Dynamic Programming

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Topic Overview

- Overview of Serial Dynamic Programming
- Serial Monadic DP Formulations
- Nonserial Monadic DP Formulations
- Serial Polyadic DP Formulations
- Nonserial Polyadic DP Formulations

Overview of Serial Dynamic Programming

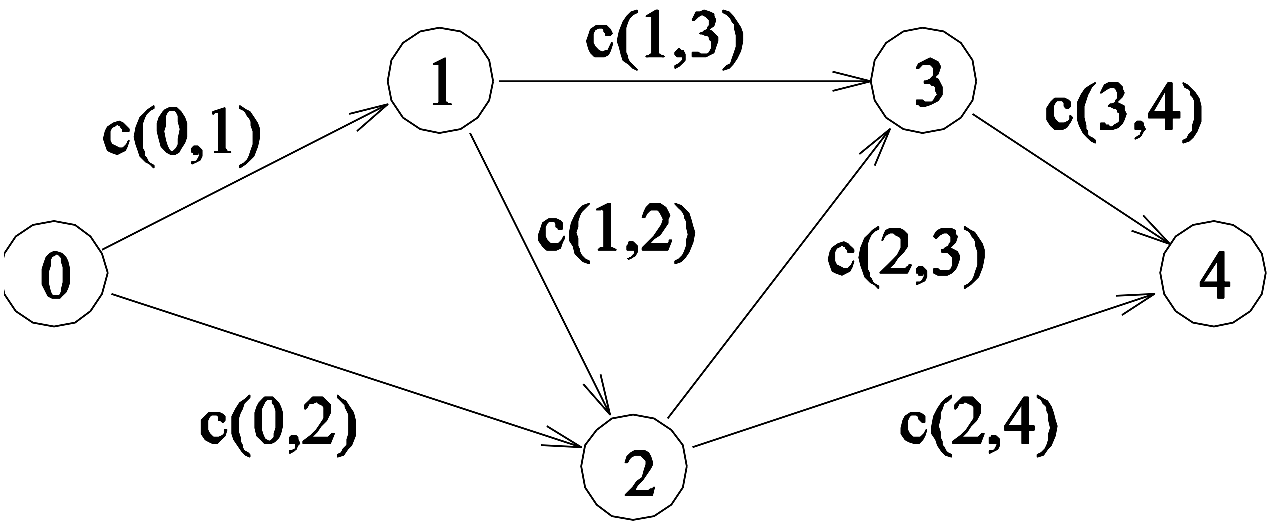
- *Dynamic programming* (DP) is used to solve a wide variety of **discrete optimization problems** such as scheduling, string-editing, packaging, and inventory management.
- **Break problems into subproblems** and **combine their solutions into solutions** to larger problems.
- In contrast to divide-and-conquer, there may be relationships across subproblems.

Dynamic Programming: Example

- Consider the problem of finding a **shortest path between a pair of vertices** in an **acyclic graph**.
- An edge connecting node *i* to node *j* has cost *c(i,j)*.
- The graph contains *n* nodes numbered *0,1,…, n-1*, and has an **edge from node *i* to node *j* only if *i* < *j***. Node 0 is **source** and node *n-1* is the **destination**.
- Let *f(x)* be the **cost of the shortest path** from node 0 to node *x*.

$$f(x) = \begin{cases} 0 & x = 0 \\ \min\limits_{0 \le j < x} \{f(j) + c(j,x)\} & 1 \le x \le n-1 \end{cases}$$

Dynamic Programming: Example



- A graph for which the shortest path between nodes 0 and 4 is to be computed.

