algorithm.
- Such a network is called a ***bitonic merging network***.
- The network contains **$\log n$ columns**. Each column contains ***n/2*** **comparators** and performs one step of the bitonic merge.
- We denote a bitonic merging network with n inputs by **⊕BM[n]**.
- Replacing the ⊕ comparators by ⊖ comparators results in a **decreasing output sequence**; such a network is denoted by **⊖BM[n]**.

Sorting Networks: Bitonic Sort



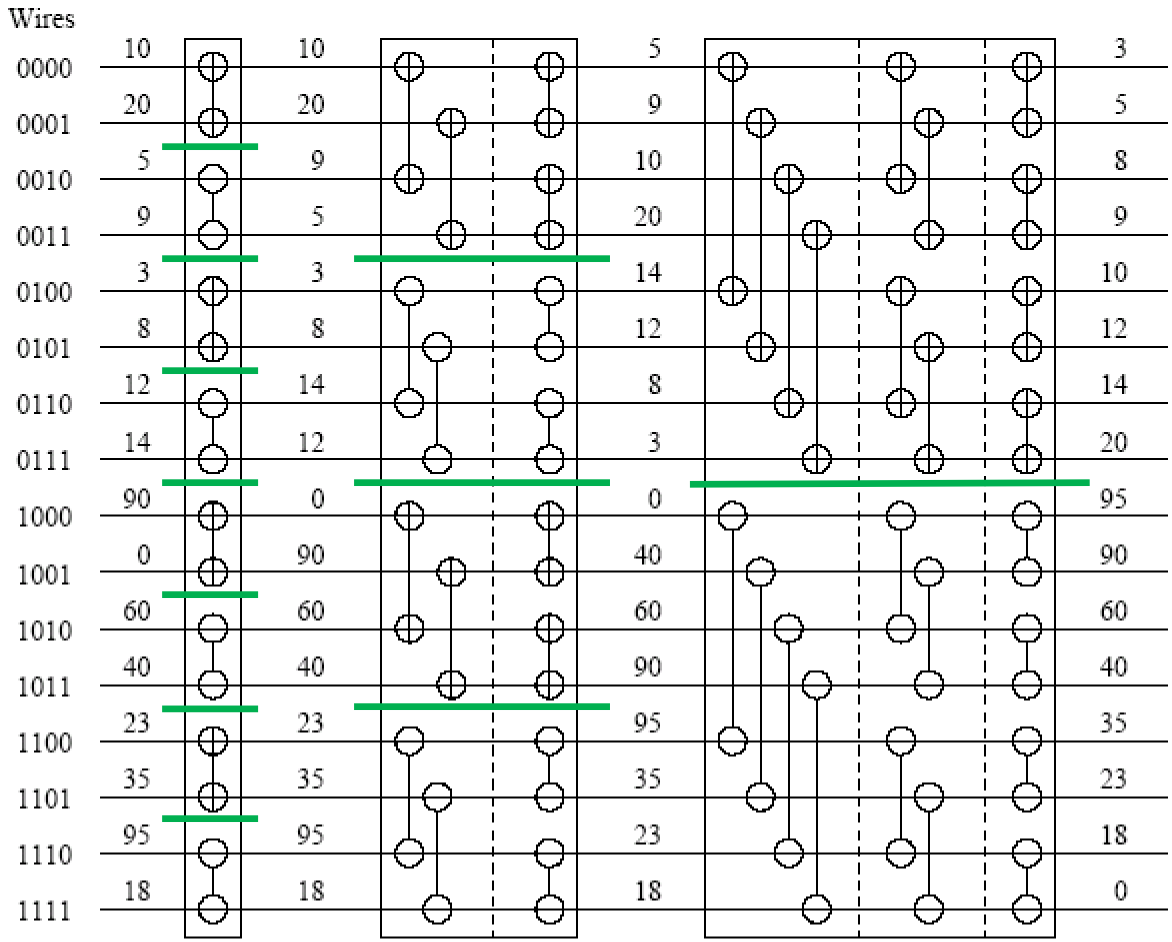
A bitonic merging network for $n = 16$. The input wires are numbered $0,1,\ldots,$ $n - 1$, and the binary representation of these numbers is shown. Each column of comparators is drawn separately; the entire figure represents a $\oplus$BM[16] bitonic merging network. The network **takes a bitonic sequence and outputs it in sorted order**.

Sorting Networks: Bitonic Sort

How do we **sort an unsorted sequence** using a bitonic merge?

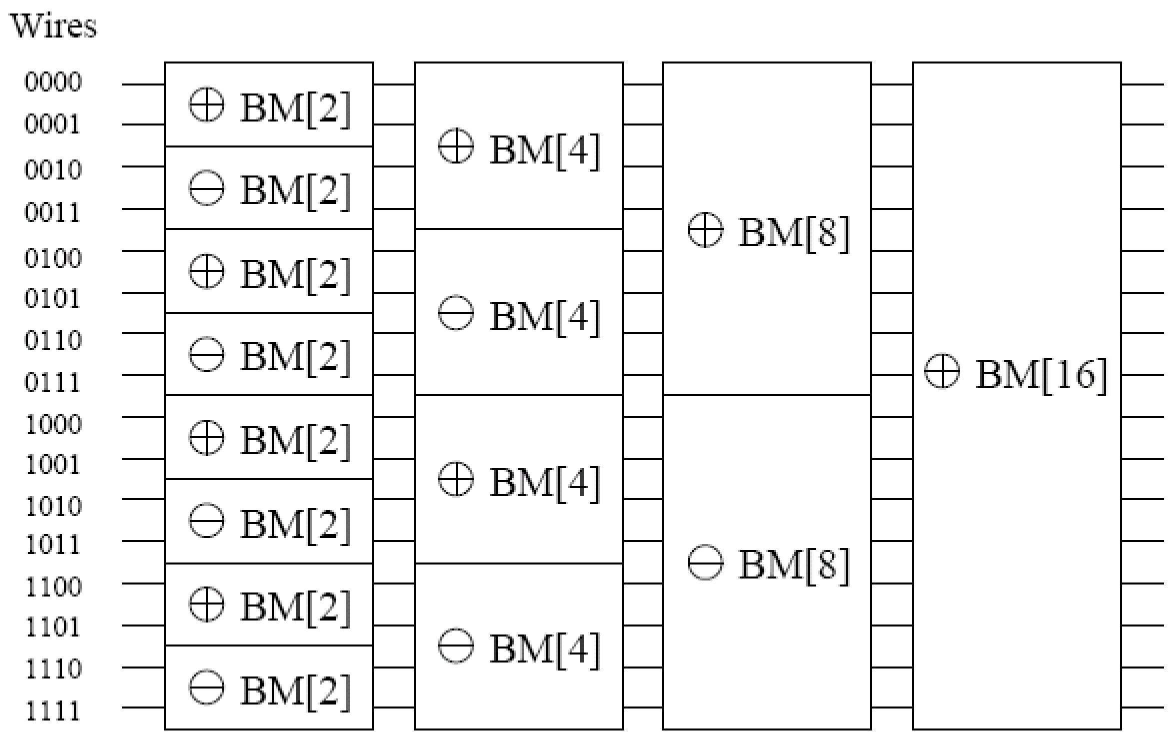
- We must **first build a single bitonic sequence** from the given sequence.
- A **sequence of length 2 is a bitonic sequence**.
- A bitonic sequence of length 4 can be built by sorting the first two elements using ⊕BM[2] and next two using ⊖BM[2].
- This process can be **repeated to generate larger bitonic sequences**.