$$f(4) = \min\{f(3) + c(3,4), f(2) + c(2,4)\}.$$

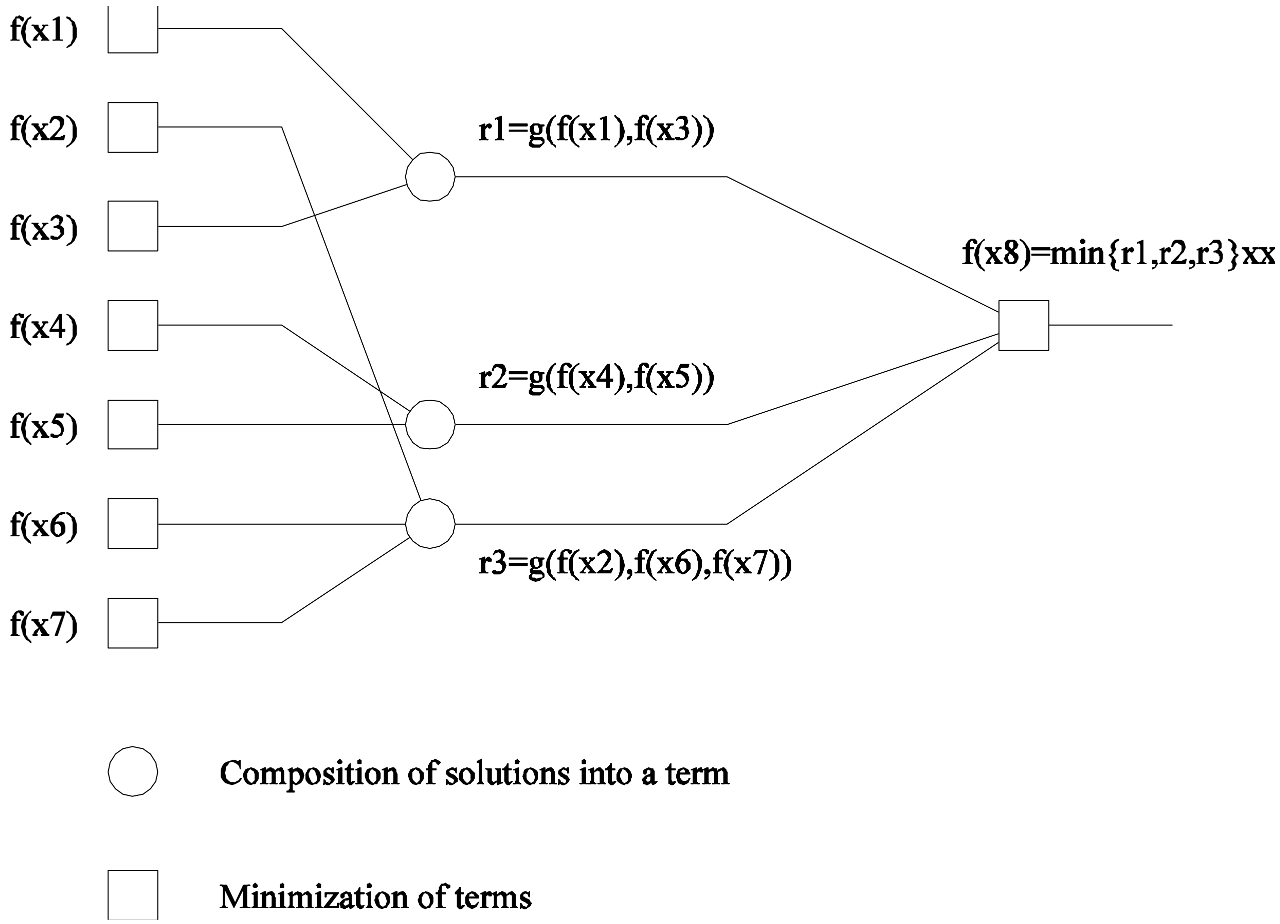
Dynamic Programming

- The solution to a DP problem is typically expressed as a **minimum (or maximum) of possible alternate solutions**.
- If r represents the **cost of a solution composed of subproblems** $x_1, x_2, \ldots, x_l$, then r can be written as

$$r = g(f(x_1), f(x_2), \ldots, f(x_l)).$$

  Here, g is the ***composition function***.
- If the **optimal solution** to each problem is determined by **composing optimal solutions to the subproblems** and **selecting the minimum** (or maximum), the formulation is said to be a **DP formulation**.

Dynamic Programming: Example



The computation and composition of subproblem solutions to solve problem $f(x_8)$.

Dynamic Programming

- The recursive DP equation is also called the *functional equation* or *optimization equation*.
- In the equation for the **shortest path problem** the composition function is $f(j) + c(j,x)$. This contains a **single recursive term** ($f(j)$). Such a formulation is called **monadic**.
- If the RHS has **multiple recursive terms**, the DP formulation is called **polyadic**.

Dynamic Programming

- The **dependencies between subproblems** can be expressed as a graph.
- If the **graph can be levelized** (i.e., solutions to problems at a level depend only on solutions to problems at the previous level), the formulation is called **serial**, else it is called **non-serial**.
- Based on these two criteria, we can classify DP formulations into **four categories** - **serial-monadic**, **serial-polyadic**, **non-serial-monadic**, **non-serial-polyadic**.
- This classification is useful since it **identifies concurrency** and dependencies that guide parallel formulations.