Sorting Networks: Bitonic Sort



The comparator network that **transforms an input sequence** of 16 unordered numbers **into a bitonic sequence**.

Sorting Networks: Bitonic Sort



A schematic representation of a network that converts an input sequence into a bitonic sequence. In this example, ⊕BM[k] and ⊖BM[k] denote bitonic merging networks of input size k that use ⊕ and ⊖ comparators, respectively. **The last merging network (⊕BM[16]) sorts the input**. In this example, $n = 16$.

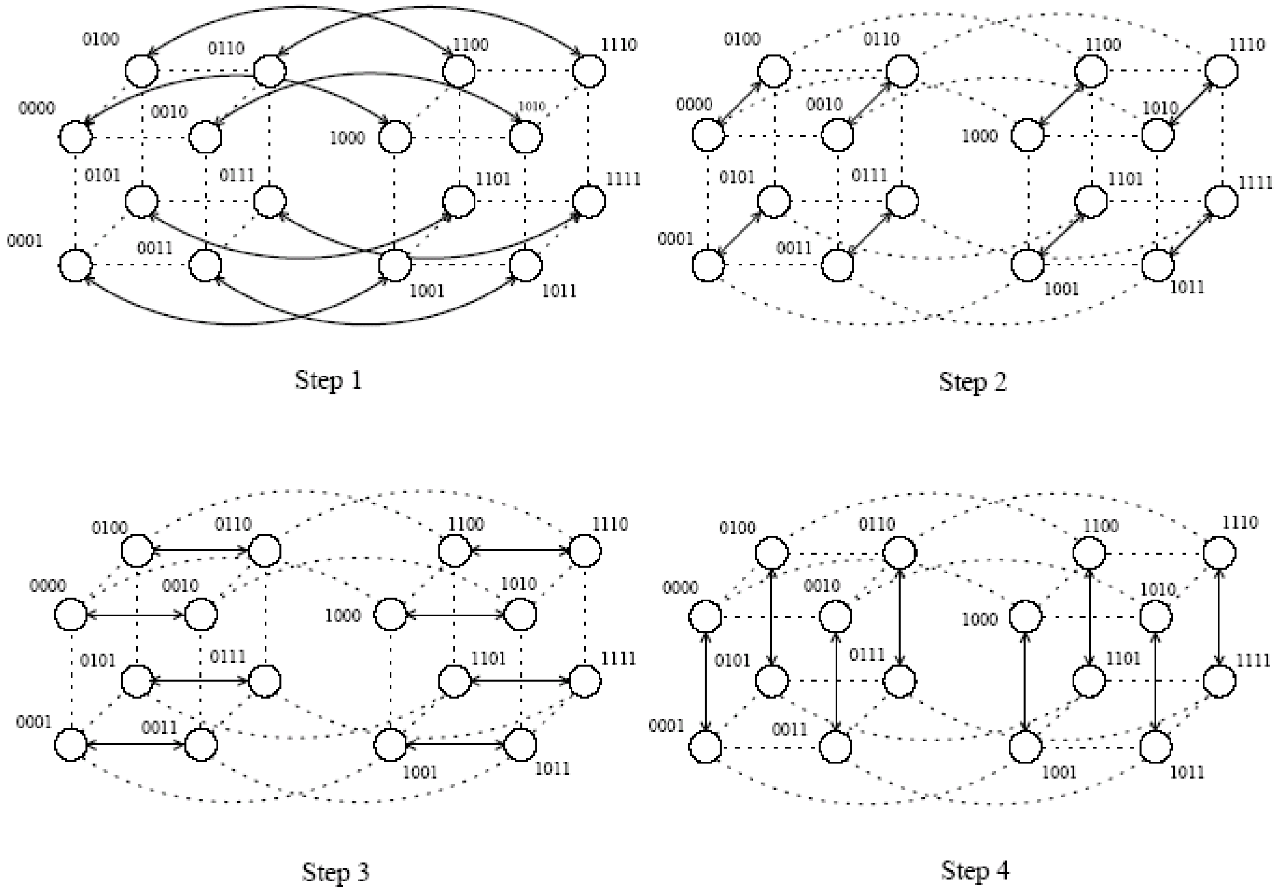
Sorting Networks: Bitonic Sort

- The **depth of the network** is d(n) = d(n/2) + log n, i.e. d(n) = $\Theta(\log^2 n)$.
- Each stage of the network contains $n/2$ comparators. A **serial implementation** of the network would have complexity $\Theta(n \log^2 n)$.

Mapping Bitonic Sort to Hypercubes

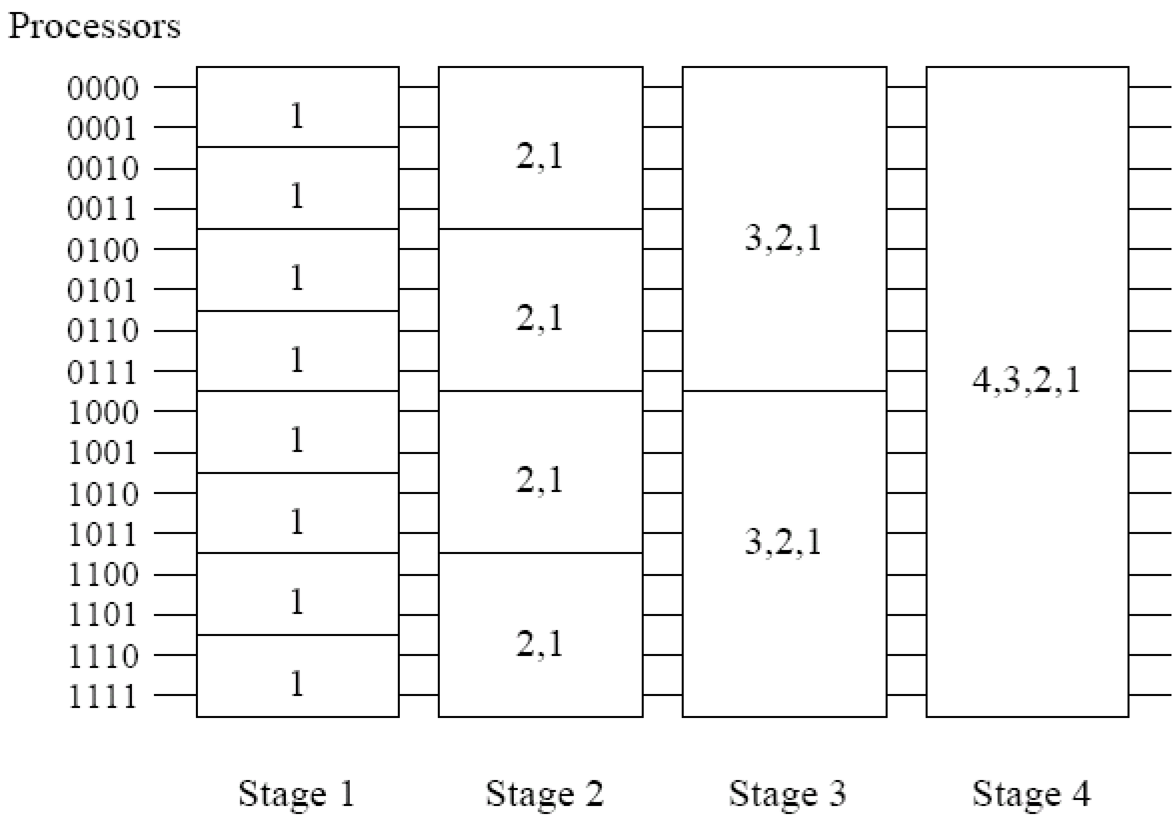
- Consider the case of one **item per processor**. The question becomes one of **how the wires in the bitonic network should be mapped to the hypercube interconnect**.
- Note from our earlier examples that the compare-exchange operation is performed between two wires only **if their labels differ in exactly one bit!**
- This implies a direct mapping of wires to processors. All communication is nearest neighbor!

Mapping Bitonic Sort to Hypercubes



Communication during the last stage of bitonic sort. Each wire is mapped to a hypercube process; each connection represents a compare-exchange between processes.

Mapping Bitonic Sort to Hypercubes



Communication characteristics of bitonic sort on a hypercube. During each stage of the algorithm, **processes communicate along the dimensions shown**.

Mapping Bitonic Sort to Hypercubes

```
procedure BITONIC_SORT(label, d)
begin
    for i := 0 to d − 1 do
        for j := i downto 0 do
            if (i + 1)^st bit of label ≠ j^th bit of label then
                comp_exchange_max(j);
            else
                comp_exchange_min(j);
end BITONIC_SORT
```

Parallel formulation of bitonic sort on a hypercube with $n = 2^d$ processes.

Mapping Bitonic Sort to Hypercubes

- During each step of the algorithm, every process performs a **compare-exchange operation** (single nearest neighbor communication of one word).
- Since each step takes $\Theta(1)$ time, the parallel time is

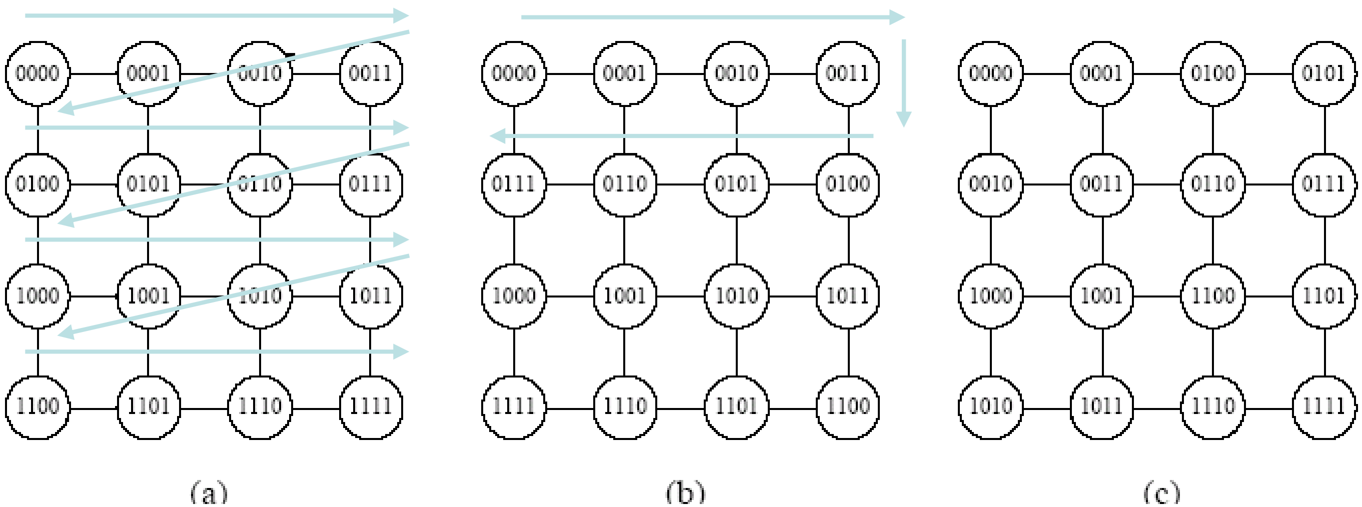
$$T_p = \Theta(\log^2 n) \qquad (2)$$

- This algorithm is **cost optimal w.r.t. its serial counterpart**, but **not w.r.t. the best sorting algorithm**.

Mapping Bitonic Sort to Meshes

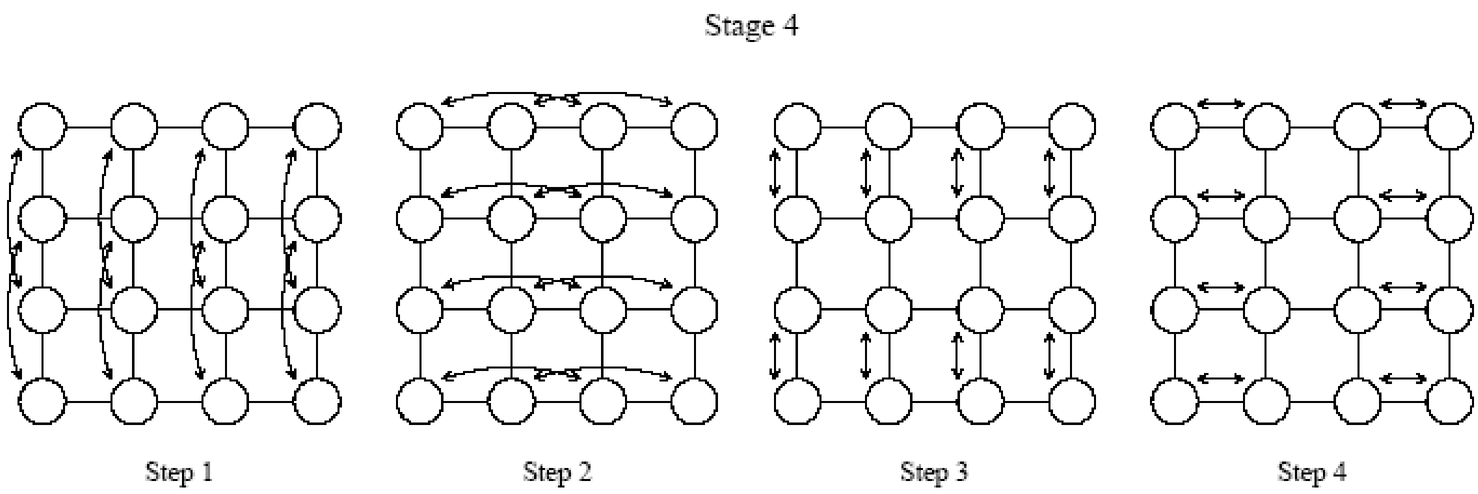
- The **connectivity of a mesh is lower** than that of a hypercube, so we must expect some **overhead** in this mapping.
- Consider the **row-major shuffled mapping** of wires to processors.