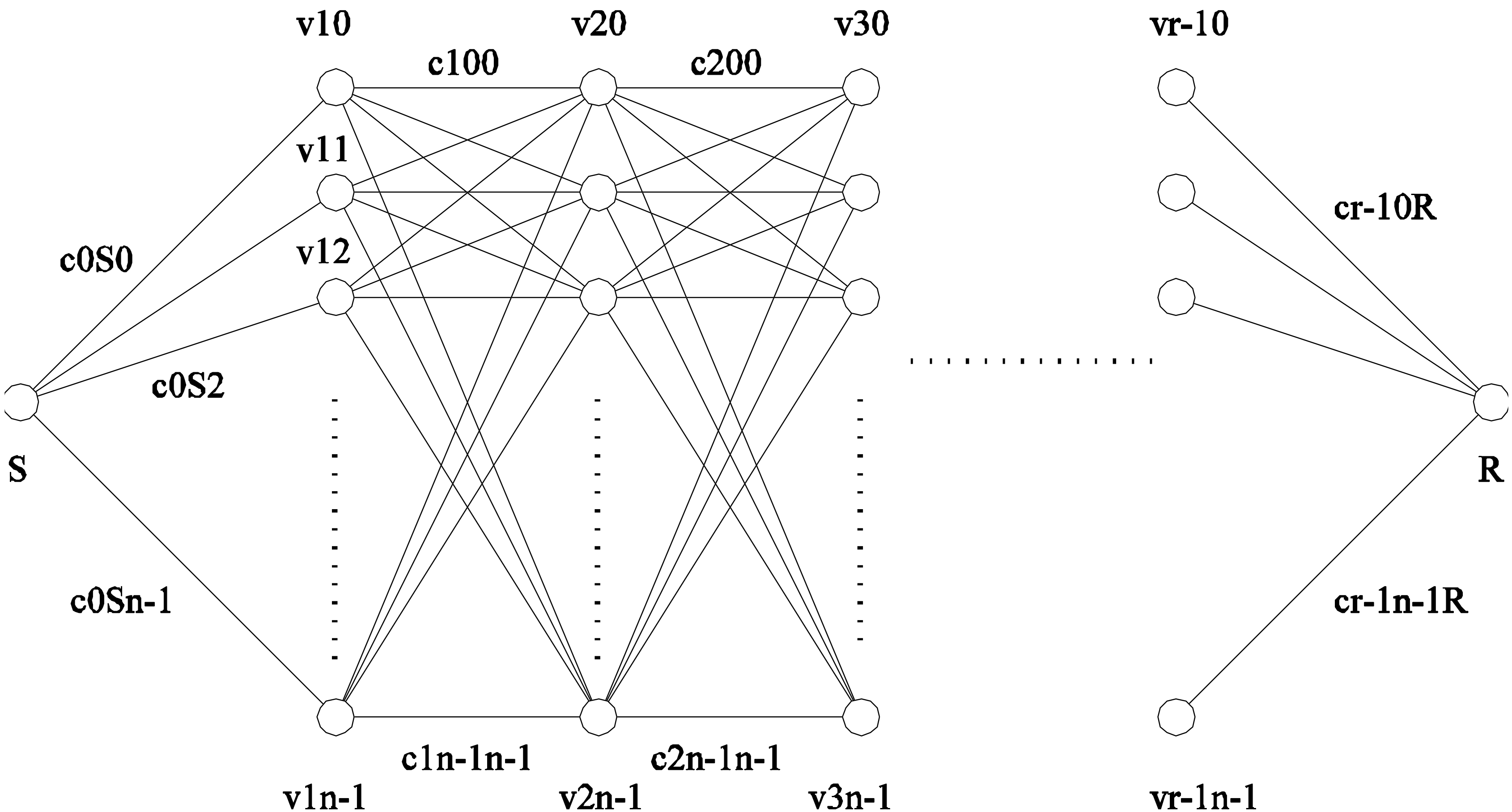
Serial Monadic DP Formulations

- It is **difficult to derive** canonical parallel formulations for the **entire class of formulations**.
- For this reason, we select two representative examples, the **shortest-path problem for a multistage graph** and the **0/1 knapsack problem**.
- We derive parallel formulations for these problems and **identify common principles** guiding design within the class.

Shortest-Path Problem

- **Special class of shortest path problem** where the graph is a weighted multistage graph of ***r* + *1* levels**.
- **Each level is assumed to have *n* nodes** and every node at level *i* is connected to every node at level *i* + *1*.
- Levels zero and *r* contain only one node, the **source and destination nodes**, respectively.
- The objective of this problem is to **find the shortest path from *S* to *R***.

Shortest-Path Problem



An example of a serial monadic DP formulation for finding the shortest path in a graph whose **nodes can be organized into levels**.

Shortest-Path Problem

- The i^{th} node at level l in the graph is labeled $\boldsymbol{v_i^l}$ and the cost of an edge connecting v_i^l to node v_j^{l+1} is labeled $\boldsymbol{c_{i,j}^l}$.
- The **cost of reaching the goal node *R* from any node** $\boldsymbol{v_i^l}$ is represented by $\boldsymbol{C_i^l}$.
- If there are n nodes at **level *l***, the vector $[C_0^l, C_1^l, \ldots, C_{n-1}^l]^T$ is referred to as C^l. Note that $C_0 = [C_0^0]$.
- We have $\boldsymbol{C_i^l = \min \{(c_{i,j}^l + C_j^{l+1}) \mid j}$ is a node at level $l + 1\}$

Shortest-Path Problem

- Since all nodes v_j^{r-1} have **only one edge** connecting them to the goal node R at level r, the cost C_j^{r-1} is equal to $c_{j,R}^{r-1}$.
- We have:

$$C^{r-1} = [c_{0,R}^{r-1}, c_{1,R}^{r-1}, \ldots, c_{n-1,R}^{r-1}].$$

Notice that this **problem is <u>serial</u> and <u>monadic</u>**.

Shortest-Path Problem

- The cost of reaching the goal node *R* from any node at level *l* is *(0 < l < r – 1)* is

$$C_0^l = \min\{(c_{0,0}^l + C_0^{l+1}), (c_{0,1}^l + C_1^{l+1}), \ldots, (c_{0,n-1}^l + C_{n-1}^{l+1})\},$$

$$C_1^l = \min\{(c_{1,0}^l + C_0^{l+1}), (c_{1,1}^l + C_1^{l+1}), \ldots, (c_{1,n-1}^l + C_{n-1}^{l+1})\},$$

$$\vdots$$

$$C_{n-1}^l = \min\{(c_{n-1,0}^l + C_0^{l+1}), (c_{n-1,1}^l + C_1^{l+1}), \ldots, (c_{n-1,n-1}^l + C_{n-1}^{l+1})\}.$$

Shortest-Path Problem

- We can express the solution to the problem as a **modified sequence of matrix-vector products**.
- **Replacing the addition operation by minimization** and the **multiplication operation by addition**, the preceding set of equations becomes:

$$C^l = M_{l,l+1} \times C^{l+1},$$

where C^l and C^{l+1} are $n \times 1$ vectors representing the cost of reaching the goal node from each node at levels l and $l + 1$.