Mapping Bitonic Sort to Meshes



Different ways of mapping the input wires of the bitonic sorting network to a mesh of processes: (a) **row-major** mapping, (b) **row-major snakelike** mapping, and (c) **row-major shuffled** mapping.

Mapping Bitonic Sort to Meshes



The last stage of the bitonic sort algorithm for $n = 16$ on a mesh, using the **row-major shuffled mapping**. During each step, process pairs compare-exchange their elements. **Arrows indicate the pairs of processes that perform compare-exchange** operations.

Block of Elements Per Processor

- Each process is assigned a **block of *n/p* elements**.
- The first step is a **local sort** of the local block.
- Each subsequent compare-exchange operation is replaced by a **compare-split** operation.
- We can effectively view the **bitonic network as having $(1 + \log p)(\log p)/2$ steps**.

Block of Elements Per Processor: Hypercube

- Initially the processes **sort their *n/p* elements** (using merge sort) in time $\Theta((n/p)\log(n/p))$ and then perform $\Theta(\log^2 p)$ compare-split steps.
- The parallel run time of this formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{comparisons}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{communication}}.$$

- Comparing to an optimal sort, the algorithm **can efficiently use up to** $p = \Theta(2^{\sqrt{\log n}})$ **processes**.
- The **isoefficiency function** due to both communication and extra work is $\Theta(p^{\log p}\log^2 p)$.

Bubble Sort and its Variants

The sequential bubble sort algorithm **compares and exchanges adjacent elements** in the sequence to be sorted:

```
procedure BUBBLE_SORT(n)
begin
    for i := n - 1 downto 1 do
        for j := 1 to i do
            compare-exchange(a_j, a_{j+1});
end BUBBLE_SORT
```

Sequential bubble sort algorithm.

Bubble Sort and its Variants

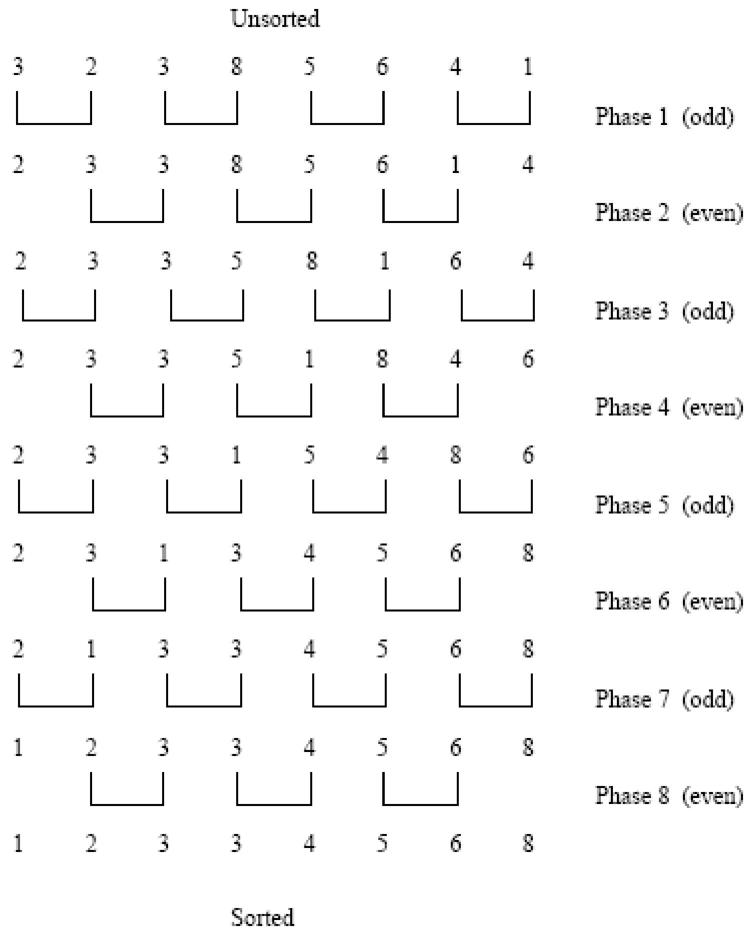
- The **complexity** of bubble sort is $\Theta(n^2)$.
- Bubble sort is **difficult to parallelize** since the algorithm has no concurrency.
- A simple variant, though, uncovers the concurrency.

Odd-Even Transposition

```
procedure ODD-EVEN(n)
begin
    for i := 1 to n do
    begin
        if i is odd then
            for j := 0 to n/2 - 1 do
                compare-exchange(a_{2j+1}, a_{2j+2});
        if i is even then
            for j := 1 to n/2 - 1 do
                compare-exchange(a_{2j}, a_{2j+1});
    end for
end ODD-EVEN
```

Sequential **odd-even transposition sort** algorithm.

Odd-Even Transposition



Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.

Odd-Even Transposition

- **After n phases** of odd-even exchanges, **the sequence is sorted**.
- Each phase of the algorithm (either odd or even) **requires $\Theta(n)$ comparisons**.
- **Serial complexity** is $\Theta(n^2)$.

Parallel Odd-Even Transposition

- Consider the one item per processor case.
- There are n iterations, in each iteration, each processor does one compare-exchange.
- The **parallel run time** of this formulation is $\Theta(n)$.
- This is **cost optimal with respect to the base serial** algorithm **but not the optimal one**.

Parallel Odd-Even Transposition

```
procedure ODD-EVEN_PAR(n)
begin
    id := process's label
    for i := 1 to n do
    begin
        if i is odd then
            if id is odd then
                compare-exchange_min(id + 1);
            else
                compare-exchange_max(id − 1);
        if i is even then
            if id is even then
                compare-exchange_min(id + 1);
            else
                compare-exchange_max(id − 1);
    end for
end ODD-EVEN_PAR
```

Parallel formulation of odd-even transposition.

Parallel Odd-Even Transposition

- Consider a **block of *n/p* elements** per processor.
- The first step is a **local sort**.
- In each subsequent step, the compare exchange operation is replaced by the **compare split** operation.
- The parallel run time of the formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta(n)}^{\text{comparisons}} + \overbrace{\Theta(n)}^{\text{communication}}.$$

Parallel Odd-Even Transposition

- The parallel formulation is **cost-optimal for** $p = O(\log n)$.
- The **isoefficiency function** of this parallel formulation is $\Theta(p2^p)$.