Shortest-Path Problem

- Matrix $M_{l,l+1}$ is an n x n matrix in which entry (i, j) stores the **cost of the edge connecting node *i* at level *l* to node *j* at level *l* + *1***.

$$M_{l,l+1} = \begin{bmatrix} c^l_{0,0} & c^l_{0,1} & \cdots & c^l_{0,n-1} \\ c^l_{1,0} & c^l_{1,1} & \cdots & c^l_{1,n-1} \\ \vdots & \vdots & & \vdots \\ c^l_{n-1,0} & c^l_{n-1,1} & \cdots & c^l_{n-1,n-1} \end{bmatrix}.$$

- The shortest path problem has been formulated as a **sequence of *r* matrix-vector products**.

Parallel Shortest-Path

- We can parallelize this algorithm using the parallel algorithms for the matrix-vector product.
- ***Θ(n)*** **processing elements** can compute each vector C^l in time *Θ(n)* and **solve the entire problem in time *Θ(rn)***.
- In many instances of this problem, the **matrix *M* may be sparse**. For such problems, it is highly desirable to use sparse matrix techniques.

0/1 Knapsack Problem

- We are given a **knapsack of capacity *c*** and a set of ***n* objects** numbered *1,2,…,n*. Each object *i* has **weight w_i** and **profit p_i**.
- Let $v = [v_1, v_2, \ldots, v_n]$ be a **solution vector** in which $v_i = 0$ if object *i* is not in the knapsack, and $v_i = 1$ if it is in the knapsack.
- The goal is to **find a subset of objects** to put into the knapsack so that

$$\sum_{i=1}^{n} w_i v_i \le c$$

(that is, the **objects fit into the knapsack**) and

$$\sum_{i=1}^{n} p_i v_i$$

is maximized (that is, the **profit is maximized**).

0/1 Knapsack Problem

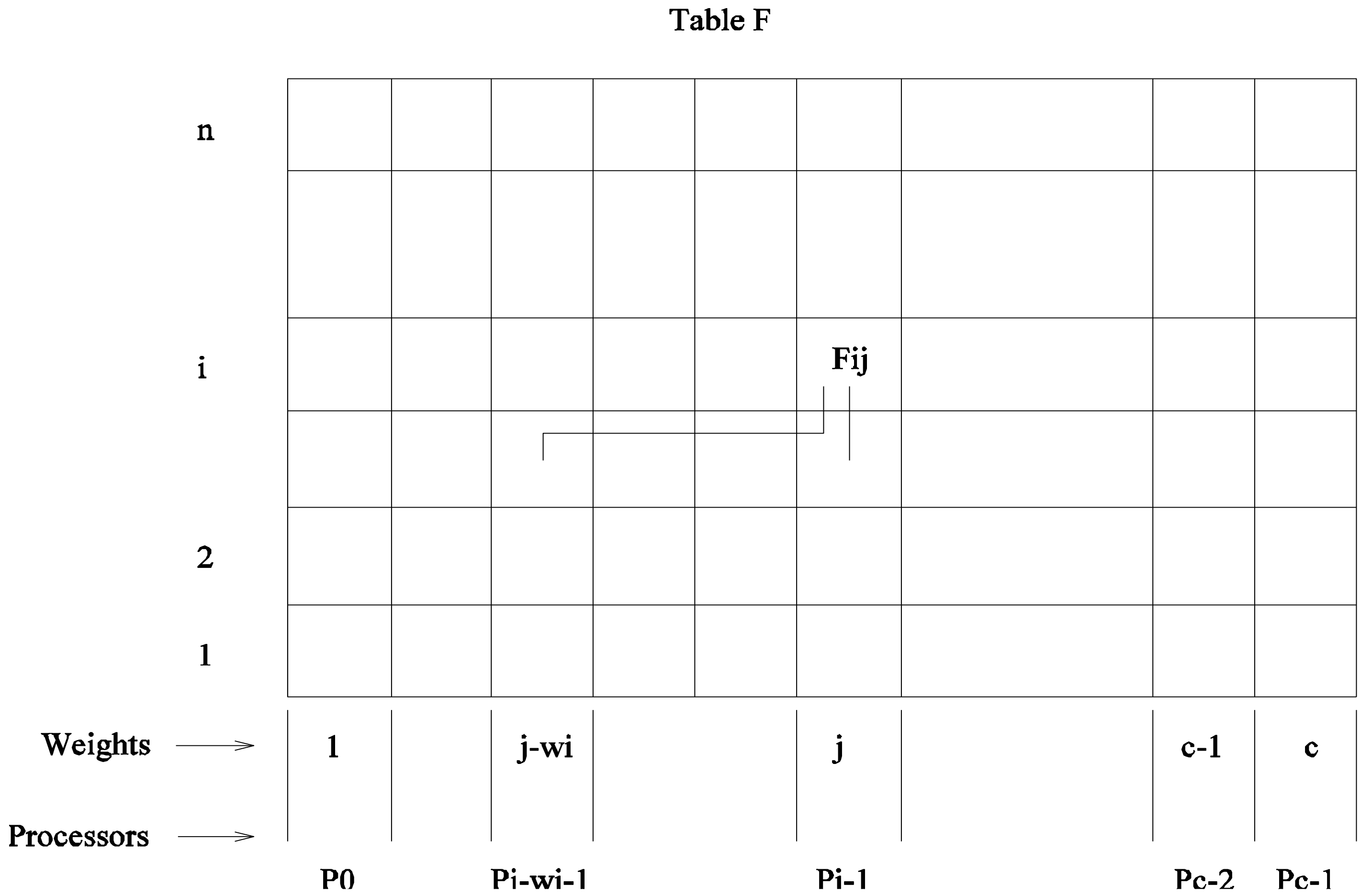
- The **naive method is to consider all *2^n* possible subsets** of the *n* objects and choose the one that fits into the knapsack and maximizes the profit.
- Let *F[i,x]* be the **maximum profit for a knapsack of capacity *x*** using only objects *{1,2,…,i}*. The DP formulation is:

$$F[i,x] = \begin{cases} 0 & x \geq 0, i = 0 \\ -\infty & x < 0, i = 0 \\ \max\{F[i-1,x], (F[i-1, x-w_i] + p_i)\} & 1 \leq i \leq n \end{cases}$$

0/1 Knapsack Problem

- Construct a **table *F* of size *n x c*** in row-major order.
- Filling **an entry in a row requires two entries from the previous row**: **one from the same column** and **one from the column offset by the weight** of the object corresponding to the row.
- **Computing each entry takes constant time**; the sequential run time of this algorithm is $\Theta(nc)$.
- The formulation is **serial-monadic**.

0/1 Knapsack Problem



Computing **entries of table *F*** for the 0/1 knapsack problem. The computation of entry *F[i,j]* requires communication with processing elements containing entries *F[i-1,j]* and *F[i-1,j-w$_i$]*.

0/1 Knapsack Problem

- Using ***c* processors** in a **PRAM**, we can derive a simple **parallel algorithm that runs in *O(n)*** time by partitioning the columns across processors.

- In a **distributed memory machine**, in the j^{th} iteration, for computing $F[j,r]$ at processing element P_{r-1}, $F[j-1,r]$ is available locally but $F[j-1,r-w_j]$ must be fetched.
- The communication operation is a **circular shift** and the time is given by $(t_s + t_w) \log c$. The **total time is therefore** $\mathbf{t_c + (t_s + t_w) \log c}$.
- Across all n iterations (rows), the parallel time is $O(n \log c)$. Note that this is **not cost optimal**.

Nonserial Monadic DP Formulations: Longest-Common-Subsequence

- Given a sequence $A = <a_1, a_2, \ldots, a_n>$, a **subsequence of *A* can be formed by deleting some entries** from *A*.
- Given two sequences $A = <a_1, a_2, \ldots, a_n>$ and $B = <b_1, b_2, \ldots, b_m>$, find **the longest sequence that is a subsequence of both *A* and *B***.
- If $A = <c,\mathbf{a},d,\mathbf{b},r,\mathbf{z}>$ and $B = <\mathbf{a},s,\mathbf{b},\mathbf{z}>$, the longest common subsequence of *A* and *B* is $<\mathbf{a},\mathbf{b},\mathbf{z}>$.

Longest-Common-Subsequence Problem

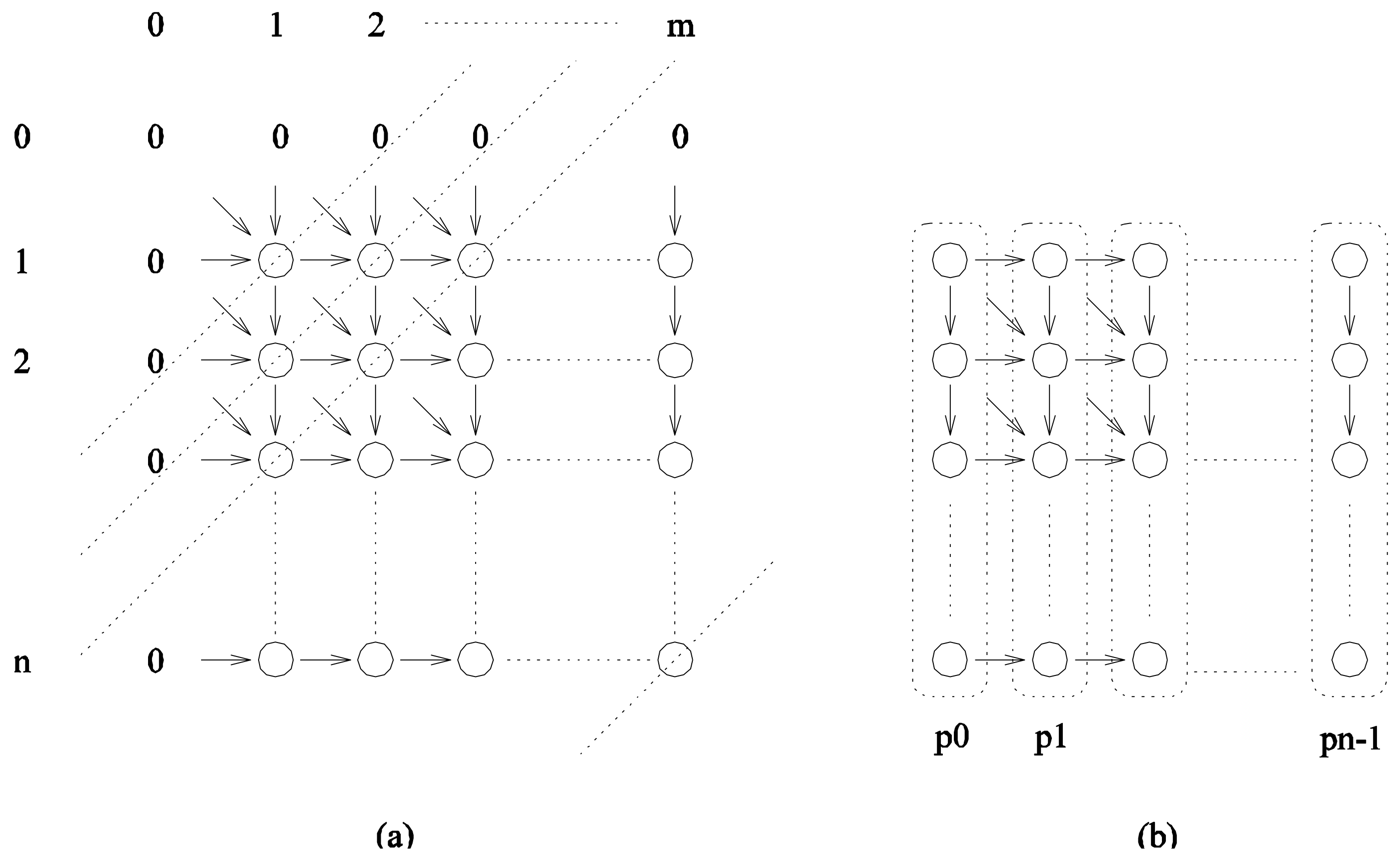
- Let ***F[i,j]*** denote the **length of the longest common subsequence** of the first *i* elements of *A* and the first *j* elements of *B*. The objective of the LCS problem is to **find *F[n,m]***.
- We can write:

$$F[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ F[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{F[i, j-1], F[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

Longest-Common-Subsequence Problem

- The algorithm computes the **two-dimensional *F* table** in a row- or column-major fashion. **The complexity is *$\Theta(nm)$***.
- Treating nodes along a diagonal as belonging to one level, **each node depends on two subproblems at the preceding level and one subproblem two levels prior**.
- This DP formulation is **<u>nonserial</u> <u>monadic</u>**.

Longest-Common-Subsequence Problem



(a) Computing entries of table   for the longest-common-subsequence problem. **Computation proceeds along the dotted diagonal lines**. (b) **Mapping elements** of the table to processing elements.

Longest-Common-Subsequence: Example

- Consider the LCS of two amino-acid sequences **H E A G A W G H E E** and **P A W H E A E**. For the interested reader, the names of the corresponding amino-acids are A: Alanine, E: Glutamic acid, G: Glycine, H: Histidine, P: Proline, and W: Tryptophan.

		H	E	A	G	A	W	G	H	E	E
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	1	1	1	1	1	1	1	1
W	0	0	0	1	1	1	2	2	2	2	2
H	0	1	1	1	1	1	2	2	3	3	3
E	0	1	2	2	2	2	2	2	3	4	4
A	0	1	2	3	3	3	3	3	3	4	4
E	0	1	2	3	3	3	3	3	3	4	5

- The *F* table for computing the LCS of the sequences. The LCS is A W H E E.

Parallel Longest-Common-Subsequence

- Table entries are **computed in a diagonal sweep** from the top-left to the bottom-right corner.
- Using ***n*** **processors** in a **PRAM**, each entry in a **diagonal can be computed in constant time**.
- For two sequences of length *n*, there are ***2n-1*** **diagonals**.

- The parallel **run time is *Θ(n)*** and the **algorithm is cost-optimal**.

Parallel Longest-Common-Subsequence

- Consider a (logical) **linear array of processors**. Processing element P_i is responsible for the $(i+1)^{th}$ column of the table.
- To compute $F[i,j]$, processing element P_{j-1} may need either $F[i-1,j-1]$ or $F[i,j-1]$ from the processing element to its left. This **communication takes time** $t_s + t_w$.
- The computation takes constant time (t_c).
- We have:

$$T_P = (2n-1)(t_s + t_w + t_c).$$

- Note that this **formulation is cost-optimal**, however, its efficiency is upper-bounded by 0.5!

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- Given **weighted graph** *G(V,E)*, Floyd's algorithm determines the cost $d_{i,j}$ of the **shortest path between each pair of nodes in *V***.
- Let $d_{i,j}^{k}$ be the minimum cost of a path from node *i* to node *j*, using only nodes $v_0, v_1, \ldots, v_{k-1}$.
- We have:

$$d_{i,j}^{k} = \begin{cases} c_{i,j} & k = 0 \\ \min\{d_{i,j}^{k-1}, (d_{i,k}^{k-1} + d_{k,j}^{k-1})\} & 0 \leq k \leq n-1 \end{cases} .$$

- **Each iteration requires time *Θ(n²)*** and the overall run time of the **sequential algorithm is *Θ(n³)***.

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- A **PRAM** formulation of this algorithm uses $\boldsymbol{n^2}$ **processors** in a logical **2D mesh**. Processor $P_{i,j}$ computes the value of $d_{i,j}^{k}$ for $k=1,2,\ldots,n$ in **constant time**.