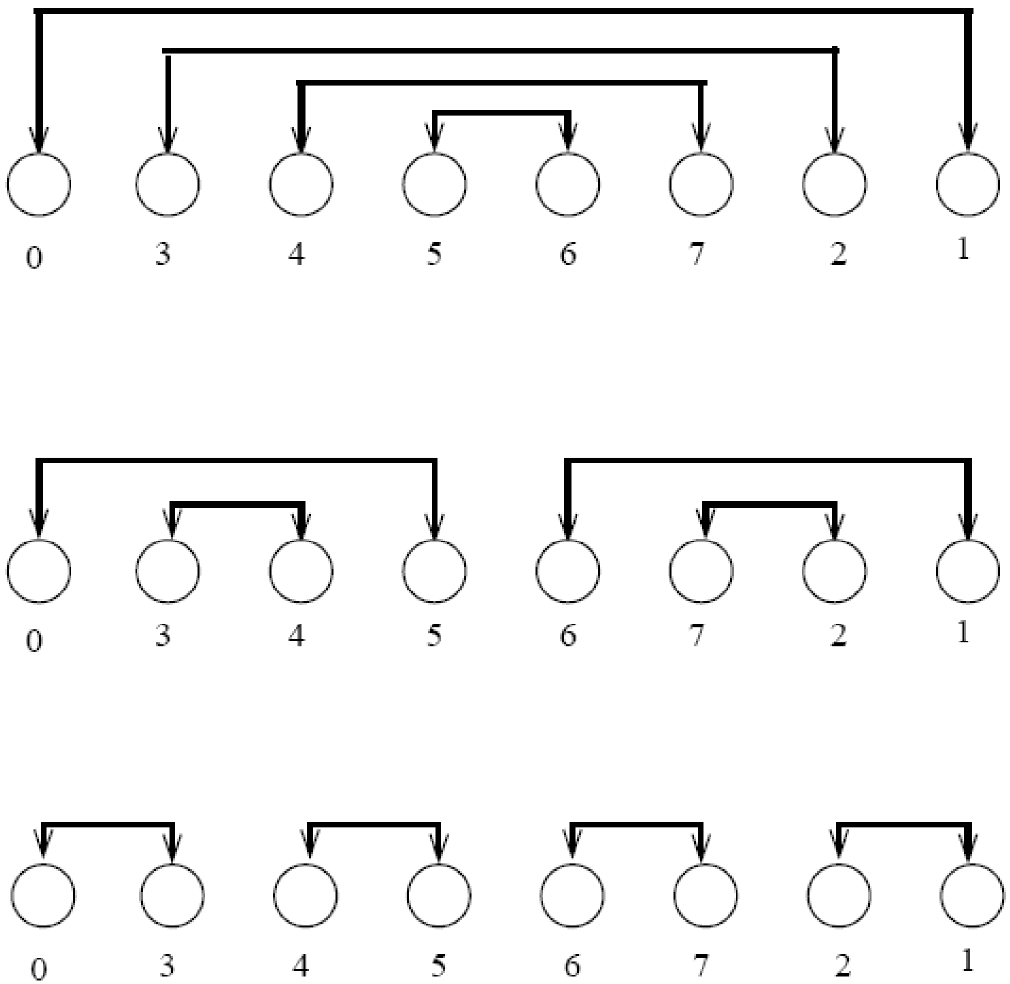
Shellsort

- Let n be the number of elements to be sorted and p be the number of processes.
- During the first phase, **processes that are far away from each other in the array compare-split their elements**.
- During the second phase, the algorithm switches to an **odd-even transposition sort**.
- Odd-even transposition is performed as long as the blocks of data are changing.

Parallel Shellsort

- Initially, each process **sorts its block** of n/p elements internally.
- Each **process is now paired with its corresponding process in the reverse order** of the array. That is, process P_i, where $i < p/2$, is paired with process P_{p-i-1}.
- A **compare-split** operation is performed.
- The **processes are split into two groups** of size $p/2$ each and the process repeated in each group.

Parallel Shellsort



An example of the first phase of parallel shellsort on an eight-process array.

Parallel Shellsort

- **Each process performs** $d = \log p$ compare-split operations.
- With $O(p)$ bisection width, each **communication can be performed in time** $\Theta(n/p)$ for a total time of $\Theta((n\log p)/p)$.
- In the **second phase**, l **odd and even phases** are performed, each requiring time $\Theta(n/p)$.
- The parallel run time of the algorithm is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right)}^{\text{first phase}} + \overbrace{\Theta\left(l\frac{n}{p}\right)}^{\text{second phase}}. \qquad (3)$$

Quicksort

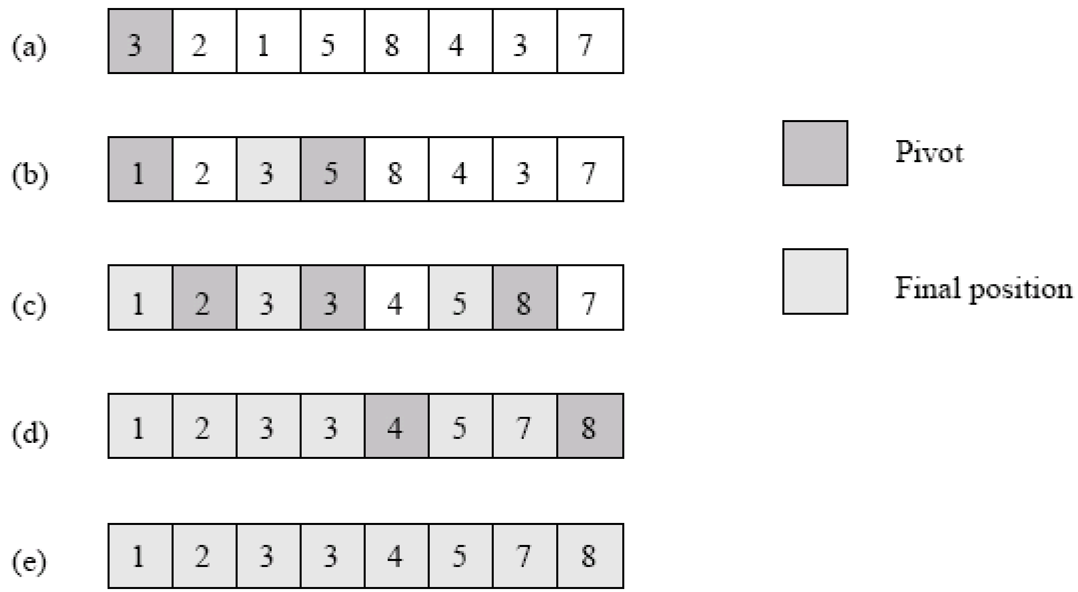
- Quicksort is one of **the most common sorting algorithms** for sequential computers because of its simplicity, low overhead, and optimal average complexity.
- Quicksort selects one of the entries in the sequence to be the **pivot** and **divides the sequence into two** - one with all elements less than the pivot and other greater.
- The **process is recursively applied** to each of the sublists.

Quicksort

```
procedure QUICKSORT (A, q, r)
begin
    if q < r then
    begin
        x := A[q];
        s := q;
        for i := q + 1 to r do
            if A[i] ≤ x then
            begin
                s := s + 1;
                swap(A[s], A[i]);
            end if
        swap(A[q], A[s]);
        QUICKSORT (A, q, s);
        QUICKSORT (A, s + 1, r);
    end if
end QUICKSORT
```

The sequential quicksort algorithm.