- The parallel runtime is $\Theta(n)$ and it is **cost-optimal**.

Nonserial Polyadic DP Formulation: Optimal Matrix-Parenthesization Problem

- When **multiplying a sequence of matrices**, the order of multiplication significantly impacts **operation count**.
- Let ***C[i,j]*** **be the optimal cost of multiplying the matrices** $A_i, \ldots A_j$.
- The chain of matrices can be expressed as a **product of two smaller chains**:

$$A_i, A_{i+1}, \ldots, A_k \quad \text{and} \quad A_{k+1}, \ldots, A_j.$$

- The chain $A_i, A_{i+1}, \ldots, A_k$ results in a matrix of dimensions $r_{i-1} \times r_k$, and the chain $A_{k+1}, \ldots, A_j$ results in a matrix of dimensions $r_k \times r_j$.
- The **cost of multiplying these two matrices is** $\boldsymbol{r_{i-1} r_k r_j}$.

Optimal Matrix-Parenthesization Problem - Example

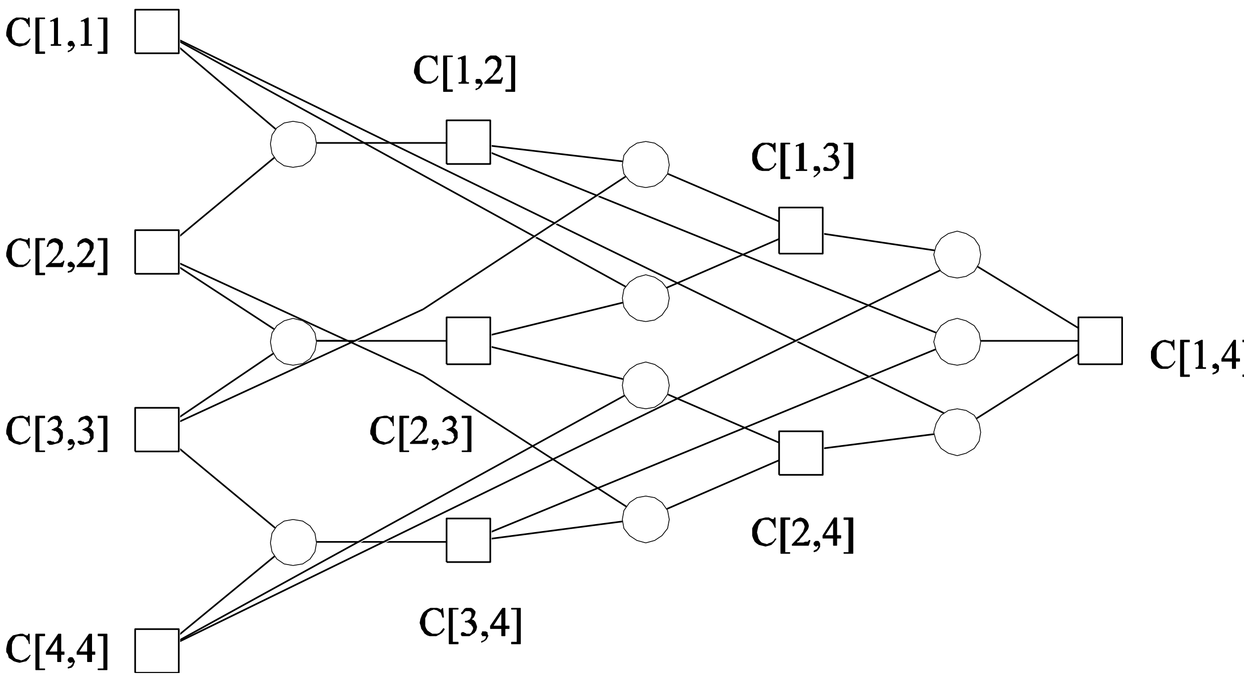
- Consider **three matrices A_1, A_2, and A_3** of dimensions 10x20, 20x30, and 30x40, respectively.
- The product of these matrices can be computed as **(A_1xA_2) x A_3** or as **A_1 x $(A_2$ x $A_3)$**.
- In $(A_1$ x A2) x A3, computing $(\mathbf{A_1xA_2})$ **requires 10·20·30** operations and yields a matrix of dimensions 10·30. Multiplying this by A_3 requires 10·30·40 additional operations. Therefore the total number of operations is 10·20·30 + 10·30·40 = **18000**.
- Similarly, computing **A_1 x (A_2xA_3)** requires 20·30·40 + 10·20·40 = **32000 operations**.
- **The first parenthesization is desirable.**

Optimal Matrix-Parenthesization Problem

- We have:

$$C[i,j] = \begin{cases} \min\limits_{i \le k < j} \{C[i,k] + C[k+1,j] + r_{i-1} r_k r_j\} & 1 \le i < j \le n \\ 0 & j = i, 0 < i \le n \end{cases}$$

Optimal Matrix-Parenthesization Problem



A nonserial polyadic DP formulation for finding an optimal matrix parenthesization for a **chain of four matrices**. A **square node represents the optimal cost** of multiplying a matrix chain. A **circle node represents a possible parenthesization**.