Quicksort



Example of the quicksort algorithm sorting a sequence of size $n = 8$.

Quicksort

- The performance of quicksort depends critically on the **quality of the pivot**.
- In the best case, the pivot divides the list in such a way that the larger of the **two lists does not have more than** $\boldsymbol{\alpha n}$ **elements** (for some constant α).
- In this case, the **complexity of quicksort** is $O(n \log n)$.

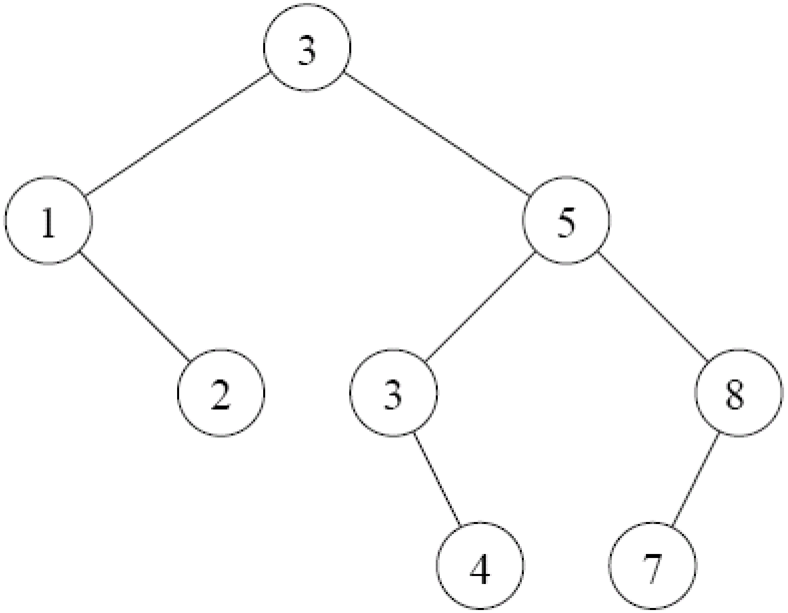
Parallelizing Quicksort

- Lets start with recursive decomposition - the **list is partitioned serially** and each of the **subproblems is handled by a different processor**.
- The time for this algorithm is **lower-bounded by $\Omega(n)$!**
- Can we **parallelize the partitioning step** - in particular, if we can use n processors to partition a list of length n around a pivot in $O(1)$ time, we have a winner.
- This is **difficult to do on real machines**, though.

Parallelizing Quicksort: PRAM Formulation

- We assume a **CRCW** (concurrent read, concurrent write) PRAM with concurrent writes resulting in an **arbitrary write succeeding**.
- The formulation works by creating **pools of processors**. Every processor is assigned to the same pool initially and has one element.
- Each processor **attempts to write its element to a common location** (for the pool).
- Each processor tries to **read back the location**. If the value read back is greater than the processor's value, it assigns itself to the **`left' pool**, else, it assigns itself to the **`right' pool**.
- Each pool performs this operation **recursively**.
- Note that the algorithm **generates a tree of pivots**. The depth of the tree is the expected parallel runtime. The **average value is** $O(\log n)$.

Parallelizing Quicksort: PRAM Formulation



A binary tree generated by the execution of the quicksort algorithm. Each level of the tree represents a different array-partitioning iteration. If **pivot selection is optimal, then the height of the tree** is $\Theta(\log n)$, which is also the **number of iterations**.

Parallelizing Quicksort: PRAM Formulation

(a)

1	2	3	4	5	6	7	8
33	21	13	54	82	33	40	72

(b) root = 4

(c)

	1	2	3	4	5	6	7	8
leftchild				1				
rightchild				5				

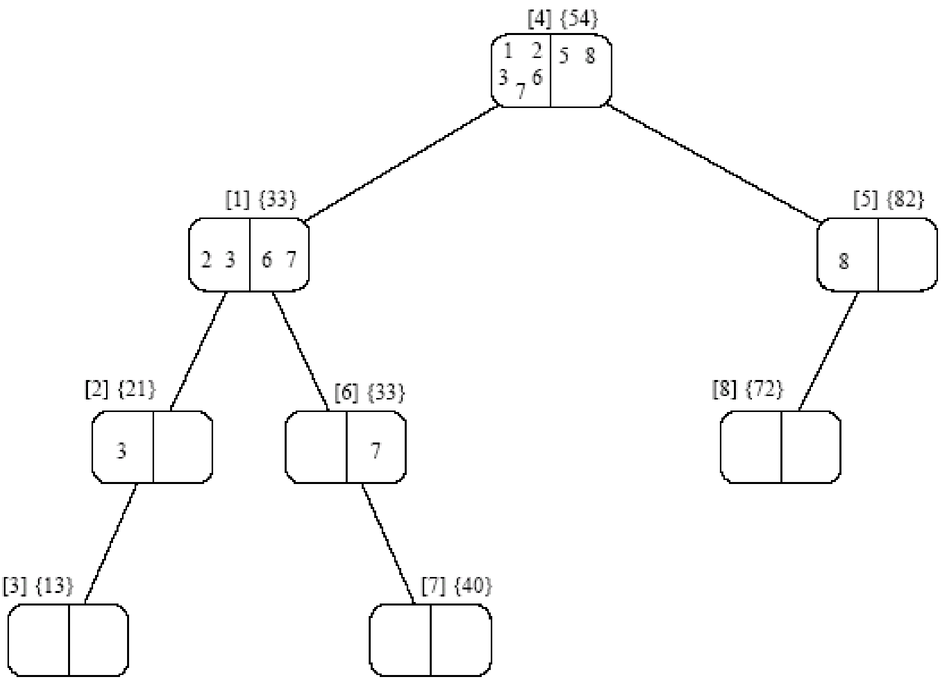
(d)

	1	2	3	4	5	6	7	8
leftchild	2			1	8			
rightchild	6			5				

(e)

	1	2	3	4	5	6	7	8
leftchild	2	3		1	8			
rightchild	6			5		7		

(f)

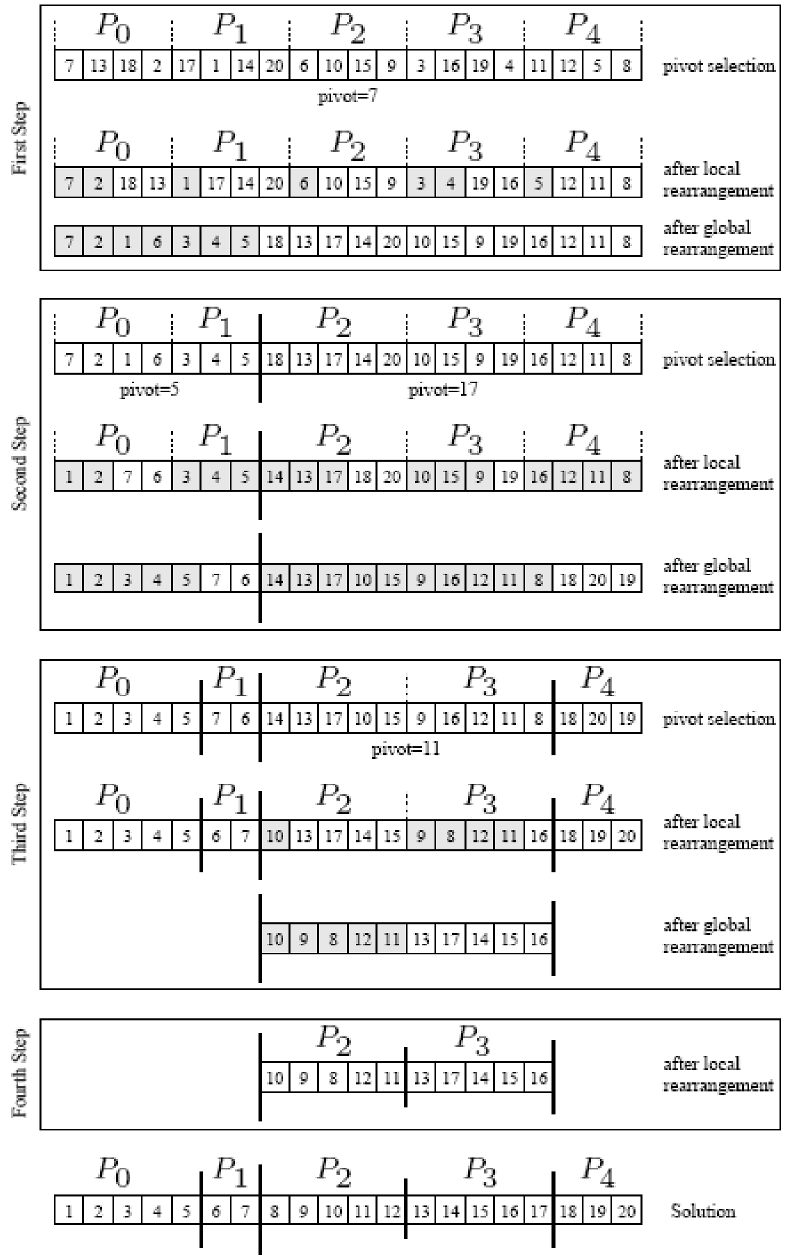


The execution of the PRAM algorithm on the array shown in (a).

Parallelizing Quicksort: Shared Address Space Formulation

- Consider a list of size n equally divided across p processors.
- A **pivot is selected by one of the processors** and made known to all processors.
- Each **processor partitions its list into two**, say L_i and U_i, based on the selected pivot.
- All of the L_i **lists are merged** and all of the U_i lists are merged **separately**.
- The set of processors is partitioned into two (in proportion of the size of lists L and U). The process is recursively applied to each of the lists.

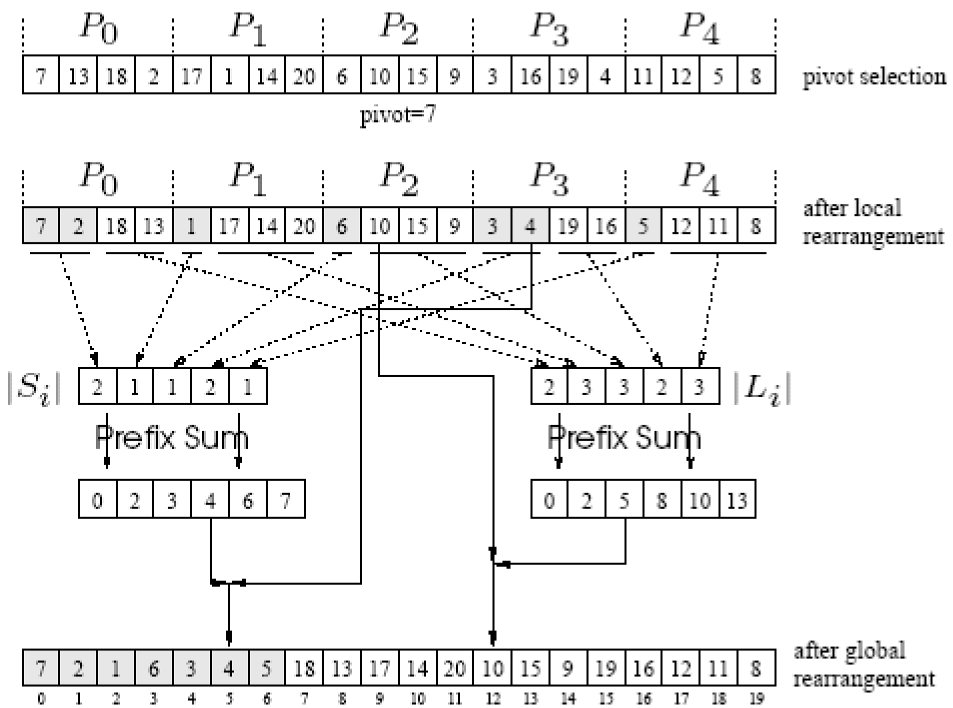
Shared Address Space Formulation



Parallelizing Quicksort: Shared Address Space Formulation

- The only thing we have not described is **the global reorganization** (merging) of local lists to form L and U.
- The problem is one of **determining the right location** for each element in the merged list.
- Each processor computes the number of elements locally less than and greater than pivot.
- It computes two **sum-scans to determine the starting location** for its elements in the merged L and U lists.
- Once it knows the starting locations, it can **write its elements safely**.

Parallelizing Quicksort: Shared Address Space Formulation



Efficient global rearrangement of the array.

Parallelizing Quicksort: Shared Address Space Formulation

- The parallel time depends on the **split and merge time**, and the **quality of the pivot**.
- The latter is an issue independent of parallelism, so we focus on the first aspect, assuming ideal pivot selection.
- The algorithm executes in four steps: (i) **determine and broadcast the pivot**; (ii) locally **rearrange the array** assigned to each process; (iii) **determine the locations in the globally rearranged array** that the local elements will go to; and (iv) **perform the global rearrangement**.
- The first step takes time $\Theta(\log p)$, the second, $\Theta(n/p)$, the third, $\Theta(\log p)$, and the fourth, $\Theta(n/p)$.
- The overall complexity of splitting an n-element array is $\Theta(n/p) + \Theta(\log p)$.

Parallelizing Quicksort: Shared Address Space Formulation

- The process **recurses until there are *p* lists**, at which point, the lists are **sorted locally**.
- Therefore, the total parallel time is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right) + \Theta(\log^2 p)}^{\text{array splits}}. \quad (4)$$

- The corresponding **isoefficiency** is $\Theta(p\log^2 p)$ due to **broadcast** and **scan** operations.

Parallelizing Quicksort: Message Passing Formulation

- A simple message passing formulation is based on the **recursive halving** of the machine.
- Assume that each processor in the lower half of a p processor ensemble is paired with a corresponding processor in the upper half.
- A designated processor selects and **broadcasts the pivot**.
- Each processor **splits its local list into two lists**, one less (L_i), and other greater (U_i) than the pivot.
- A processor in the low half of the machine **sends its list U_i to the paired processor** in the other half. The paired processor sends its list L_i.
- It is easy to see that after this step, **all elements less than the pivot are in the low half** of the machine and **all elements greater than the pivot are in the high half**.