Optimal Matrix-Parenthesization Problem

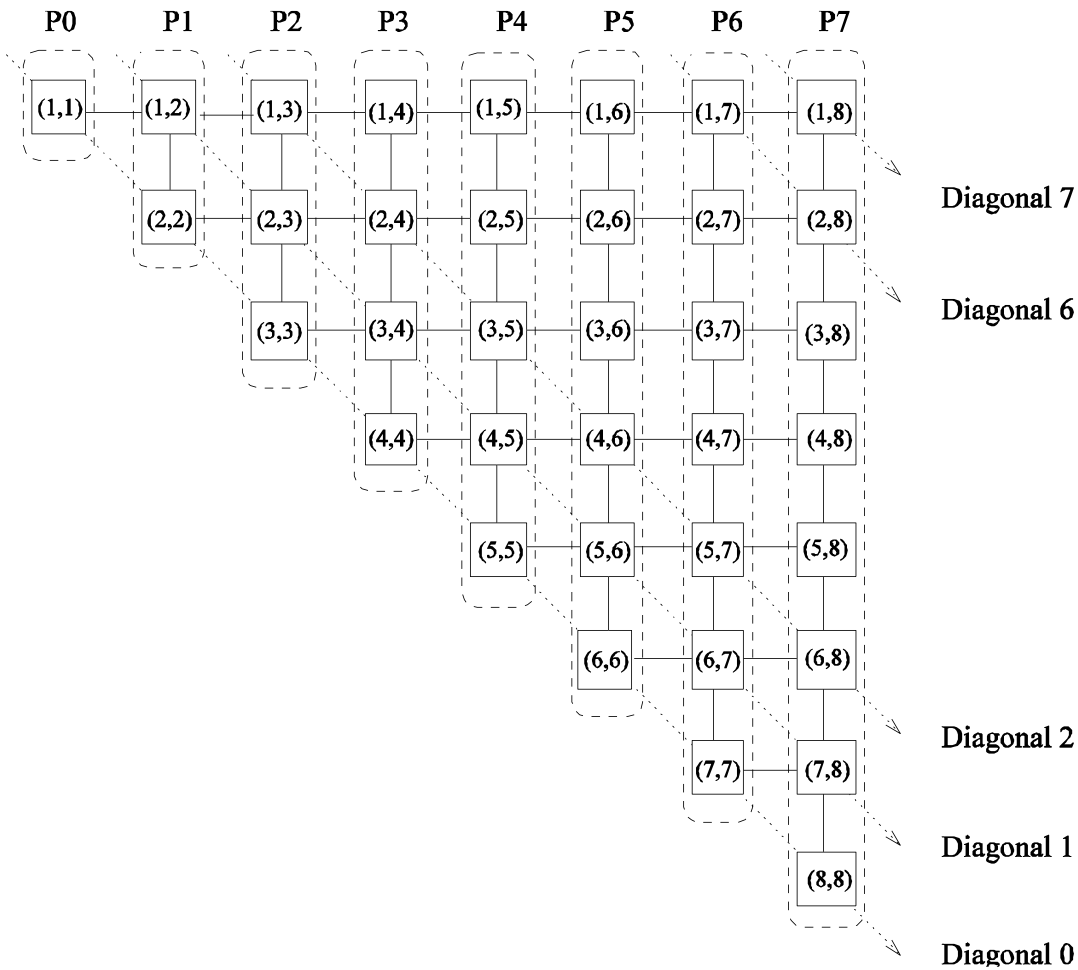
- The goal of **finding *C[1,n]*** is accomplished in a **bottom-up fashion**.
- Visualize this by thinking of **filling in the *C* table** diagonally. Entries in diagonal *l* corresponds to the cost of multiplying matrix chains of length *l+1*.
- The value of *C[i,j]* is computed as ***min{C[i,k] + C[k+1,j] +*** $\boldsymbol{r_{i-1}r_kr_j}$***}***, where ***k* can take values from *i* to *j-1***.
- Computing *C[i,j]* requires that we evaluate *(j-i)* terms and select their minimum.
- The computation of each term takes time t_c, and the **computation of *C[i,j]* takes time** $\boldsymbol{(j-i)t_c}$. Each entry in diagonal *l* can be computed in time lt_c.

Optimal Matrix-Parenthesization Problem

- The algorithm computes ***(n-1)* chains of length two**. This takes time $(n-1)t_c$; **computing *n-2* chains of length three** takes time $(n-2)2t_c$. In **the final step, the algorithm computes one chain of length *n*** in time $1(n-1)t_c$.

- It follows that the **serial time is $\Theta(n^3)$**.

Optimal Matrix-Parenthesization Problem



The **diagonal order of computation** for the optimal matrix-parenthesization problem.

Parallel Optimal Matrix-Parenthesization Problem

- Consider a **logical ring of processors**. In step *l*, each processor computes a single element belonging to the l^{th} diagonal.
- On computing the assigned value of the element in table *C*, **each processor sends its value to all other processors using an all-to-all broadcast**.
- The **next value** can then be **computed locally**.
- The total time required to compute the entries along diagonal *l* is $\boldsymbol{lt_c+t_s \log n+t_w(n-1)}$.
- The corresponding parallel time is given by:

$$T_P = \sum_{l=1}^{n-1} (lt_c + t_s \log n + t_w(n-1)),$$

$$= \frac{(n-1)(n)}{2} t_c + t_s(n-1)\log n + t_w(n-1)^2.$$