Parallelizing Quicksort: Message Passing Formulation

- The above **process is recursed** until each processor has its own local list, which is **sorted locally**.
- The time for a single reorganization is $\Theta(\log p)$ for broadcasting the pivot element, $\Theta(n/p)$ for splitting the locally assigned portion of the array, $\Theta(n/p)$ for exchange and local reorganization.
- We note that this **time is identical to that of the corresponding shared address** space formulation.
- It is important to remember that the **reorganization of elements is a bandwidth sensitive** operation.

Bucket and Sample Sort

- In Bucket sort, the range $[a,b]$ of **input numbers is divided** into m equal sized intervals, called **buckets**.
- Each element is placed in its appropriate bucket.
- If the numbers are uniformly divided in the range, the buckets can be expected to **have roughly identical number of elements**.
- Elements in the buckets are **locally sorted**.
- The run time of this algorithm is $\Theta(n\log(n/m))$.

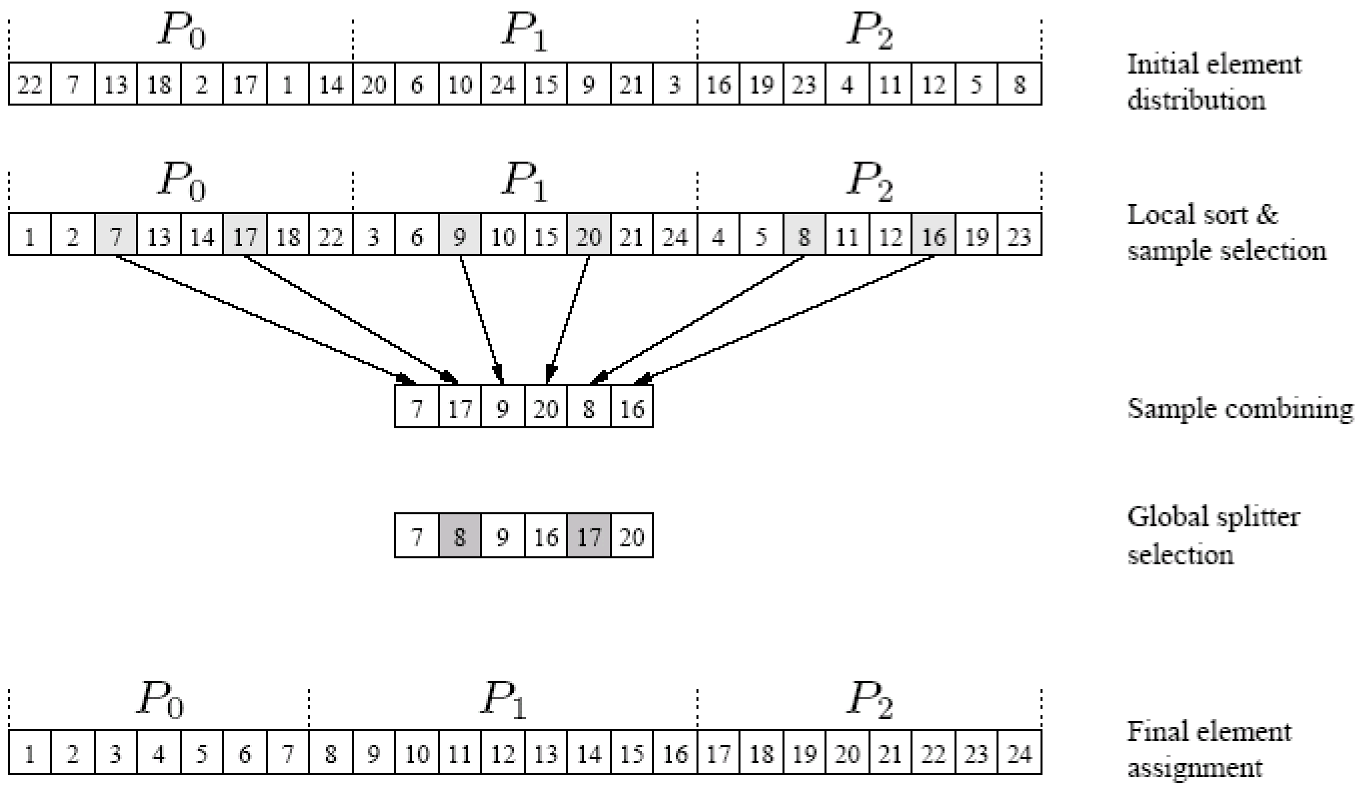
Parallel Bucket Sort

- Parallelizing bucket sort is relatively simple. We can select $m = p$.
- In this case, **each processor has a range of values** it is responsible for.
- Each processor **runs through its local list** and assigns each of its elements to the appropriate processor.
- The elements are **sent to the destination processors** using a single **all-to-all personalized communication**.
- Each processor **sorts all the elements** it receives.

Parallel Bucket and Sample Sort

- The critical aspect of the above algorithm is one of **assigning ranges to processors**. This is done by suitable splitter selection.
- The splitter selection method divides the n elements into m blocks of size n/m each, and sorts each block by using quicksort.
- From each sorted block it **chooses $m - 1$ evenly spaced elements**.
- The $m(m - 1)$ **elements selected** from all the blocks represent the sample used to **determine the buckets**.
- This scheme guarantees that the number of elements ending up in each bucket is less than $2n/m$.

Parallel Bucket and Sample Sort



An example of the execution of sample sort on an array with 24 elements on three processes.

Parallel Bucket and Sample Sort

- The **splitter selection** scheme can itself be **parallelized**.
- Each processor **generates the** $p - 1$ **local splitters** in parallel.
- All processors share their splitters using a single **all-to-all broadcast** operation.
- Each processor **sorts** the $p(p - 1)$ elements it receives and **selects** $p - 1$ **uniformly spaces splitters** from them.

Parallel Bucket and Sample Sort: Analysis

- The internal sort of n/p elements requires time $\Theta((n/p)\log(n/p))$, and the selection of $p - 1$ sample elements requires time $\Theta(p)$.
- The time for an all-to-all broadcast is $\Theta(p^2)$, the time to internally sort the $p(p - 1)$ sample elements is $\Theta(p^2 \log p)$, and selecting $p - 1$ evenly spaced splitters takes time $\Theta(p)$.
- Each process can *insert* these $p - 1$splitters in its local sorted block of size n/p by performing $p - 1$ binary searches in time $\Theta(p\log(n/p))$.
- The time for reorganization of the elements is $O(n/p)$.

Parallel Bucket and Sample Sort: Analysis

- The total time is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(p^2\log p\right)}^{\text{sort sample}} + \overbrace{\Theta\left(p\log\frac{n}{p}\right)}^{\text{block partition}} + \overbrace{\Theta(n/p).}^{\text{communication}} \qquad (5)$$

- The isoefficiency of the formulation is $\Theta(p^3 \log p)$.